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HEAT AND RESOLVENT KERNELS AND A FUNDAMENTAL SOLUTION FOR THE OCTONIONIC HEISENBERG LAPLACIAN

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ABSTRACT. We give an integral representation of the fundamental solution of the heat and resolvent equations associated to the Kohn's Laplacian on the octonionic Heisenberg group. Moreover, the fundamental solution for the octonionic Heisenberg Laplacian is found.

1. Introduction. The heat equation is known as a fundamental equation in mathematical physics and occur in many branches of physics, in applied mathematics as well as in engineering (cf. [4, 12, 25]). In harmonic analysis, this equation has known interesting developments.

For a given Lie group G whose associated Lie algebra is generated by some left-invariant vector fields $\nabla_1, \dots, \nabla_n$, the heat equation and the inverse problem can be defined respectively by

$$(1.1) \quad \frac{\partial}{\partial t} u = \left(\sum_{i=1}^n \nabla_i^2 \right) u \text{ on } G \times \mathbb{R}^+ \quad \text{and} \quad \left(\sum_{i=1}^n \nabla_i^2 \right) v = \delta_0 \text{ on } G.$$

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Key words: Octonionic Heisenberg group. Kohn's Laplacian, fundamental solution, heat kernel, resolvent kernel, Whittaker function.

An explicit usable expression of the corresponding heat kernel and fundamental solution is always desirable.

The purpose of this paper is to discuss some complex spectral objects associated with the Kohn's Laplacian $\mathcal{L}$ on the octonionic Heisenberg group. Namely, the heat, the resolvent and the Green kernels are derived. Moreover, the fundamental solution for $\mathcal{L}$ is obtained.

This paper gives an extension of the results in [2, 18, 21, 22] and [28] for classical Heisenberg and quaternionic Heisenberg groups to the case of the octonionic Heisenberg group and therefore to all nilpotent part of the Iwasawa decomposition of rank one semisimple Lie groups.

On the analogy with the complex and quaternion point of view, there exists a wide literature regarding operator theory over the octonions that can be found in [13, 16, 19, 27] and references therein. The octonion algebra is the largest division real algebra in which the complex field has non-central embeddings [3, 14]. A nice description of the algebra of octonions is given by Baez [3].

The Heisenberg group was studied extensively and systematically by many authors during the last decade, while the investigation of the octonionic Heisenberg group remained incomplete. The corresponding results of the octonionic Heisenberg group are much more complicated than the Heisenberg group because the octonionic field is not commutative and not associative. We hope, that the octonionic Heisenberg group will play an important role in the future, similar to the role played by the Heisenberg group in different aspects of mathematics.

The first aim of the present paper is to construct an explicit expression of the heat kernel for the Kohn's Laplacian $\mathcal{L}$ on the octonionic Heisenberg group by using the geometric variational method introduced in [4] by Calin et al. This method was initially applied for the Heisenberg Laplacian; see [6].

The second aim of this paper is to use the explicit formula for the heat kernel to derive an integral representation of the resolvent kernel. More precisely, one can use the well known formula connecting the resolvent and the heat kernels [7, p.56]

$$(1.2) \quad (\zeta - \mathcal{L})^{-1} = \int_0^\infty e^{-\zeta s} e^{s\mathcal{L}} ds; \quad \Re(\zeta) < 0,$$

to find the resolvent kernel associated with the Kohn's Laplacian $\mathcal{L}$ for the octonionic Heisenberg group $\mathbb{H}_O$. We prove that its expression is given in terms of the Whittaker function $W_{\kappa, \mu}(z)$.

The third purpose of this paper is to construct the fundamental solution for the Kohn's Laplacian $\mathcal{L}$ on the octonionic Heisenberg group $\mathbb{H}_O$.

The rest of the paper is organized as follows: In Section 2, we introduce the octonionic Heisenberg group $\mathbb{H}_O$ and its corresponding Lie algebra. Also, the associated Kohn's Laplacian $\mathcal{L}$ and some of its properties are given. In Sections 3 and 4, we discuss and state our main results giving integral representations of the corresponding heat and resolvent kernels. In Section 5, the fundamental solution for $\mathcal{L}$ is derived. Finally, in Section 6, we compute the Green function.

