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A GENERAL EXISTENCE PRINCIPLE FOR FIXED POINT THEOREMS IN ONE PARAMETER CASE OF SOFT D_a-METRIC SPACE

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Abstract. In this article we define orbit and related terminologies for soft sets. We prove some fixed point results for soft D_a- metric space in a specific orbital case and extending various general principles for fixed point theorems in D-metric space to soft D_a-metric spaces.

1. Introduction. Studies involves for obtaining various generalizations in metric spaces are pioneer works in field of mathematical reasearch involving uncertainity problems. The major reaon for this generalizations is that wide classification of metric spaces provides powerfull tool in dealing with uncertainities in various branches of mathematics such as approximation theory, variational inequalities and so on. B. C. Dhage [3] introduced the concept of D-metrics and developed various properties and results in their later attempts. The concept of soft set theory was introduced by Molodtsov [8] in 1999 and it became an

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important tool to deal with mathematical uncertainities. Das and Samanta [2] in 2013 devolped a new calss of metric space named soft metric based on soft points of soft sets. Fixed point results of D-metric spaces and soft metric spaces can be seen in [1, 5, 6, 7] and in other related papers of the same authors. The scope of soft D-metric was first investigated by Gundus Aras et al. [7] and they proposed the notion of Δ-distance on a complete soft D-metric and gave a fixed point theorem on it.

In this paper we are introducing the concept of orbit for the case soft points and then we are defining soft D_a-metric for the single parameter a as in [5]. And for the single parameter case of soft D_a-metric space we establish several fixed point results and its general principles with valid proofs.

The main results of this paper are devided in to two sections. In section 3, we define orbit and realted terminologies. Later we discusses an important general fixed point principle and some general results as corollaries. Section 4 deals with α_a-condensing maps and fixed point results involving α_a-condensing maps in soft D_a-metric space.

2. Priliminaries.

Definition 2.1 ([8]). *A pair (F, E) is called a soft set over X, where F is a mapping given by $F : E \longrightarrow P(X)$. That is we can understand a soft set as a parametrized family of subsets of the set X. For each $a \in E, F(a)$ denotes the set of $a-$elements or $a-$approximate elements of the soft set (F, E).*

Definition 2.2 ([3]). *Let X be a non empty set. A function $D : X^3 \longrightarrow [0, \infty)$ is called a $D-$ metric if the following conditions are satisfied:*

(1) $D(x, y, z) \geq 0$ *for all* $x, y, z \in X$ *and equality holds if and only if* $x = y = z$,

(2) $D(x, y, z) = D(x, z, y) = D(y, x, z) = \cdots$

(3) $D(x, y, z) \leq D(x, y, u) + D(x, u, z) + D(u, y, z)$, *for all* $x, y, z, u \in X$

Then the pair (X, D) is called a D-metric space.

Let us denote the absolute soft set by $\tilde{X}$ and the collection of all soft points of $\tilde{X}$ by $SP(\tilde{X})$. E be a non empty set of parameters and $R(E^*)$ denotes the set of all non negative soft real numbers.

Definition 2.3 ([7]). *A mapping $D : SP(\tilde{X}) \times SP(\tilde{X}) \times SP(\tilde{X}) \longrightarrow R(E^*)$ is called a soft D-metric on the soft set $\tilde{X}$ that satisfies the following conditions, for each soft points $x_a, y_b, z_c, u_d \in SP(\tilde{X})$,*

(1) $D(x_a, y_b, z_c) \geq \tilde{0}$ *and equality holds if and only if* $x_a = y_b = z_c$ *(coincidence)*

(2) $D(x_a, y_b, z_c) = D(y_b, x_a, z_c) = D(x_a, z_c, y_b) = \cdots$ *(Symmetry)*

(3) $D(x_a, y_b, z_c) \leq D(x_a, y_b, u_d) + D(x_a, u_d, z_c) + D(u_d, y_b, z_c)$

Then the set of $\tilde{X}$ *with a soft D-metric is called a soft D-metric space and denoted by* $(\tilde{X}, D, E)$.

Definition 2.4 ([7]). *Let* $(\tilde{X}, D, E)$ *be a soft D-metric space. A soft sequence* $\left\{x^n_{a_n}\right\}$ *converges to* $x_b \in SP(\tilde{X})$ *if for each* $\tilde{\epsilon} > \tilde{0}$*, there exists* $n_0 \in N$ *such that* $\forall m, n > n_0, D(x^n_{a_n}, x^m_{a_m}, x_b) < \tilde{\epsilon}$.

Definition 2.5 ([7]). *A soft sequence* $\left\{x^n_{a_n}\right\}$ *is called a Cauchy sequence if for each* $\tilde{\epsilon} > \tilde{0}$*, there exists* $n_0 \in N$ *such that* $\forall m > n, p > n_0$, $D(x^n_{a_n}, x^m_{a_m}, x^p_{a_p}) < \tilde{\epsilon}$.

Definition 2.6 ([7]). *A soft metric space is complete if every Cauchy sequence is convergent.*

Definition 2.7 ([4]). *Let* $f : X \longrightarrow X$ *The orbit of* f *at the point* $x \in X$ *is the set* $O(x) = \left\{x, fx, f^2x, \dots\right\}$.

Definition 2.8 ([4]). *An orbit of* x *is said to be bounded if there exists a constant* $K > 0$ *such that* $D(u, v, w) \leq K$ *for all* $u, v, w \in O(x)$*. The constant* K *is called* ***D-*** *bound of* $(O(x))$.

Definition 2.9 ([4]). *The D-metric is said to be* f*-orbitally bounded if* $O(x)$ *is bounded for each* $x \in X$.

