Serdica
Mathematical Journal
Сердика
Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://serdica.math.bas.bg
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **50** (2024), 251–268

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY MILLER–ROSS-TYPE POISSION DISTRIBUTION SERIES

Aditya Lagad, Rajkumar N. Ingle, Pinninti Thirupathi Reddy

Communicated by B. Draganov

Abstract. In this work, we introduce and investigate a new class of analytic functions in the open unit disc $\mathbb{U}$ with negative coefficients defined by the Miller Ross function. The object of the present paper is to determine coefficient estimates, distortion bounds, radii of starlike and convexity, extreme points, Hadamard product and closure property belonging to this class.

1. Introduction. There are many branches of complex analysis, but Geometric Function Theory is one of the most important. Basically, it deals with the geometric properties of analytic functions. Complex analysis is widely used in various fields of mathematics, such as pure and applied mathematics. In the literature, several researchers have studied certain geometric properties for some special classes of univalent functions such as problems for studying the geometric properties (including univalency, starlikeness, or convexity) of some classes

https://doi.org/10.55630/serdica.2024.50.251-268

2020 *Mathematics Subject Classification:* 30C45.
Key words: analytic, starlike, convex, distortion, convolution.

of analytic functions (in the unit disk) associated with some special functions have always been attracted by several researchers. In recent years, the distribution of random variables has attracted excessive interest. Probability density functions perform an essential role in statistics and the concept of probability, particularly for distributions. There are numerous forms of distribution from situations of real existence, together with the binomial distribution, Poisson distribution and hypergeometric distribution. In the theory of geometric functions, simple distribution, along with Pascal, Poisson, logarithmic, binomial, beta negative binomial has been partially studied from a theoretical point of view (see [1, 5, 15, 16]) and two parameters of the Mittag-Leffler-type probability distribution (see [6, 21, 13]). Let us now recall some known definitions and results in Geometric Function Theory.

Let $\mathcal{A}$ denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of $\mathcal{A}$ consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of $\mathcal{A}$ consisting of functions $u(z)$ which are all univalent in $\mathbb{U}$. A function $u \in \mathcal{A}$ is a starlike function of the order ξ, $0 \leq \xi < 1$, if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \xi, \quad z \in \mathbb{U}. \tag{1.2}$$

We denote this class with $S^*(\xi)$. A function $u \in \mathcal{A}$ is a convex function of the order ξ, $0 \leq \xi < 1$, if it satisfies

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \xi, \quad z \in \mathbb{U}. \tag{1.3}$$

We denote this class with $K(\xi)$. Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in $\mathbb{U}$ respectively. For $u \in \mathcal{A}$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{1.4}$$

their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * u)(z), \quad (z \in \mathbb{U}). \tag{1.5}$$

Note that $u * g \in \mathcal{A}$.

Let T denotes the class of functions analytic in $\mathbb{U}$ that are of the form

$$u(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \quad (z \in \mathbb{U}) \tag{1.6}$$

and let $T^*(\xi) = T \cap S^*(\xi)$, $C(\xi) = T \cap K(\xi)$. The class $T^*(\xi)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [20].

Miller and Ross proposed the following special function in their monograph (p. 314, [12]), which is called now the Miller-Ross function defined as

$$E_{\nu,c}(z) = z^{\nu} e^{cz} \gamma^*(\nu, cz),$$

where γ^* is the incomplete gamma function. Using the properties of the incomplete gamma functions the Miller-Ross function can easily be written as

$$E_{\nu,\ c}(z) = z^{\nu} \sum_{k=0}^{\infty} \frac{(cz)^k}{\Gamma(k+\nu+1)}; \quad z, c, \nu \in \mathbb{C}. \tag{1.7}$$

In this paper, we shall restrict our attention to the case of real-valued $c > 0$ and $z \in \mathbb{U}$. It is clear that the Miller Ross function $E_{\nu,\ c}(z)$ does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of Miller-Ross function [4]:

$$\begin{aligned} E_{\nu,\ c}(z) &= z^{1-\nu} \Gamma(\nu+1) E_{\nu,\ c}(z) \\ &= z + \sum_{k=2}^{\infty} \frac{c^{k-1} \Gamma(\nu+1)}{\Gamma(k+\nu)} z^k \end{aligned} \tag{1.8}$$

For $c, \nu \in \mathbb{C}$, we can write the following

$$\begin{aligned} E_{\nu,\ c}(1) - 1 &= \sum_{k=2}^{\infty} \frac{c^{k-1} \Gamma(\nu+1)}{\Gamma(k+\nu)}, \\ E'_{\nu,\ c}(1) - 1 &= \sum_{k=2}^{\infty} \frac{k\ c^{k-1} \Gamma(\nu+1)}{\Gamma(k+\nu)}, \\ E''_{\nu,\ c}(1) &= \sum_{k=2}^{\infty} \frac{k(k-1) c^{k-1} \Gamma(\nu+1)}{\Gamma(k+\nu)}. \end{aligned}$$

In recent years, a large literature has evolved on the use of distribution series such as Poisson, Pascal, Borel, etc., in geometric function theory. Many researchers have examined some important features in the field of geometric function theory, such as coefficient estimates, inclusion relations, and conditions of being in some known classes, using different probability distributions, see for example [10, 11, 12, 13, 14, 15].