2. Octonionic Heisenberg Laplacian. We denote by $\mathbb{O}$ the algebra of octonions. $\mathbb{O}$ has a basis $\{e_j\}$ satisfying the relation $e_i e_0 = e_0 e_i$, $i \in \{0, \dots, 7\}$ and

$$(2.1) \quad e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_j, \quad i, j \in \{1, \dots, 7\},$$

where δ_{ij} is the Kronecker delta and ε_{ijk} are completely skew-symmetric in i, j, k , and equal the value $+1$ for $(i, j, k) \in \Lambda$ (cf. [3, 5]), where

$$(2.2) \quad \Lambda = \{(1, 2, 3), (1, 4, 5), (1, 7, 6), (2, 4, 6), (2, 5, 7), (3, 4, 7), (3, 6, 5)\}.$$

ε_{ijk} is nonzero only when (i, j, k) is a permutation of triples in Λ .

We write an octonionic number as $x = \sum_{k=0}^7 x_k e_k$. The conjugate of x is $\bar{x} = x_0 - \sum_{k=1}^7 x_k e_k$, and $\Re x = x_0$, $\Im x = \sum_{k=1}^7 x_k e_k$. The set of all purely imaginary octonions span a 7 dimension subspace of $\mathbb{O}$, denoted $\Im \mathbb{O}$. $\mathbb{O}$ is also equipped with the inner product and norm of $\mathbb{R}^8$. For $x, y \in \mathbb{O}$, we have

$$\langle x, y \rangle = \sum_{k=0}^7 x_k y_k \quad \text{and} \quad |x|^2 = \langle x, x \rangle.$$

We define the octonionic Heisenberg group $\mathbb{H}_O := \mathbb{O} \times \Im \mathbb{O}$ to be the noncommutative group equipped with the multiplication "·" given by

$$(2.3) \quad (x, u) \cdot (y, v) = \left(x + y; u + v + \frac{1}{2} \Im(\bar{x}y) \right)$$

for all $x, y \in \mathbb{O}$ and $u, v \in \Im \mathbb{O}$. Topologically, $\mathbb{H}_O := \mathbb{O} \times \Im \mathbb{O}$ is the $\mathbb{R}^8 \times \mathbb{R}^7 \cong \mathbb{R}^{15}$. Moreover, it is a Lie group of dimension 15, whose center is $\{0\} \times \mathbb{R}^7 \sim \{0\} \times \Im \mathbb{O}$. The Haar measure on $\mathbb{H}_O$ coincides with the Lebesgue measure on $\mathbb{R}^8 \times \mathbb{R}^7$ which is denoted by $dm(x, u)$.

The Lie algebra of $\mathbb{H}_O$ is generated by the left-invariant vector fields