3. Main results. The main points concerned in this section is providing a definition of orbit for the case of soft sets and detailed study of fixed point results under the soft D_a-metric background on a space endowed with softpoints. All the Theorems and Corollaries are occupied for the case of D-metric space. And here we are extending them for the case of soft D_a-metric space.

Definition 3.1. *Let* $\tilde{X}$ *be an absolute soft set and* $(\tilde{X}, D_a, E)$ *be a soft* D_a*-metric space. Let* $\tilde{f}$ *be a self map on* $SP(\tilde{X})$*. The orbit of* $\tilde{f}$ *at the point* x_a *is the set* $O(x_a) = \left\{x_a, \tilde{f}x_a, \tilde{f}^2x_a, \dots\right\}$.

Definition 3.2. *An orbit of* x_a *is said to be bounded if there exist a constant* $K_a > \tilde{0}$ *such that* $D_a(\alpha_a, \beta_a, \gamma_a) \leq K_a$ *for all* $\alpha_a, \beta_a, \gamma_a \in O(x_a)$*. The constant* K_a *is called a* D_a*-bound of* $O(x_a)$.

Definition 3.3. *A soft D_a-metric space $(\tilde{X}, D_a, E)$ is said to be $\tilde{f}$ orbitally bounded if $O(x_a)$ is bounded for each $x_a \in SP(\tilde{X})$.*

Definition 3.4. *An orbit $O(x_a)$ is called $\tilde{f}$ orbitally complete if every D_a cauchy sequence in $O(x_a)$ converges to a point in $SP(\tilde{X})$.*

Lemma 3.1. *Let $\left\{x^n_{a_n}\right\}$ be a bounded soft sequence with D_a bound K_a satisfying*

$$D_a(x^n_{a_n}, x^{n+1}_{a_{n+1}}, x^m_{a_m}) \leq \lambda^n_a K_a$$

for all positive integers $m > n$, and some $\tilde{0} \leq \lambda_a < \tilde{1}$. Then $\left\{x^n_{a_n}\right\}$ is D_a-Cauchy.

Theorem 3.1. *Let $(\tilde{X}, D_a, E)$ be a soft D_a-metric space, $\tilde{f}$ be a self map of $SP(\tilde{X})$. Suppose that there exists an $x_a \in SP(\tilde{X})$ such that $O(x_a)$ is D_a bounded and $\tilde{f}$-orbitally complete. Suppose also that $\tilde{f}$ satisfies*

$$D_a(\tilde{f}\tilde{x}, \tilde{f}\tilde{y}, \tilde{f}\tilde{z}) \leq \lambda_a \max\left\{D_a(\tilde{x}, \tilde{y}, \tilde{z}), D_a(\tilde{x}, \tilde{f}\tilde{x}, \tilde{z})\right\} \tag{3.1}$$

for $\tilde{x}, \tilde{y}, \tilde{z} \in \overline{O(x_a)}$ for some $\tilde{0} \leq \lambda_a < \tilde{1}$. Then $\tilde{f}$ has a unique fixed point in $SP(\tilde{X})$.

P r o o f. Suppose that there exist an n such that $\tilde{x}_{a_n} = \tilde{x}_{a_{n+1}}$. Then $\tilde{f}$ has $\tilde{x}_{a_n}$ as a fixed point in $SP(\tilde{X})$. Therefore we may assume that all of the $\tilde{x}_{a_n}$ are distinct.

We wish to show that, for any positive integers $m, n, m > n$, that

$$D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+2}}, \tilde{x}_{a_m}) \leq \lambda^n_a K_a,$$

where K_a is the D_a bound of $O(x_a)$. the proof is by induction. From (3.1), for any m,

$$\begin{aligned} D_a(\tilde{x}_{a_1}, \tilde{x}_{a_2}, \tilde{x}_{a_m}) &\leq \lambda_a \max\left\{D_a(\tilde{x}_{a_0}, \tilde{x}_{a_1}, \tilde{x}_{a_{m-1}}), D_a(\tilde{x}_{a_0}, \tilde{x}_{a_1}, \tilde{x}_{a_{m-1}})\right\} \\ &\leq \lambda_a K_a. \end{aligned} \tag{3.2}$$

Again using (3.1)

$$D_a(\tilde{x}_a, \tilde{x}_{a_3}, \tilde{x}_{a_m}) \leq \lambda_a \max\left\{D_a(\tilde{x}_{a_2}, \tilde{x}_{a_3}, \tilde{x}_{a_{m-1}}), D_a(\tilde{x}_{a_1}, \tilde{x}_{a_2}, \tilde{x}_{a_{m-1}})\right\}. \tag{3.3}$$

Using (3.2)

$$D_a(\tilde{x}_{a_2}, \tilde{x}_{a_3}, \tilde{x}_{a_m}) \leq \lambda_a \max\left\{D_a(\tilde{x}_{a_2}, \tilde{x}_{a_3}, \tilde{x}_{a_{m-1}}), \lambda_a K_a\right\}. \tag{3.4}$$

Inequality (3.4) can be regarded as a recursion formula in m. Therefore

$$\begin{aligned} D_a(\tilde{x}_{a_2}, \tilde{x}_{a_3}, \tilde{x}_{a_m}) &\leq \lambda_a \max\left\{\lambda_a \max\left\{D_a(\tilde{x}_{a_2}, \tilde{x}_{a_3}, \tilde{x}_{a_{m-2}}), \lambda_a K_a\right\}, \lambda_a K_a\right\} \\ &\leq \lambda_a^2 K_a. \end{aligned} \tag{3.5}$$