We now recall that a discrete random variable X whose probability mass function is given by

$$P[X = i] = \frac{e^{-m}m^i}{i!}, \quad i = 0, 1, 2 \ldots, \quad m > 0$$

is said to have a Poisson distribution with parameter m.

Recently, Porwal and Dixit [17] introduced Mittag–Leffler-type Poisson distribution and obtain moments, moment generating function. Bajpai [3] introduced Mittag–Leffler-type Poisson distribution series. Lately Srivastava et al.[22] introduced the Poisson distribution which is a two-parameter Mittag–Leffler-type Poisson distribution. Motivated by results on connections between various subclasses of analytic univalent functions using special functions and distribution series [2, 7, 21, 18, 19, 8] we obtain coefficient inequalities, distortion theorem, radii of starlike, convex, convex linear combination and convolution property for the Miller Ross-type Poisson distribution series to be in classes.

The applications of the Miller–Ross function is a powerful tool in theoretical research and practical applications involving univalent functions and complex analysis. Its versatility in providing insights into coefficient bounds, growth, and distortion properties makes it an essential component in these fields.

The contributions of the Miller-Ross function to science and mathematics are substantial. It has enhanced our understanding of univalent functions, provided valuable tools for analyzing geometric and analytical properties, and has had practical implications in various fields such as fluid dynamics, signal processing, and computational analysis.

First, we recall the definition of the Miller Ross-type distribution.

The probability mass function of the Miller Ross-type Poisson distribution is given by

$$P_{\nu,c}(m,k) = \frac{m^\nu (cm)^k}{E_{\nu,c}(m)\Gamma(k+\nu+1)}, \quad k = 0, 1, 2, \ldots, \tag{1.9}$$

where $\nu > -1, c > 0$ and $E_{\nu,\ c}(z)$ is Miller-Ross function given in (1.7).

The Miller Ross-type Poisson distribution series is defined by

$$\mathbb{F}^m_{\nu,c}(z) = z + \sum_{k=2}^{\infty} \frac{m^\nu (cm)^{k-1}}{\Gamma(k+\nu)E_{\nu,c}(m)} z^k, \quad z \in \mathbb{U}. \tag{1.10}$$

(see [18], see also [22]). Furthermore, using the convolution (or Hadamard product), we define

$$\begin{aligned} \mathbb{K}^m_{\nu,c}u(z) &= \mathbb{F}^m_{\nu,c}(z) * u(z) \\ &= z + \sum_{k=2}^{\infty} \frac{m^\nu (cm)^{k-1}}{\Gamma(k+\nu)E_{\nu,c}(m)} a_k z^k \\ &= z + \sum_{k=2}^{\infty} \Phi^\nu_c(k,m) a_k z^k, \end{aligned} \tag{1.11}$$

where

$$\Phi^\nu_c(k,m) = \frac{m^\nu (cm)^{k-1}}{\Gamma(k+\nu)E_{\nu,c}(m)}. \tag{1.12}$$

Inspired by the work of [9, 11, 14], we introduce the new subclass involving Miller–Ross-type poission distribution series $\mathbb{K}^m_{\nu,c}u(z)$, as below:

Definition 1.1. *For $0 \le \omega < 1$, $0 \le \sigma < 1$ and $0 < \varsigma < 1$, we let $TS^m_{\nu,c}(\omega,\sigma,\varsigma)$ be the subclass of u consisting of functions of the form (1.6) and its geometrical condition satisfy*

$$\left| \frac{\omega \left((\mathbb{K}^m_{\nu,c}u(z))' - \frac{\mathbb{K}^m_{\nu,c}u(z)}{z} \right)}{\sigma (\mathbb{K}^m_{\nu,c}u(z))' + (1-\omega)\frac{\mathbb{K}^m_{\nu,c}u(z)}{z}} \right| < \varsigma, \; z \in \mathbb{U}$$

where $\mathbb{K}^m_{\nu,c}u(z)$, is given by (1.11).

2. Coefficient inequality. In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $TS^m_{\nu,c}(\omega,\sigma,\varsigma)$.

Theorem 2.1. *Let the function u be defined by (1.6). Then $u \in TS^m_{\nu,c}(\omega,\sigma,\varsigma)$ if and only if*

$$\sum_{k=2}^{\infty} [\omega(k-1) + \varsigma(k\sigma + 1 - \omega)]\Phi^\nu_c(k,m) a_k \le \varsigma(\sigma + (1-\omega)), \tag{2.1}$$

where $0 < \varsigma < 1, 0 \leq \omega < 1, 0 \leq \sigma < 1$ *and* $0 \leq \vartheta < 1$. *The result* (2.1) *is sharp for the function*

$$u(z) = z - \frac{\varsigma(\sigma + (1-\omega))}{[\omega(k-1) + \varsigma(k\sigma + 1 - \omega)]\Phi_c^\nu(k,m)} z^k, \ k \geq 2.$$