$$\begin{aligned}
X_0 &= \frac{\partial}{\partial x_0} + \frac{1}{2} \left(-x_1 \frac{\partial}{\partial u_1} - x_2 \frac{\partial}{\partial u_2} - x_3 \frac{\partial}{\partial u_3} - x_4 \frac{\partial}{\partial u_4} - x_5 \frac{\partial}{\partial u_5} - x_6 \frac{\partial}{\partial u_6} - x_7 \frac{\partial}{\partial u_7} \right); \\
X_1 &= \frac{\partial}{\partial x_1} + \frac{1}{2} \left(x_0 \frac{\partial}{\partial u_1} + x_3 \frac{\partial}{\partial u_2} - x_2 \frac{\partial}{\partial u_3} + x_5 \frac{\partial}{\partial u_4} - x_4 \frac{\partial}{\partial u_5} - x_7 \frac{\partial}{\partial u_6} + x_6 \frac{\partial}{\partial u_7} \right); \\
X_2 &= \frac{\partial}{\partial x_2} + \frac{1}{2} \left(-x_3 \frac{\partial}{\partial u_1} + x_0 \frac{\partial}{\partial u_2} + x_1 \frac{\partial}{\partial u_3} + x_6 \frac{\partial}{\partial u_4} + x_7 \frac{\partial}{\partial u_5} - x_4 \frac{\partial}{\partial u_6} - x_5 \frac{\partial}{\partial u_7} \right); \\
X_3 &= \frac{\partial}{\partial x_3} + \frac{1}{2} \left(x_2 \frac{\partial}{\partial u_1} - x_1 \frac{\partial}{\partial u_2} + x_0 \frac{\partial}{\partial u_3} + x_7 \frac{\partial}{\partial u_4} - x_6 \frac{\partial}{\partial u_5} + x_5 \frac{\partial}{\partial u_6} - x_4 \frac{\partial}{\partial u_7} \right); \\
X_4 &= \frac{\partial}{\partial x_4} + \frac{1}{2} \left(-x_5 \frac{\partial}{\partial u_1} - x_6 \frac{\partial}{\partial u_2} - x_7 \frac{\partial}{\partial u_3} + x_0 \frac{\partial}{\partial u_4} + x_1 \frac{\partial}{\partial u_5} + x_2 \frac{\partial}{\partial u_6} + x_3 \frac{\partial}{\partial u_7} \right); \\
X_5 &= \frac{\partial}{\partial x_5} + \frac{1}{2} \left(x_4 \frac{\partial}{\partial u_1} - x_7 \frac{\partial}{\partial u_2} + x_6 \frac{\partial}{\partial u_3} - x_1 \frac{\partial}{\partial u_4} + x_0 \frac{\partial}{\partial u_5} - x_3 \frac{\partial}{\partial u_6} + x_2 \frac{\partial}{\partial u_7} \right); \\
X_6 &= \frac{\partial}{\partial x_6} + \frac{1}{2} \left(x_7 \frac{\partial}{\partial u_1} + x_4 \frac{\partial}{\partial u_2} - x_5 \frac{\partial}{\partial u_3} - x_2 \frac{\partial}{\partial u_4} + x_3 \frac{\partial}{\partial u_5} + x_0 \frac{\partial}{\partial u_6} - x_1 \frac{\partial}{\partial u_7} \right); \\
X_7 &= \frac{\partial}{\partial x_7} + \frac{1}{2} \left(-x_6 \frac{\partial}{\partial u_1} + x_5 \frac{\partial}{\partial u_2} + x_4 \frac{\partial}{\partial u_3} - x_3 \frac{\partial}{\partial u_4} - x_2 \frac{\partial}{\partial u_5} + x_1 \frac{\partial}{\partial u_6} + x_0 \frac{\partial}{\partial u_7} \right)
\end{aligned}$$

and the canonical vector fields $T_j = \frac{\partial}{\partial u_j}$ for $j = 1, \dots, 7$.

The commutation relations between the generators are

$$(2.4) \quad [X_i, X_j] = -\varepsilon_{ijk} T_k, \quad [X_0, X_j] = T_j,$$

where ε_{ijk} is given in (2.1).

The center of the Lie algebra $Lie(\mathbb{H}_O)$ of $\mathbb{H}_O$ is the real span of $\{T_1, T_2, \dots, T_7\}$, i.e.,

$$\forall X \in Lie(\mathbb{H}_O), \quad [X, T_j] = 0; \quad j = 1, \dots, 7.$$

Hence $\mathbb{H}_O$ is a 2-step nilpotent Lie group.

We define the Kohn's Laplace operator $\mathcal{L}$ as a Casimir operator on $\mathbb{H}_O$, by considering

$$\mathcal{L} = - \sum_{i=0}^7 X_i^2.$$

Explicitly, $\mathcal{L}$ takes the following form:

$$\begin{aligned} \mathcal{L} = & - \sum_{k=0}^7 \frac{\partial^2}{\partial x_k^2} - \frac{1}{4} |x|^2 \left(\sum_{k=1}^7 \frac{\partial^2}{\partial u_k^2} \right) \\ & - \sum_{l=1}^7 \left(x_0 \frac{\partial^2}{\partial x_l \partial u_l} - x_l \frac{\partial^2}{\partial x_0 \partial u_l} \right) - \sum_{(i,j,k)} x_i \left(\frac{\partial^2}{\partial x_j \partial u_k} - \frac{\partial^2}{\partial x_k \partial u_j} \right), \end{aligned}$$

where (i, j, k) is a permutation of triples in Λ .