Assume the induction hypothesis. Then from (3.1),

$$\begin{aligned} &D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+2}}, \tilde{x}_{a_m}) \\ &\qquad \leq \lambda_a \max\left\{D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+2}}, \tilde{x}_{a_{m-1}}), D_a\left(\tilde{x}_{a_n}, \tilde{x}_{a_{n+1}}, \tilde{x}_{a_{m-1}}\right)\right\} \\ &\qquad \leq \lambda_a \max\left\{D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+2}}, \tilde{x}_{a_{m-1}}), \lambda_a^n K_a\right\} \end{aligned} \tag{3.6}$$

Inequality (3.6) can be regarded as a recursion formula in m. Therefore,

(3.7)

$$\begin{aligned} &D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+2}}, \tilde{x}_{a_m}) \\ &\qquad \leq \lambda_a \max\left\{\lambda_a \max\left\{D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+2}}, \tilde{x}_{a_{m-2}}), \lambda_a^n K_a\right\}, \lambda_a^n K_a\right\} \\ &\qquad = \max\left\{\lambda_a^2 D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+2}}, \tilde{x}_{a_{m-2}}), \lambda_a^{n+2} K_a, \lambda_a^{n+1} K_a\right\} \\ &\qquad = \max\left\{\lambda_a^2 D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+2}}, \tilde{x}_{a_{m-2}}), \lambda_a^{n+1} K_a\right\} \\ &\qquad \leq \max\left\{\lambda_a^2 \lambda_a \max\left\{D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+2}}, \tilde{x}_{a_{m-3}}), \lambda_a^{n+1} K_a\right\}, \lambda_a^{n+1} K_a\right\} \\ &\qquad = \max\left\{\lambda_a^3 D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+2}}, \tilde{x}_{a_{m-3}}), \lambda_a^{n+1} K_a\right\} \leq \cdots \\ &\qquad \leq \max\left\{\lambda_a^n D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+2}}, \tilde{x}_{a_{m-n}}), \lambda_a^{n+1} K_a\right\} \\ &\qquad \leq \max\left\{\lambda_a^n \lambda_a \max\left\{D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+2}}, \tilde{x}_{a_{m-n-1}}), \lambda_a^{n+1} K_a\right\}, \lambda_a^{n+1} K_a\right\} \\ &\qquad = \lambda_a^{n+1} K_a, \end{aligned}$$

and $\left\{\tilde{x}_{a_n}^n\right\}$ is D_a-Cauchy by Lemma 3.1. Since $SP(\tilde{X})$ is x_a-orbitally complete, there exist a $\tilde{p_a} \in SP(\tilde{X})$ with $\lim \tilde{x}_{a_n}^n = \tilde{p_a}$.

In (3.1) set $\tilde{x} = \tilde{x}_{a_n}, \tilde{z} = \tilde{p_a}$ to obtain

$$D_a(\tilde{x}_{a_{n+1}}, \tilde{x}_{a_{n+1}} \tilde{f} \tilde{p_a}) \leq \lambda_a \max\left\{D_a(\tilde{x}_{a_n}, \tilde{x}_{a_{n+1}}, \tilde{p_a}), D_a(\tilde{x}_{a_n}, \tilde{x}_{a_{n+1}}, \tilde{p_a})\right\} \tag{3.8}$$

Taking the limit of (3.8) as $n \longrightarrow \infty$ yields

$$D_a(\tilde{p_a}, \tilde{p_a}, \tilde{f}\tilde{p_a}) \leq \lambda_a D_a(\tilde{p_a}, \tilde{p_a}, \tilde{p_a}) = \tilde{0}$$

and $\tilde{p_a} = \tilde{f}\tilde{p_a}$.

To prove uniqueness, suppose that $\tilde{q_a}$ is also a fixed point of $\tilde{f}$. Then from (3.1)

(3.9)

$$\begin{aligned} D_a(\tilde{p_a}, \tilde{p_a}, \tilde{q_a}) = D_a(\tilde{f}\tilde{p_a}, \tilde{f}\tilde{p_a}, \tilde{f}\tilde{q_a}) &\leq \lambda_a \max\left\{D_a\left(\tilde{p_a}, \tilde{p_a}, \tilde{q_a}\right), D_a(\tilde{p_a}, \tilde{f}\tilde{p_a}, \tilde{q_a})\right\} \\ &= \lambda_a D_a(\tilde{p_a}, \tilde{p_a}, \tilde{q_a}) \end{aligned}$$

which implies that $\tilde{p_a} = \tilde{q_a}$. □

Corollary 3.1. *Let $\tilde{f}$ be a selfmap of a complete and bounded soft D_a-metric space $SP(\tilde{X})$ satisfying*

$$D_a(\tilde{f}\tilde{x}_a, \tilde{f}\tilde{y}_a, \tilde{f}\tilde{z}_a) \leq \lambda_a D_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a) \tag{3.10}$$

for all $\tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in SP(\tilde{X})$, for some $\tilde{0} \leq \lambda_a < \tilde{1}$. Then $\tilde{f}$ has a unique fixed point $\tilde{p_a}$, and $\tilde{f}$ is continuous at $\tilde{p_a}$.