P r o o f. Suppose that the inequality (2.1) holds true and $|z| = 1$. Then we obtain

$$\begin{aligned}
&\left|\omega\left((\mathbb{K}_{\nu,c}^m u(z))' - \frac{\mathbb{K}_{\nu,c}^m u(z)}{z}\right)\right| - \varsigma\left|\sigma\left(\mathbb{K}_{\nu,c}^m u(z))' + (1-\omega)\frac{\mathbb{K}_{\nu,c}^m u(z)}{z}\right)\right| \\
&= \left|-\omega\sum_{k=2}^{\infty}(k-1)\Phi_c^\nu(k,m)a_k z^{k-1}\right| \\
&\quad - \varsigma\left|\sigma + (1-\omega) - \sum_{k=2}^{\infty}(k\sigma + 1 - \omega)\Phi_c^\nu(k,m)a_k z^{k-1}\right| \\
&\leq \sum_{k=2}^{\infty}[\omega(k-1) + \varsigma(k\sigma + 1 - \omega)]\Phi_c^\nu(k,m)a_k - \varsigma(\sigma + (1-\omega)) \\
&\leq 0.
\end{aligned}$$

Hence, by maximum modulus principle,$u \in TS_{\nu,c}^m(\omega,\sigma,\varsigma)$. Now assume that $u \in TS_{\nu,c}^m(\omega,\sigma,\varsigma)$ so that

$$\left|\frac{\omega\left((\mathbb{K}_{\nu,c}^m u(z))' - \frac{\mathbb{K}_{\nu,c}^m u(z)}{z}\right)}{\sigma(\mathbb{K}_{\nu,c}^m u(z))' + (1-\omega)\frac{\mathbb{K}_{\nu,c}^m u(z)}{z}}\right| < \varsigma, \ z \in \mathbb{U}$$

Hence,

$$\left|\omega\left((\mathbb{K}_{\nu,c}^m u(z))' - \frac{\mathbb{K}_{\nu,c}^m u(z)}{z}\right)\right| < \varsigma\left|\sigma\left(\mathbb{K}_{\nu,c}^m u(z))' + (1-\omega)\frac{\mathbb{K}_{\nu,c}^m u(z)}{z}\right)\right|.$$

Therefore, we get

$$\begin{aligned}
&\left|-\sum_{k=2}^{\infty}\omega(k-1)\Phi_c^\nu(k,m)a_n z^{k-1}\right| \\
&< \varsigma\left|\sigma + (1-\omega) - \sum_{k=2}^{\infty}(k\sigma + 1 - \omega)\Phi_c^\nu(k,m)a_k z^{k-1}\right|.
\end{aligned}$$

Thus,

$$\sum_{k=2}^{\infty}[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi_c^{\nu}(k,m)a_k \leq \varsigma(\sigma+(1-\omega))$$

and this completes the proof. □

Corollary 2.2. *Let the function* $u \in TS_{\nu,c}^{m}(\omega,\sigma,\varsigma)$*. Then*

$$a_k \leq \frac{\varsigma(\sigma+(1-\omega))}{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi_c^{\nu}(k,m)}z^k, \;\; k \geq 2.$$

3. Distortion and covering theorem. We introduce the growth and distortion theorems for the functions in the class $TS_{\nu,c}^{m}(\omega,\sigma,\varsigma)$.

Theorem 3.1. *Let the function* $u \in TS_{\nu,c}^{m}(\omega,\sigma,\varsigma)$*. Then*

$$|z| - \frac{\varsigma(\sigma+(1-\omega))}{\Phi_c^{\nu}(2,m)[\omega+\varsigma(2\sigma+1-\omega)]}|z|^2 \leq |u(z)| \leq |z| + \frac{\varsigma(\sigma+(1-\omega))}{\Phi_c^{\nu}(2,m)[\omega+\varsigma(2\sigma+1-\omega)]}|z|^2.$$

The result is sharp and attained

$$u(z) = z - \frac{\varsigma(\sigma+(1-\omega))}{\Phi_c^{\nu}(2,m)[\omega+\varsigma(2\sigma+1-\omega)]}z^2.$$

Proof.

$$|u(z)| = \left|z - \sum_{k=2}^{\infty} a_k z^k\right| \leq |z| + \sum_{k=2}^{\infty} a_k |z|^k \leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k.$$

By Theorem 2.1, we get

$$\sum_{k=2}^{\infty} a_k \leq \frac{\varsigma(\sigma+(1-\omega))}{[\omega+\varsigma(2\sigma+1-\omega)]\Phi_c^{\nu}(k,m)}. \tag{3.1}$$

Thus,

$$|u(z)| \leq |z| + \frac{\varsigma(\sigma+(1-\omega))}{\Phi_c^{\nu}(2,m)[\omega+\varsigma(2\sigma+1-\omega)]}|z|^2.$$