The Kohn Laplacian $\mathcal{L}$ can be rewritten in terms of the standard Laplacian on $\mathbb{R}^8$ and the one on $\mathbb{R}^7$

$$\Delta_{\mathbb{R}_x^8} = \sum_{r=0}^7 \frac{\partial^2}{\partial x_r^2} \quad \text{and} \quad \Delta_{\mathbb{R}_u^7} = \sum_{s=1}^7 \frac{\partial^2}{\partial u_s^2}$$

respectively, as follows

$$(2.5) \quad \mathcal{L} = -\Delta_{\mathbb{R}_x^8} - \left\langle J(\vec{\partial}_u)x, \vec{\partial}_x \right\rangle - \frac{1}{4} |x|^2 \Delta_{\mathbb{R}_u^7},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product and where $\vec{\partial}_x$ and $\vec{\partial}_u$ denotes respectively the gradient vectors on $\mathbb{R}^8$ and $\mathbb{R}^7$ given by

$$\vec{\partial}_x = \left(\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_7} \right) \quad \text{and} \quad \vec{\partial}_u = \left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_7} \right),$$

and where $J(\vec{\partial}_u)$ is the operator matrix given by

$$J(\vec{\partial}_u) := \begin{pmatrix} 0 & \partial_{u_1} & \partial_{u_2} & \partial_{u_3} & \partial_{u_4} & \partial_{u_5} & \partial_{u_6} & \partial_{u_7} \\ -\partial_{u_1} & 0 & \partial_{u_3} & -\partial_{u_2} & \partial_{u_5} & -\partial_{u_4} & -\partial_{u_7} & \partial_{u_6} \\ -\partial_{u_2} & -\partial_{u_3} & 0 & \partial_{u_1} & \partial_{u_6} & \partial_{u_7} & -\partial_{u_4} & -\partial_{u_5} \\ -\partial_{u_3} & \partial_{u_2} & -\partial_{u_1} & 0 & \partial_{u_7} & -\partial_{u_6} & \partial_{u_5} & -\partial_{u_4} \\ -\partial_{u_4} & -\partial_{u_5} & -\partial_{u_6} & -\partial_{u_7} & 0 & \partial_{u_1} & \partial_{u_2} & \partial_{u_3} \\ -\partial_{u_5} & \partial_{u_4} & -\partial_{u_7} & \partial_{u_6} & -\partial_{u_1} & 0 & -\partial_{u_3} & \partial_{u_2} \\ -\partial_{u_6} & \partial_{u_7} & \partial_{u_4} & -\partial_{u_5} & -\partial_{u_2} & \partial_{u_3} & 0 & -\partial_{u_1} \\ -\partial_{u_7} & -\partial_{u_6} & \partial_{u_5} & \partial_{u_4} & -\partial_{u_3} & -\partial_{u_2} & \partial_{u_1} & 0 \end{pmatrix}.$$

Remark 2.1. Note that $J(u)$ is an skew-symmetric matrix whose square is given by

$$J(u)^2 = -|u|^2 I_8,$$

for all $u \in \mathbb{R}^7$, where I_8 is the identity matrix on $\mathbb{R}^8$.

The operator $\mathcal{L}$ acting on $D(\mathcal{L}) := C_0^\infty(\mathbb{H}_O)$, the space of complex-valued C^∞ - functions with a compact support in $\mathbb{H}_O$ as its natural regular domain, is a densely defined symmetric operator. Precisely, we have

$$(2.6) \quad \int_{\mathbb{H}_O} (\mathcal{L}f) \bar{g} \, dm = \int_{\mathbb{H}_O} f (\overline{\mathcal{L}g}) \, dm; \quad f, g \in D(\mathcal{L}).$$

Let $(X_i)_{0 \leq i \leq 7}$ be the left-invariant vector fields on $\mathbb{H}_O$ given above. Then, we have

$$\mathcal{L} = - \sum_{i=0}^7 X_i^2,$$

implying

$$\int_{\mathbb{H}_O} (\mathcal{L}f) \bar{g} \, dm = - \sum_{i=0}^7 \int_{\mathbb{H}_O} (X_i^2 f) \bar{g} \, dm.$$