Proof. In (3.10) set $\tilde{y} = \tilde{f}\tilde{x}$ to obtain (3.1). Then, from theorem 3.1, f has a unique fixed point p. To prove continuity, let $\left\{\tilde{z}^n_{a_n}\right\} \subset SP(\tilde{X})$ with $\lim \tilde{z}^n_{a_n} = \tilde{p_a}$. From (3.10),

$$D_a(\tilde{p_a}, \tilde{p_a}, \tilde{f}\tilde{z}_{a_n}) = D_a(\tilde{f}\tilde{p_a}, \tilde{f}p_a, \tilde{f}\tilde{z}_{a_n}) \leq \lambda_a D_a(\tilde{p_a}, \tilde{p_a}, \tilde{f}\tilde{z}_{a_n}). \tag{3.11}$$

Taking the limit as $n \longrightarrow \infty$ gives $\limsup D_a(\tilde{p_a}, \tilde{p_a}, \tilde{f}\tilde{z}_{a_n}) = \tilde{0}$, and $\liminf D_a(\tilde{p_a}, \tilde{p_a}, \tilde{f}\tilde{z}_{a_n}) = \tilde{0}$ which implies that $\lim \tilde{f}\tilde{z}_{a_n} = \tilde{p_a} = \tilde{f}\tilde{p_a}$, and $\tilde{f}$ is continuous at $\tilde{p_a}$. □

Corollary 3.2. *Let $\tilde{f}$ be a selfmap of a complete and bounded soft D_a-metric space satisfying the condition that there exists a positive integer m*

$$D_a(\tilde{f}^m\tilde{x}, \tilde{f}^m\tilde{x}, \tilde{f}^m\tilde{z}) \leq \lambda_a D_a(\tilde{x}, \tilde{y}, \tilde{z}) \tag{3.12}$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}$, for some $\tilde{0} \leq \lambda < \tilde{1}$. Then $\tilde{f}$ has a unique fixed point $\tilde{p_a}$ and $\tilde{f}$ is orbitally continuous at $\tilde{p_a}$.

Proof. Define $\tilde{T} = \tilde{f}^m$. Then (3.12) reduces to (3.10), and $\tilde{T}$ has a unique fixed point $\tilde{p_a}$ by previous corollary; i.e., $\tilde{p_a} = \tilde{T}\tilde{p_a} = \tilde{f}^m\tilde{p_a}$. Thus $\tilde{f}\tilde{p_a} = \tilde{f}^{m+1}\tilde{p_a} = \tilde{T}(\tilde{f}\tilde{p_a})$, and $\tilde{f}\tilde{p_a}$ is also a fixed point of $\tilde{T}$. Uniqueness implies that $\tilde{f}\tilde{p_a} = \tilde{p_a}$, and $\tilde{p_a}$ is a fixed point of $\tilde{f}$.

For the continuity, let $\{\tilde{z}_{a_n}\} \subset O\left(\tilde{f}\right)$, with $\lim \tilde{z}_{a_n} = \tilde{p_a}$. From (3.12),

$$D_a(\tilde{f}^m\tilde{p_a}, \tilde{f}^m\tilde{p_a}, \tilde{f}^m\tilde{z}_{a_n}) \leq \lambda_a D_a(\tilde{p_a}, \tilde{p_a}, \tilde{z}_{a_n}) \tag{3.13}$$

Taking the limit as $n \longrightarrow \infty$ shows that $\lim \tilde{f}^m\tilde{z}_{a_n} = \tilde{p_a} = \tilde{f}^m\tilde{p_a}$, and $\tilde{f}^m$ is $\tilde{f}$orbitally continuous at $\tilde{p_a}$. But since each $\tilde{z}_{a_n} \in O(\tilde{f}), \lim \tilde{f}^m\tilde{z}_{a_n} = \tilde{f}\tilde{z}_{a_{n+m-1}}$, and $\tilde{f}$ is $\tilde{f}$-orbitally continuous at $\tilde{p_a}$. □

Corollary 3.3. *Let $M\tilde{f}$ be a self map of $SP(\tilde{X})$,$SP(\tilde{X})$ an $\tilde{f}$-orbitally bounded and soft D_a-metric space satisfying*

$$D_a(\tilde{f}\tilde{x}_a, \tilde{f}\tilde{y}_a, \tilde{f}\tilde{z}_a) \leq \alpha_a \left[\frac{\tilde{1} + D_a(\tilde{x}_a, \tilde{f}\tilde{x}_a, \tilde{z}_a)}{\tilde{1} + D_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a)}\right] D_a(\tilde{y}_a, \tilde{f}\tilde{y}_a, \tilde{z}_a) + \beta_a D_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a) \tag{3.14}$$

for all $\tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in SP(\tilde{X}), \alpha_a + \beta_a < \tilde{1}$. *Then* $\tilde{f}$ *has a unique fixed point* $\tilde{p_a}$ *and* $\tilde{f}$ *is continuous at* $\tilde{p_a}$.

P r o o f. In (3.14) set $\tilde{y} = \tilde{f}\tilde{x}$ to obtain

$$\begin{aligned} D_a(\tilde{f}\tilde{x}_a, \tilde{f}^2\tilde{x}_a, \tilde{f}\tilde{z}_a) \leq & \alpha_a D_a(\tilde{f}\tilde{x}_a, \tilde{f}^2\tilde{x}_a, \tilde{z}_a) + \beta_a D_a(\tilde{x}_a, \tilde{f}\tilde{x}_a, \tilde{z}_a) \\ \leq & \lambda_a \max\left\{ D_a(\tilde{f}\tilde{x}_a, \tilde{f}^2\tilde{x}_a, \tilde{z}_a), D_a(\tilde{x}_a, \tilde{f}\tilde{x}_a, \tilde{z}_a) \right\} \end{aligned} \tag{3.15}$$