Also,

$$
\begin{aligned}
|u(z)| &\geq |z| - \sum_{k=2}^{\infty} a_k |z|^k \\
&\geq |z| - |z|^2 \sum_{k=2}^{\infty} a_k \\
&\geq |z| - \frac{\varsigma(\sigma + (1-\omega))}{\Phi_c^{\nu}(2,m)[\omega + \varsigma(2\sigma + 1 - \omega)]} |z|^2.
\end{aligned}
$$

□

Theorem 3.2. *Let $u \in TS_{\nu,c}^{m}(\omega, \sigma, \varsigma)$. Then*

$$
1 - \frac{2\varsigma(\sigma + (1-\omega))}{\Phi_c^{\nu}(2,m)[\omega + \varsigma(2\sigma + 1 - \omega)]} |z| \leq |u'(z)| \leq 1 + \frac{2\varsigma(\sigma + (1-\omega))}{\Phi_c^{\nu}(2,m)[\omega + \varsigma(2\sigma + 1 - \omega)]} |z|
$$

with equality for

$$
u(z) = z - \frac{2\varsigma(\sigma + (1-\omega))}{\Phi_c^{\nu}(2,m)[\omega + \varsigma(2\sigma + 1 - \omega)]} z^2.
$$

Proof. Notice that

$$
\begin{aligned}
&\Phi_c^{\nu}(2,m)[\omega + \varsigma(2\sigma + 1 - \omega)] \sum_{k=2}^{\infty} k a_k \\
&\leq \sum_{k=2}^{\infty} n[\omega(k-1) + \varsigma(k\sigma + 1 - \omega)] \Phi_c^{\nu}(k,m) a_k \\
&\leq \varsigma(\sigma + (1-\omega)), \qquad (3.2)
\end{aligned}
$$

from Theorem 2.1. Thus,

$$
\begin{aligned}
|u'(z)| &= \left| 1 - \sum_{k=2}^{\infty} k a_k z^{k-1} \right| \\
&\leq 1 + \sum_{k=2}^{\infty} k a_k |z|^{k-1} \\
&\leq 1 + |z| \sum_{k=2}^{\infty} k a_k
\end{aligned}
$$

$$\leq 1+|z|\frac{2\varsigma(\sigma+(1-\omega))}{\Phi_c^{\nu}(2,m)[\omega+\varsigma(2\sigma+1-\omega)]}. \tag{3.3}$$

On the other hand

$$\begin{aligned}|u'(z)| &= \left|1-\sum_{k=2}^{\infty}ka_k z^{k-1}\right| \\ &\geq 1-\sum_{k=2}^{\infty}ka_k|z|^{k-1} \\ &\geq 1-|z|\sum_{k=2}^{\infty}ka_k \\ &\geq 1-|z|\frac{2\varsigma(\sigma+(1-\omega))}{\Phi_c^{\nu}(2,m)[\omega+\varsigma(2\sigma+1-\omega)]}. \qquad (3.4)\end{aligned}$$

Combining (3.3) and (3.4), we get the result. □

4. Radii of starlikeness, convexity and close-to-convexity. In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $TS_{\nu,c}^{m}(\omega,\sigma,\varsigma)$.

Theorem 4.1. *Let $u \in TS_{\nu,c}^{m}(\omega,\sigma,\varsigma)$. Then u is starlike in $|z|<R_1$ of order $\wp$, $0\leq\wp<1$, where*

$$R_1=\inf_k\left\{\frac{(1-\wp)(\omega(k-1)+\varsigma(k\sigma+1-\omega))\Phi_c^{\nu}(k,m)}{(k-\wp)\varsigma(\sigma+(1-\omega))}\right\}^{\frac{1}{k-1}},\ k\geq 2. \tag{4.1}$$

P r o o f. u is starlike of order $\wp, 0\leq\wp<1$ if

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\}>\wp.$$

Thus, it is enough to show that

$$\left|\frac{zu'(z)}{u(z)}-1\right|=\left|\frac{-\sum\limits_{k=2}^{\infty}(k-1)a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k z^{k-1}}\right|\leq\frac{\sum\limits_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k|z|^{k-1}}.$$

Thus,

$$\left|\frac{zu'(z)}{u(z)}-1\right|\leq 1-\wp \quad \text{if} \quad \sum_{k=2}^{\infty}\frac{(k-\wp)}{(1-\wp)}a_k|z|^{k-1}\leq 1. \tag{4.2}$$

Hence, by Theorem 2.1, (4.2) will be true if

$$\frac{k-\wp}{1-\wp}|z|^{k-1} \leq \frac{(\omega(k-1)+\varsigma(k\sigma+1-\omega))\Phi_c^{\nu}(k,m)}{\varsigma(\sigma+(1-\omega)}$$

or if

$$|z| \leq \left[\frac{(1-\wp)(\omega(k-1)+\varsigma(k\sigma+1-\omega))\Phi_c^{\nu}(k,m)}{(k-\wp)\varsigma(\sigma+(1-\omega))}\right]^{\frac{1}{k-1}}, k \geq 2. \tag{4.3}$$