where $\lambda_a = \alpha_a + \beta_a < \tilde{1}$, and (3.1) is satisfied. The conclusion follows from Theorem(3.1). To prove the continuity of $\tilde{f}$ at $\tilde{p_a}$, let $\left\{\tilde{z}^n_{a_n}\right\} \subset SP(\tilde{X})$ with $\lim \tilde{z}_{a_n} = \tilde{p_a}$. In (3.14) set $\tilde{x}_a = \tilde{z}_a = \tilde{p_a}, \tilde{y}_a = \tilde{z}_{a_n}$, to obtain

$$\begin{aligned} D_a(\tilde{p_a}, \tilde{f}\tilde{z}_{a_n}, \tilde{p_a}) &\leq D_a(\tilde{f}\tilde{p_a}, \tilde{f}\tilde{z}_{a_n}, \tilde{f}\tilde{p_a}) \\ &\leq \alpha_a \left[\frac{\tilde{1} + D_a(\tilde{p_a}, \tilde{f}\tilde{p}_a, p_a)}{\tilde{1} + D_a(\tilde{p_a}, \tilde{f}\tilde{z}_{a_n}, \tilde{p_a})} \right] D_a(\tilde{z}_{a_n}, \tilde{f}\tilde{z}_{a_n}, \tilde{p_a}) \\ &\quad + \beta_a D_a(\tilde{p_a}, \tilde{f}\tilde{z}_{a_n}, \tilde{p_a}) \\ &\leq \alpha_a D_a(\tilde{z}_{a_n}, \tilde{f}\tilde{z}_{a_n}, \tilde{p_a}) + \beta_a D_a(\tilde{p_a}, \tilde{z}_{a_n}, \tilde{p_a}). \end{aligned} \tag{3.16}$$

Taking the lim sup of both sides of (3.16) as $n \longrightarrow \infty$ yields

$$D_a(\tilde{p_a}, \limsup \tilde{f}\tilde{z}_{a_n}, \tilde{p_a}) \leq \alpha_a D_a(\tilde{p_a}, \limsup \tilde{f}\tilde{z}_{a_n}, \tilde{p_a}), \tag{3.17}$$

which implies that $\limsup \tilde{f}\tilde{z}_{a_n} = \tilde{p_a}$. Similarly, taking the lim inf of both sides of (3.16) as $n \longrightarrow \infty$ yields

$$D_a(\tilde{p_a}, \liminf \tilde{f}\tilde{z}_{a_n}, \tilde{p_a}) \leq \alpha_a D_a(p_a, \liminf \tilde{f}\tilde{z}_{a_n}, \tilde{p_a}), \tag{3.18}$$

which implies that $\liminf \tilde{f}\tilde{z}_{a_n} = \tilde{p_a}$. Therefore $\lim \tilde{f}\tilde{z}_{a_n} = \tilde{p_a} = \tilde{f}\tilde{p_a}$, and $\tilde{f}$ is continuous at $\tilde{p_a}$. □

Corollary 3.4. *Let* $\tilde{f}$ *be a self map of an* $\tilde{f}$ *orbitally bounded and complete soft* D_a *metric space* $SP(\tilde{X})$, m *a fixed positive integer. Suppose that* $\tilde{f}$ *satisfies*

$$\begin{aligned} D_a(\tilde{f}^m\tilde{x}_a, \tilde{f}^m\tilde{y}_a, \tilde{f}^m\tilde{z}_a) \leq & \alpha_a \left[\frac{\tilde{1} + D_a(\tilde{x}_a, \tilde{f}^m\tilde{x}_a, \tilde{z}_a)}{\tilde{1} + D_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a)} \right] D_a(\tilde{y}_a, \tilde{f}^m\tilde{y}_a, \tilde{z}_a) \\ & + \beta_a D_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a) \end{aligned} \tag{3.19}$$

for all $\tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in SP(\tilde{X}), \alpha_a + \beta_a < \tilde{1}$. *Then* $\tilde{f}$ *has a unique fixed point* $\tilde{p_a}$ *and* $\tilde{f}$ *is* $\tilde{f}$ *orbitally continuous at* $\tilde{p_a}$.

P r o o f. Set $\tilde{T} = \tilde{f}^m$. then $\tilde{T}$ satisfies (3.14). Therefore $\tilde{T}$ has a unique fixed point at $\tilde{p_a}$, and is continuous at $\tilde{p_a}$. □

4. α_a-conndensing.

Definition 4.1. *For any set $O(x_a)$ in a soft D_a-metric space $SP(\tilde{X})$, the D_a diameter of $O(x_a), \delta_a O(x_a)$ is defined by*

$$\delta_a O(x_a) = \sup_{\tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in O(x_a)} D_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a).$$

The measure of non compactness of an orbit $O(x_a)$ in a soft D_a-metric space $SP(\tilde{X})$ is a non negative real number $\alpha_a(O(x_a))$ defined by

(4.1)

$$\alpha_a(O(x_a)) = \inf \left\{ \gamma_a > \tilde{0} : O(x_a) = \bigcup_{i=1}^{n} O(x_{ai}) \text{ for which } \delta_a(O(x_{ai})) \leq \gamma_a \text{ for } i = 1, 2, \ldots, n \right\}$$

Definition 4.2. *A self map $\tilde{f}$ of $SP(\tilde{X})$ is called α_a-condensing, if for any bounded orbit $O(x_a)$ in $SP(\tilde{X})$ is bounded*

$$\alpha_a(\tilde{f}(O(x_a))) < \alpha_a(O(x_a))$$

if $\alpha_a(O(x_a)) > \tilde{0}$.

Lemma 4.1. *Let $\tilde{f} : SP(\tilde{X}) \longrightarrow SP(\tilde{X})$, $SP(\tilde{X})$ an $\tilde{f}$-orbitally bounded and complete soft D_a-metric space be α_a-condensing. Then $\overline{O(x_a)}$ is compact for each $x_a \in SP(\tilde{X})$.*

Theorem 4.1. *Let $\tilde{f}$ be a continuous compact selfmap of a bounded soft D_a metric space $SP(\tilde{X})$ satisfying*

$$D_a(\tilde{f}^r \tilde{x}_a, \tilde{f}^s \tilde{y}_a, \tilde{f}^t \tilde{z}_a) < \delta_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a) \text{ for each } \tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in SP(\tilde{X}), \tag{4.2}$$

with two of $\{\tilde{x}_a, \tilde{y}_a, \tilde{z}_a\}$ distinct, where r, s, t are fixed positive integers. Then $\tilde{f}$ has a unique fixed point in $SP(\tilde{X})$.

Proof. Since $\tilde{f}$ is compact, there exist a compact subset $SP(\tilde{Y})$ of $SP(\tilde{X})$ containing $\tilde{f}(SP(\tilde{X}))$. Then $\tilde{f}(SP(\tilde{Y})) \subset \tilde{Y}$ and $\tilde{A} := \cap_{n=1}^{\infty} \tilde{f}^n SP(\tilde{Y})$ is a non empty compact $\tilde{f}$ -invariant subset of $SP(\tilde{X})$ which is mapped by $\tilde{f}$ onto itself. $\tilde{A}$ has the same properties with respect with respect to $\tilde{f}^r, \tilde{f}^s$, and $\tilde{f}^t$. Suppose that $\delta(\tilde{A}) > \tilde{0}$. Since $\tilde{A}$ is compact there exist $\tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in \tilde{A}$ such that $\delta(\tilde{A}) = D_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a)$. Since $\tilde{f}\tilde{A} = \tilde{A}$, there exist $\tilde{x}'_a, \tilde{y}'_a$ and $\tilde{z}'_a$ in $\tilde{A}$ such that $\tilde{x}_a = \tilde{f}^r \tilde{x}'_a, \tilde{y}_a = \tilde{f}^s \tilde{y}'_a$ and $\tilde{z}_a = \tilde{f}^t \tilde{z}'_a$. Then from (4.2),

$$\delta(\tilde{A}) = D_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a) = D_a(\tilde{f}^r \tilde{x}'_a, \tilde{f}^s \tilde{y}'_a, \tilde{f}^t \tilde{z}'_a) < \delta(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a) = \delta(\tilde{A}), \tag{4.3}$$

a contradiction. Therefore $\tilde{A}$ consists of a single point, which is a fixed point of $\tilde{f}$.

Suppose $\tilde{p}_a$ and $\tilde{q}_a$ are fixed points of $\tilde{f}, \tilde{p}_a \neq \tilde{q}_a$.Then from (4.2)

$$\tilde{0} < D_a(\tilde{p}_a, \tilde{p}_a, \tilde{q}_a) = D_a(\tilde{f}^r\tilde{p}_a, \tilde{f}^s\tilde{p}_a, \tilde{f}^t\tilde{q}_a) < D_a(\tilde{p}_a, \tilde{p}_a, \tilde{q}_a) \tag{4.4}$$

a contradiction. Therefore the fixed point is unique. □

Corollary 4.1. *Let $SP(\tilde{X})$ be a compact D_a-metric space, $\tilde{f}$, a continuous selfmap of $SP(\tilde{X})$ satisfying*

(4.5)

$$D_a(\tilde{f}\tilde{x}_a, \tilde{f}\tilde{y}_a, \tilde{f}\tilde{z}_a) < \max \left\{ \begin{array}{l} D_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a), D_a(\tilde{x}_a, \tilde{f}\tilde{x}_a, \tilde{z}_a), \\ D_a(\tilde{y}_a, \tilde{f}\tilde{y}_a, \tilde{z}_a), D_a(\tilde{x}_a, \tilde{f}\tilde{y}_a, \tilde{z}_a), \\ D_a(\tilde{y}_a, \tilde{f}\tilde{x}_a, \tilde{z}_a) \end{array} \right\} D_a(\tilde{p}_a, \tilde{p}_a, \tilde{q}_a)$$

for all $\tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in SP(\tilde{X})$ with $\tilde{x}_a \neq \tilde{f}\tilde{x}_a, \tilde{y}_a \neq \tilde{f}\tilde{y}_a, \tilde{z}_a \neq \tilde{f}\tilde{z}_a$. Then $\tilde{f}$ has a unique fixed $\tilde{p_a}$ in $SP(\tilde{X})$.

P r o o f. Inequality (4.5) implies that $D_a(\tilde{f}\tilde{x}_a, \tilde{f}\tilde{y}_a, \tilde{f}\tilde{z}_a) < \delta_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a)$, and the existense and uniqueness of a fixed point $\tilde{p_a}$ follows from theorem (4.4).