The theorem follows easily from (4.3). □

Theorem 4.2. *Let $u \in TS_{\nu,c}^{m}(\omega,\sigma,\varsigma)$. Then u is convex in $|z| < R_2$ of order $\wp, 0 \leq \wp < 1$, where*

$$R_2 = \inf_k \left\{\frac{(1-\wp)(\omega(k-1)+\varsigma(k\sigma+1-\omega))\Phi_c^{\nu}(k,m)}{k(k-\wp)\varsigma(\sigma+(1-\omega))}\right\}^{\frac{1}{k-1}}, \quad k \geq 2. \tag{4.4}$$

Proof. u is convex of order $\wp, 0 \leq \wp < 1$ if

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \wp.$$

Thus, it is enough to show that

$$\left|\frac{zu''(z)}{u'(z)}\right| = \left|\frac{-\sum_{k=2}^{\infty} k(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty} k a_k z^{k-1}}\right| \leq \frac{\sum_{k=2}^{\infty} k(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty} k a_k|z|^{k-1}}.$$

Thus,

$$\left|\frac{zu''(z)}{u'(z)}\right| \leq 1-\wp \quad \text{if} \quad \sum_{k=2}^{\infty}\frac{k(k-\wp)}{(1-\wp)}a_k|z|^{k-1} \leq 1. \tag{4.5}$$

Hence, by Theorem 2.1, (4.5) will be true if

$$\frac{k(k-\wp)}{1-\wp}|z|^{k-1} \leq \frac{(\omega(k-1)+\varsigma(k\sigma+1-\omega))\Phi_c^{\nu}(k,m)}{\varsigma(\sigma+(1-\omega)}$$

or if

$$|z| \leq \left[\frac{(1-\wp)(\omega(k-1)+\varsigma(k\sigma+1-\omega))\Phi_c^{\nu}(k,m)}{k(k-\wp)\varsigma(\sigma+(1-\omega))}\right]^{\frac{1}{k-1}}, k \geq 2. \tag{4.6}$$

The theorem follows easily from (4.6). □

Theorem 4.3. *Let $u \in TS^m_{\nu,c}(\omega,\sigma,\varsigma)$. Then u is close-to-convex in $|z| < R_3$ of order $\wp$, $0 \leq \wp < 1$, where*

$$R_3 = \inf_n \left\{ \frac{(1-\wp)(\omega(k-1)+\varsigma(k\sigma+1-\omega))\Phi^\nu_c(k,m)}{k\varsigma(\sigma+(1-\omega))} \right\}^{\frac{1}{k-1}}, \ k \geq 2. \tag{4.7}$$

Proof. u is close-to-convex of order $\wp, 0 \leq \wp < 1$ if

$$\Re\left\{u'(z)\right\} > \wp.$$

Thus, it is enough to show that

$$|u'(z)-1| = \left| -\sum_{k=2}^{\infty} k a_k z^{n-1} \right| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus,

$$|u'(z)-1| \leq 1-\wp \quad \text{if} \quad \sum_{k=2}^{\infty} \frac{k}{(1-\wp)} a_k |z|^{k-1} \leq 1. \tag{4.8}$$

Hence, by Theorem 2.1, (4.8) will be true if

$$\frac{k}{1-\wp}|z|^{k-1} \leq \frac{(\omega(k-1)+\varsigma(k\sigma+1-\omega))\Phi^\nu_c(k,m)}{\varsigma(\sigma+(1-\omega)}$$

or if

$$|z| \leq \left[\frac{(1-\wp)(\omega(k-1)+\varsigma(k\sigma+1-\omega))\Phi^\nu_c(k,m)}{k\varsigma(\sigma+(1-\omega))} \right]^{\frac{1}{k-1}}, n \geq 2. \tag{4.9}$$

The theorem follows easily from (4.9). □

5. Extreme points. In the following theorem, we obtain extreme points for the class $TS^m_{\nu,c}(\omega,\sigma,\varsigma)$.

Theorem 5.1. *Let $u_1(z) = z$ and*

$$u_k(z) = z - \frac{\varsigma(\sigma+(1-\omega))}{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi^\nu_c(k,m)} z^k, \ \textit{for } n = 2,3,\ldots$$

Then $u \in TS^m_{\nu,c}(\omega,\sigma,\varsigma)$ if and only if it can be expressed in the form

$$u(z) = \sum_{k=1}^{\infty} \theta_k u_k(z), \ \textit{where } \theta_k \geq 0 \textit{ and } \sum_{k=1}^{\infty} \theta_k = 1.$$

P r o o f. Assume that $u(z)=\sum\limits_{k=1}^{\infty}\theta_k u_k(z)$, hence we get

$$u(z)=z-\sum_{k=2}^{\infty}\frac{\varsigma(\sigma+(1-\omega))\theta_n}{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi_c^{\nu}(k,m)}z^k.$$

Now, $u\in TS_{\nu,c}^{m}(\omega,\sigma,\varsigma)$, since

$$\sum_{k=2}^{\infty}\frac{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi_c^{\nu}(k,m)}{\varsigma(\sigma+(1-\omega))}\times\frac{\varsigma(\sigma+(1-\omega))\theta_n}{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi_c^{\nu}(k,m)}$$

$$=\sum_{k=2}^{\infty}\theta_k=1-\theta_1\leq 1.$$

Conversely, suppose $u\in TS_{\nu,c}^{m}(\omega,\sigma,\varsigma)$. Then we show that u can be written in the form $\sum\limits_{k=1}^{\infty}\theta_k u_k(z)$.