For continuity, let $\{\tilde{z}_{a_n}\} \subset SP(\tilde{X})$ with $\tilde{z}_{a_n} \neq \tilde{p_a}$ for each n and $\lim \tilde{z}_{a_n} = \tilde{p_a}$. From(4.5)

$$D_a(\tilde{p}_a, \tilde{p}_a, \tilde{f}\tilde{z}_{a_n}) = D_a(\tilde{f}\tilde{p}_a, \tilde{f}\tilde{p}_a, \tilde{f}\tilde{z}_{a_n}) < D_a(\tilde{p}_a, \tilde{f}\tilde{p}_a, \tilde{z}_{a_n}) \tag{4.6}$$

taking the limit as $n \longrightarrow \infty$ implies that $\tilde{f}$ is continuous at p_a. □

Theorem 4.2. *Let $\tilde{f}$ be an $\tilde{f}$-orbitally continuous α_a-condensing selfmap of a complete bounded soft D_a-metric space $SP(\tilde{X})$. Let $\tilde{x}_a \in SP(\tilde{X})$. If (4.2) holds on $\overline{O(x_a)}$, then $\tilde{f}$ has a unique fixed point $\tilde{p_a} \in \overline{O(x_a)}$, and $\lim\limits_n \tilde{f}^n\tilde{x}_a = \tilde{p_a}$ for each $\tilde{x}_a \in \overline{O(x_a)}$.*

P r o o f. From (4.3), $\overline{O(x_a)}$ is compact. Since $\tilde{f}$ is a continuous α_a condensing selfmap of $\overline{O(x_a)}$, $\tilde{f}$ is compact. Now apply Theorem 4.1. □

Corollary 4.2. *Let $\tilde{f}$ be a continuous α_a-condensing selfmap of a complete bounded soft D_a-metric space $SP(\tilde{X})$ satisfying (4.5) for all $\tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in SP(\tilde{X})$ with $\tilde{x}_a \neq \tilde{f}\tilde{x}_a, \tilde{y}_a \neq \tilde{f}\tilde{y}_a, \tilde{z}_a \neq \tilde{f}\tilde{z}_a$. Then $\tilde{f}$ has a unique fixed point $\tilde{p_a}$ in $SP(\tilde{X})$.*

From Corollary 4.1, $D_a(\tilde{f}\tilde{x}_a, \tilde{f}\tilde{y}_a, \tilde{f}\tilde{z}_a) < \delta_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a)$ and Theorem 4.2 proves the Corollary.

Theorem 4.3. *Let $\tilde{f}$ be a selfmap of a soft D_a-metric space $SP(\tilde{X})$. Suppose that there exist a point $x_a \in SP(\tilde{X})$ with $\overline{O(x_a)}$ bounded and complete. Suppose that $\tilde{f}$ is continuous and α_a-condensing on $\overline{O(x_a)}$ and satisfies (4.2) for each $\tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in \overline{O(x_a)}$ with two of $\{\tilde{x}_a, \tilde{y}_a, \tilde{z}_a\}$ distinct, and $\tilde{x}_a \neq \tilde{f}\tilde{x}_a, \tilde{y}_a \neq \tilde{f}\tilde{y}_a, \tilde{z}_a \neq \tilde{f}\tilde{z}_a$. Then $\tilde{f}$ has a fixed point in $\overline{O(x_a)}$.*

Proof. By Lemma 4.1 $\overline{O(x_a)}$ is compact. If there exists some integer n for which $\tilde{f}^n\tilde{x}_a = \tilde{f}^{n+1}\tilde{x}_a$, then $\tilde{f}$ has a fixed point in $\overline{O(x_a)}$. Assume that $\tilde{f}^n\tilde{x}_a \neq \tilde{f}^{n+1}\tilde{x}_a$ each n. Note that $\tilde{f}$ restricted to $\overline{O(x_a)}$ is a continuous compact selfmap of $\overline{O(x_a)}$. Suppose $\tilde{u}_a \neq \tilde{f}\tilde{u}_a$ for each cluster point u_a of $\overline{O(x_a)}$. Then $\tilde{f}$ satisfies condition (4.2) for all $\tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in \overline{O(x_a)}$, with two of $\{\tilde{x}_a, \tilde{y}_a, \tilde{z}_a\}$ distinct. Therefore, by Theorem 4.2, $\tilde{f}$ restricted to $\overline{O(x_a)}$ has a unique fixed point $p_a \in \overline{O(x_a)}$. This contradicts the assumption that $\tilde{u}_a \neq \tilde{f}\tilde{u}_a$ for each cluster point $\tilde{u}_a$ of $\overline{O(x_a)}$. Therefore $\tilde{u}_a = \tilde{f}\tilde{u}_a$ for some cluster point $\tilde{u}_a \in \overline{O(x_a)}$. □

The proofs of theorem (4.1),(4.2),and (4.3) are very similar to their metric space counterparts in [9], and [4], but given here for completeness.

Theorem 4.4. *Let $\tilde{f}$ be a selfmap of $SP(\tilde{X})$, and $\tilde{f}$-orbitally bounded and complete soft D_a-metric space. Suppose that $\tilde{f}$ is α_a-condensing, $\tilde{f}$-orbitally bounded and continuous and satisfies*

(4.7)

$$\begin{aligned} D_a(\tilde{f}\tilde{x}_a, \tilde{f}\tilde{y}_a, \tilde{f}\tilde{z}_a) &\leq \alpha_a \left[\frac{\tilde{1} + D_a(\tilde{x}_a, \tilde{f}\tilde{x}_a, \tilde{z}_a)}{\tilde{1} + D_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a)}\right] D_a(\tilde{y}_a, \tilde{f}\tilde{y}_a, \tilde{z}_a) + \beta_a D_a(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a) \\ &= \tilde{M}(\tilde{x}_a, \tilde{y}_a, \tilde{z}_a) \end{aligned}$$

for all $\tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in SP(\tilde{X})$ with $\tilde{x}_a \neq \tilde{f}\tilde{x}_a, \tilde{y}_a \neq \tilde{f}\tilde{y}_a, \tilde{z}_a \neq \tilde{f}\tilde{z}_a$, where $\alpha_a, \beta_a > \tilde{0}, \alpha_a + \beta_a \leq \tilde{1}$. Then $\tilde{f}$ has a unique fixed point $p_a \in \tilde{X}$ and $\tilde{f}$ is continuous at p_a.