Now, $u\in TS_{\nu,c}^{m}(\omega,\sigma,\varsigma)$ implies from Theorem 2.1

$$a_k\leq\frac{\varsigma(\sigma+(1-\omega))}{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi_c^{\nu}(k,m)}.$$

Setting

$$\theta_n=\frac{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi_c^{\nu}(k,m)}{\varsigma(\sigma+(1-\omega))}a_k,\quad k=2,3,\ldots$$

and $\theta_1=1-\sum\limits_{k=2}^{\infty}\theta_k$, we obtain $u(z)=\sum\limits_{k=1}^{\infty}\theta_k u_k(z)$. □

6. Hadamard product. In the following theorem, we obtain the convolution result for functions belongs to the class $TS_{\nu,c}^{m}(\omega,\sigma,\varsigma)$.

Theorem 6.1. *Let* $u,g\in TS(\omega,\sigma,\varsigma,\vartheta)$. *Then* $u*g\in TS(\omega,\sigma,\zeta,\vartheta)$ *for*

$$u(z)=z-\sum_{k=2}^{\infty}a_kz^k,\ g(z)=z-\sum_{k=2}^{\infty}b_kz^k\ \text{and}\ (u*g)(z)=z-\sum_{k=2}^{\infty}a_kb_kz^k,$$

where

$$\zeta\geq\frac{\varsigma^2(\sigma+(1-\omega))\omega(k-1)}{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]^2\Phi_c^{\nu}(k,m)-\varsigma^2(\sigma+(1-\omega))(k\sigma+1-\omega)}.$$

Proof. $u \in TS^m_{\nu,c}(\omega,\sigma,\varsigma)$ and so

$$\sum_{k=2}^{\infty} \frac{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi^{\nu}_{c}(k,m)}{\varsigma(\sigma+(1-\omega))} a_k \leq 1, \tag{6.1}$$

and

$$\sum_{k=2}^{\infty} \frac{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi^{\nu}_{c}(k,m)}{\varsigma(\sigma+(1-\omega))} b_k \leq 1. \tag{6.2}$$

We have to find the smallest number ζ such that

$$\sum_{k=2}^{\infty} \frac{[\omega(k-1)+\zeta(k\sigma+1-\omega)]\Phi^{\nu}_{c}(k,m)}{\zeta(\sigma+(1-\omega))} a_k b_k \leq 1. \tag{6.3}$$

By Cauchy-Schwarz inequality

$$\sum_{k=2}^{\infty} \frac{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi^{\nu}_{c}(k,m)}{\varsigma(\sigma+(1-\omega))} \sqrt{a_k b_k} \leq 1. \tag{6.4}$$

Therefore, it is enough to show that

$$\frac{[\omega(k-1)+\zeta(k\sigma+1-\omega)]\Phi^{\nu}_{c}(k,m)}{\zeta(\sigma+(1-\omega))} a_k b_k \leq \frac{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi^{\nu}_{c}(k,m)}{\varsigma(\sigma+(1-\omega))} \sqrt{a_k b_k}.$$

That is

$$\sqrt{a_n b_n} \leq \frac{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\zeta}{[\omega(k-1)+\zeta(k\sigma+1-\omega)]\varsigma}. \tag{6.5}$$

From (6.4)

$$\sqrt{a_k b_k} \leq \frac{\varsigma(\sigma+(1-\omega))}{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi^{\nu}_{c}(k,m)}.$$

Thus it is enough to show that

$$\frac{\varsigma(\sigma+(1-\omega))}{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\Phi^{\nu}_{c}(k,m)} \leq \frac{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]\zeta}{[\omega(k-1)+\zeta(k\sigma+1-\omega)]\varsigma},$$

which simplifies to

$$\zeta \geq \frac{\varsigma^2(\sigma+(1-\omega))\omega(k-1)}{[\omega(k-1)+\varsigma(k\sigma+1-\omega)]^2\Phi^{\nu}_{c}(k,m) - \varsigma^2(\sigma+(1-\omega))(k\sigma+1-\omega)}.$$

□

7. Closure theorems. We shall prove the following closure theorems for the class $TS_{\nu,c}^{m}(\omega,\sigma,\varsigma)$.

Theorem 7.1. *Let* $u_j \in TS_{\nu,c}^{m}(\omega,\sigma,\varsigma), j = 1, 2, \ldots, s$. *Then*

$$g(z) = \sum_{j=1}^{s} c_j u_j(z) \in TS_{\nu,c}^{m}(\omega,\sigma,\varsigma).$$

For $u_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k$, *where* $\sum_{j=1}^{s} c_j = 1$.

Proof.

$$\begin{aligned} g(z) &= \sum_{j=1}^{s} c_j u_j(z) \\ &= z - \sum_{k=2}^{\infty} \sum_{j=1}^{s} c_j a_{k,j} z^k \\ &= z - \sum_{k=2}^{\infty} e_k z^k, \end{aligned}$$

where $e_k = \sum_{j=1}^{s} c_j a_{k,j}$. Thus $g(z) \in TS_{\nu,c}^{m}(\omega,\sigma,\varsigma)$ if

$$\sum_{k=2}^{\infty} \frac{[\omega(k-1) + \varsigma(k\sigma + 1 - \omega)]\Phi_c^{\nu}(k,m)}{\varsigma(\sigma + (1-\omega))} e_k \leq 1,$$

that is, if

$$\begin{aligned} &\sum_{k=2}^{\infty} \sum_{j=1}^{s} \frac{[\omega(k-1) + \varsigma(k\sigma + 1 - \omega)]\Phi_c^{\nu}(k,m)}{\varsigma(\sigma + (1-\omega))} c_j a_{k,j} \\ =&\sum_{j=1}^{s} c_j \sum_{k=2}^{\infty} \frac{[\omega(k-1) + \varsigma(k\sigma + 1 - \omega)]\Phi_c^{\nu}(k,m)}{\varsigma(\sigma + (1-\omega))} a_{k,j} \\ \leq&\sum_{j=1}^{s} c_j = 1. \end{aligned}$$

□

Theorem 7.2. *Let $u, g \in TS^m_{\nu,c}(\omega, \sigma, \varsigma)$. Then*

$$h(z) = z - \sum_{k=2}^{\infty}(a_k^2 + b_k^2)z^k \in TS^m_{\nu,c}(\omega, \sigma, \varsigma), \textit{where}$$

$$\zeta \geq \frac{2\omega(k-1)\varsigma^2(\sigma + (1-\omega))}{[\omega(k-1) + \varsigma(k\sigma + 1 - \omega)]^2 \Phi_c^\nu(k, m) - 2\varsigma^2(\sigma + (1-\omega))(k\sigma + 1 - \omega)}.$$

Proof. Since $u, g \in TS^m_{\nu,c}(\omega, \sigma, \varsigma)$, so Theorem 2.1 yields

$$\sum_{k=2}^{\infty}\left[\frac{(\omega(k-1) + \varsigma(k\sigma + 1 - \omega))\Phi_c^\nu(k, m)}{\varsigma(\sigma + (1-\omega))}a_k\right]^2 \leq 1$$

and

$$\sum_{k=2}^{\infty}\left[\frac{(\omega(k-1) + \varsigma(k\sigma + 1 - \omega))\Phi_c^\nu(k, m)}{\varsigma(\sigma + (1-\omega))}b_k\right]^2 \leq 1.$$

We obtain from the last two inequalities

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{(\omega(k-1) + \varsigma(k\sigma + 1 - \omega))\Phi_c^\nu(k, m)}{\varsigma(\sigma + (1-\omega))}\right]^2 (a_k^2 + b_k^2) \leq 1. \tag{7.1}$$

But $h(z) \in TS(\omega, \sigma, \zeta, q, m)$, if and only if

$$\sum_{k=2}^{\infty}\frac{[\omega(k-1) + \zeta(k\sigma + 1 - \omega)]\Phi_c^\nu(k, m)}{\zeta(\sigma + (1-\omega))}(a_k^2 + b_k^2) \leq 1, \tag{7.2}$$

where $0 < \zeta < 1$, however (7.1) implies (7.2) if

$$\frac{[\omega(k-1) + \zeta(k\sigma + 1 - \omega)]\Phi_c^\nu(k, m)}{\zeta(\sigma + (1-\omega))} \leq \frac{1}{2}\left[\frac{(\omega(k-1) + \varsigma(k\sigma + 1 - \omega))\Phi_c^\nu(k, m)}{\varsigma(\sigma + (1-\omega))}\right]^2.$$

Simplifying, we get

$$\zeta \geq \frac{2\omega(k-1)\varsigma^2(\sigma + (1-\omega))}{[\omega(k-1) + \varsigma(k\sigma + 1 - \omega)]^2 \Phi_c^\nu(k, m) - 2\varsigma^2(\sigma + (1-\omega))(k\sigma + 1 - \omega)}.$$

□

Acknowledgements. The authors record their sincere thanks to the Editor and the referees for their valuable comments to revise the paper in present form.

REFERENCES

[1] S. Altinkaya, S. Yalcin. Poisson distribution series for certain subclasses of starlike functions with negative coefficients. *An. Univ. Oradea Fasc. Mat.* **24**, *2* (2017), no. 2, 5–8.

[2] S. Altinkaya, S. Yalçin, S. Çakmak. Some connections between various classes of analytic functions associated with the power series distribution. *Sakarya University J. Sci.* **23**, *5* (2019), 982–985, `https://doi.org/10.16984/saufenbilder.552957`.

[3] D. Bajpai. A study of univalent functions associated with distortion series and q-calculus. M. Phil. thesis, CSJM Univerity, Kanpur, India, 2016.