Proof. If $\alpha_a + \beta_a < \tilde{1}$, the result follows from Corollary (3.6). Therefore we assume that $\alpha_a + \beta_a = \tilde{1}$. Let $x_a \in SP(\tilde{X})$ and define $x_{a_{n+1}} = \tilde{f}\tilde{x}_{a_n}, n \geq 0$. From Lemma 4.1 it follows that $\overline{O(x_a)}$ is compact. Obviously $\tilde{f} : \overline{O(x_a)} \longrightarrow \overline{O(x_a)}$.

Case I. There exist some $\tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in \overline{O(x_a)}$ for which $\tilde{M} = \tilde{0}$. Then $\tilde{y}_a = \tilde{f}\tilde{y}_a = \tilde{z}_a = \tilde{x}_a$, and $\tilde{y}_a$ is a fixed point of $\tilde{f}$. Inequality (4.7) implies uniqueness.

Case II. $\tilde{M} \neq \tilde{0}$ for all $\tilde{x}_a, \tilde{y}_a, \tilde{z}_a \in \overline{O(x_a)}$. Define a function $\tilde{F} : (\overline{O(x_a)})^3 \longrightarrow$

$[0,\infty)$ by

$$\tilde{F}(\tilde{x}_a,\tilde{y}_a,\tilde{z}_a)=\frac{D_a(\tilde{f}\tilde{x}_a,\tilde{f}\tilde{y}_a,\tilde{f}\tilde{z}_a)}{M(\tilde{x}_a,\tilde{y}_a,\tilde{z}_a)}. \tag{4.8}$$

The function $\tilde{F}$ is well defined on $(\overline{O(x_a)})^3$ since $\tilde{M}\neq\tilde{0}$ on $\overline{O(x_a)}$.
Since $\tilde{F}$ is continuous on $\overline{O(x_a)}$, it attains its maximum value at some point $(\tilde{u}_a,\tilde{u}_a,\tilde{u}_a)\in\overline{O(x_a)}$. We call this maximum value $\tilde{c}$. From (4.7) it follows that $\tilde{0}<\tilde{c}<\tilde{1}$. Therefore

(4.9)

$$\begin{aligned}D_a(\tilde{f}\tilde{x}_a,\tilde{f}\tilde{y}_a,\tilde{f}\tilde{z}_a)&\\ &\leq \tilde{c}M(\tilde{x}_a,\tilde{y}_a,\tilde{z}_a)\\ &\leq \alpha'_a\left[\frac{\tilde{1}+D_a(\tilde{x}_a,\tilde{f}\tilde{x}_a,\tilde{z}_a)}{\tilde{1}+D_a(\tilde{x}_a,\tilde{x}_a,\tilde{z}_a)}\right]D_a(\tilde{y}_a,\tilde{f}\tilde{y}_a,\tilde{z}_a)+\beta'_aD_a(\tilde{x}_a,\tilde{y}_a,\tilde{z}_a)\end{aligned}$$

for all $\tilde{x}_a,\tilde{y}_a,\tilde{z}_a\in\overline{O(x_a)}$, where $\alpha'_a=\tilde{c}\alpha_a>\tilde{0},\beta'_a=\tilde{c}\beta_a>\tilde{0}$ and $\alpha'_a+\beta'_a=\tilde{c}(\alpha_a+\beta_a)<\tilde{1}$. Since $\overline{O(x_a)}$ is compact, it is bounded anf complete. The result follows from Corollary 3.3. □

Corollary 4.3. *Let $\tilde{f}$ be a selfmap of a complete and $\tilde{f}$-orbitally bounded D_a-metric space. Suppose that $\tilde{f}$ is α_a-condensing and $\tilde{f}$-orbitally continuous. Let m be a positive integer. Suppose that $\tilde{f}$ satisfies*

$$\begin{aligned}D_a(\tilde{f}^m\tilde{x}_a,\tilde{f}^m\tilde{y}_a,\tilde{f}^m\tilde{z}_a)\leq&\alpha_a\left[\frac{\tilde{1}+D_a(\tilde{x}_a,\tilde{f}^m\tilde{x}_a,\tilde{z}_a)}{\tilde{1}+D_a(\tilde{x}_a,\tilde{y}_a,\tilde{z}_a)}\right]D_a(\tilde{y}_a,\tilde{f}^m\tilde{y}_a,\tilde{z}_a)\\ &+\beta_aD_a(\tilde{x}_a,\tilde{y}_a,\tilde{z}_a)\end{aligned} \tag{4.10}$$

for all $\tilde{x}_a,\tilde{y}_a,\tilde{z}_a\in SP(\tilde{X})$ for which the right hand side of (4.10) is not zero, where $\alpha_a,\beta_a>\tilde{0},\alpha_a+\beta_a\leq\tilde{1}$. Then $\tilde{f}$ has a unique fixed point p_a and $\tilde{f}$ is $\tilde{f}$ orbitally continuous at $\tilde{p_a}$.

Proof. Set $\tilde{T}=\tilde{f}^m$. then $\tilde{T}$ satisfies (4.7), and the existence and uniqueness of the fixed point p_a, for $\tilde{T}$, follows from (4.7). It then follows that $\tilde{p_a}$ is the unique fixed point for $\tilde{f}$. The continuity argument is the same as that used in the proof of Corollary (3.4). □

Corollary 4.4. *Let $\tilde{f}$ be a continuous selfmap of a compact D_a-metric space satisfying (4.7). Then $\tilde{f}$ has a unique fixed point $\tilde{p_a}$, and $\tilde{f}$ is continuous at $\tilde{p_a}$.*

This results is an immediate consequence of Theorem 4.4.

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