[4] S. S. Eker, S. Ece. Geometric properties of the Miller-Ross functions. *Iran. J. Sci. Technol. Trans. A Sci.* **46**, *2* (2022), 631–636.

[5] S. M. El-Deeb, T. Bulboacă, J. Dziok. Pascal distribution series connected with certain subclasses of univalent functions. *Kyungpook Math. J.* **59**, **2** (2019), 301–314.

[6] S. M. El-Deeb, G. Murugusundaramoorthy, A. Alburaikan. Bi-Bazilevič functions based on the Mittag-Leffler-type Borel distribution associated with Legendre polynomials. *J. Math. Comput. Sci.* **24**, *3* (2022), 173–183, `http://dx.doi.org/10.22436/jmcs.024.03.05`.

[7] B. A. Frasin, S. Porwal, F. Yousef. Subclasses of starlike and convex functions associated with Mittag- Leffler-type Poisson distribution series. *Montes Taurus J. Pure Appl. Math.* **3**, *3* (2021), 147–154, Article ID: MTJPAM-D-20-00044.

[8] B. A. Frasin. Subclass of analytic functions with negative coefficients related with Miller-Ross-type Poission distribution series. *Acta Univ. Sapientiae Math.* **15**, *1* (2023), 109–122.

[9] B. A. Frasin, L.-I. Cotirla. On Miller-Ross-type Poisson distribution series. *Mathematics* **11**, *18* (2023), 3989, https://doi.org/10.3390/math11183989.

[10] S. Kazımoğlu, E. Deniz, M. Çağlar. Partial sums of the Bessel-Struve kernel function. 3rd International Conference on Mathematical and Related Sciences: Current Trend and Developments, ICMRS'2020, Online meeting, Turkey, November 20–22, 2020, 267–275.

[11] S. Kazımoğlu. Partial Sums of The Miller-Ross Function, Turkish Journal of Science **6**, *3* (2021), 167–173, https://dergipark.org.tr/en/download/article-file/2097482.

[12] K. S. Miller, B. Ross. An introduction to the fractional calculus and fractional differential equations. Wiley-Intersci. Publ. New York, John Wiley & Sons, Inc., 1993.

[13] G. Murugusundaramoorthy, S. M. El-Deeb. Second Hankel determinant for a class of analytic functions of the Mittag-Leffler-type Borel distribution related with Legendre polynomials. *TWMS J. App. and Eng. Math.* **12**, *4* (2022), 1247–1258, https://jaem.isikun.edu.tr/web/images/articles/vol.12.no.4/09.pdf.

[14] G. Murugusundaramoorthy, H. O. Guney, D. Breaz. Starlike Functions of the Miller-Ross-Type Poisson Distribution in the Janowski Domain. *Mathematics* **12**, *6* (2024), 795, https://doi.org/10.3390/math12060795.

[15] W. Nazeer, Q. Mehmood, S. M. Kang, A. Ul Haq. An application of binomial distribution series on certain analytic functions. *J. Comput. Anal. Appl.* **26**, *1* (2019), 11–17.

[16] S. Porwal, M. Kumar. A unified study on starlike and convex functions associated with Poisson distribution series. *Afr. Mat.* **27**, *5–6* (2016), 1021–1027.

[17] S. Porwal, K. K. Dixit. On Mittag-Leffler type Poisson distribution. *Afr. Math.* **28**, *1–2* (2017), 29–34.

[18] B. Şeker, S. S. Eker, B. Çekiç. On subclasses of analytic functions associated with Miller-Rosstype Poisson distribution series. *Sahand Commun. Math. Anal.* **19**, *4* (2022), 69–79.

[19] B. Şeker, S. S. Eker, B. Çekiç. Certain subclasses of analytic functions associated with Miller- Ross-type Poisson distribution series. *Honam Math. J.* **44**, *4* (2022), 504–512.

[20] H. Silverman. Univalent functions with negative coefficients. *Proc. Amer. Math. Soc.* **51** (1975), 109–116.

[21] H. M. Srivastava, S. M. El-Deeb. Fuzzy differential subordinations based upon the Mittag-Leffler type borel distribution. *Symmetry* **13**, *6* (2021), 1023, https://doi.org/10.3390/sym13061023.

[22] H. M. Srivastava, B. Şeker, S. S. Eker, B. Çekiç. A class of Poisson distributions based upon a two parameter Mittag-Leffler type function. *J. Nonlinear Convex Anal.* **24**, *2* (2023), 475–485.

Aditya Lagad
Department of Mathematics
N.E.S. Science College
Nanded – 431 602, Maharashtra, India
e-mail: lagadac@gmail.com

Rajkumar N. Ingle
Department of Mathematics
Bahirji Smarak Mahavidyalay
Basamathnagar – 431 512, Maharashtra, India
e-mail: ingleraju@gmail.com

Pinninti Thirupathi Reddy
Department of Mathematics
DRK Institute of Science and Technology
Bowarmpet – 500 043, Hyderabad, Telangana, India
e-mail: reddypt2@gmail.com

Received May 31, 2024
Accepted September 14, 2024