Set $I = (i_1, \dots, i_n) \in \mathbb{N}^n$ and $|I| = i_1 + i_2 + \dots + i_n$, the formula [11, p.21]

$$\int (X^I f) g = (-1)^{|I|} \int f (X^I g),$$

where $X^I = X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}$ and where the X_j^I s are left-invariant differential operators on a Lie group G , help us to obtain

$$\begin{aligned} \int_{\mathbb{H}_O} (\mathcal{L}f) \bar{g} \, dm &= - \sum_{i=0}^7 \int_{\mathbb{H}_O} (X_i^2 f) \bar{g} \, dm \\ &= - \sum_{i=0}^7 \int_{\mathbb{H}_O} f (\overline{X_i^2 g}) \, dm = \int_{\mathbb{H}_O} f (\overline{\mathcal{L}g}) \, dm, \end{aligned}$$

which is exactly the symmetry propriety of $\mathcal{L}$.

By considering the quadratic form defined by

$$(2.7) \quad Q(f, g) = \langle \mathcal{L}f, g \rangle, \quad f, g \in D(\mathcal{L}),$$

and using also the formula [11, p.21], we get

$$Q(f, f) = \langle \mathcal{L}f, f \rangle = \int_{\mathbb{H}_O} (\mathcal{L}f) \bar{f} \, dm = - \sum_{i=0}^7 \int_{\mathbb{H}_O} (X_i X_i f) \bar{f} \, dm$$

$$= \sum_{i=0}^7 \int_{\mathbb{H}_O} (X_i f) (\overline{X_i f}) dm = \sum_{i=0}^7 \|X_i f\|_{L^2(\mathbb{H}_O, dm)}^2 \geq 0.$$

It follows that $\mathcal{L}$ is a positive operator on $L^2(\mathbb{H}_O, dm)$.

Following the theory of operators associated with forms, the operator $\mathcal{L}$ admits a unique self-adjoint extension (Friedrichs extension) which is also a non-negative operator (see [24, p.107] for the general theory). Here this extension will be also denoted by $\mathcal{L}$.

Positivity of $\mathcal{L}$ ensures that its spectrum is a closed unbounded set contained in the positive real-axis. Then, by the general spectral theory for unbounded self-adjoint operator, $\mathcal{L}$ admits a spectral decomposition $\{E_\lambda\}_{\lambda=-\infty}^\infty$ [8, 24].

The Kohn Laplacian $\mathcal{L}$ can be also expressed in the divergence form as follows

$$\mathcal{L} = \operatorname{div} \left(M \vec{\nabla} \right),$$

where “div” is the vector operator divergence and $\vec{\nabla}$ is the gradient vector on $\mathbb{R}^{15}$, i.e.,

$$\vec{\nabla} = \left(\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_7} \right)^t,$$

and M is the non-degenerate symmetric matrix given by

$$M := \left(\begin{array}{c|c} I_8 & \zeta(x) \\ \hline \zeta(x)^t & (|x|^2/4) I_7 \end{array} \right).$$

Here I_7 and I_8 are the identity matrices on $\mathbb{R}^7$ and $\mathbb{R}^8$ respectively and $\zeta(x)$ is the matrix given by

$$\zeta(x) := \frac{1}{2} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ -x_0 & -x_3 & x_2 & -x_7 & x_6 & -x_5 & x_4 \\ x_3 & -x_0 & -x_1 & -x_6 & -x_7 & x_4 & x_5 \\ -x_2 & x_1 & -x_0 & x_5 & -x_4 & x_7 & x_6 \\ x_7 & x_6 & -x_5 & -x_0 & x_3 & -x_2 & -x_1 \\ -x_6 & x_7 & x_4 & -x_3 & -x_0 & x_1 & -x_2 \\ x_5 & -x_4 & x_7 & x_2 & -x_1 & -x_0 & -x_3 \\ -x_4 & -x_5 & -x_6 & x_1 & x_2 & x_3 & -x_0 \end{pmatrix}.$$

Using Sylvester's criterion [15], we prove that M is a positive definite matrix. Hence, the operator $\mathcal{L}$ is elliptic.

The positivity of our operator $\mathcal{L}$ also ensures that the semi group $e^{-t\mathcal{L}}$ is well defined. Computing its integral kernel will be the goal of the next section.

3. Heat kernel on the octonionic Heisenberg group. We introduce a time parameter $t \geq 0$ and we consider the fundamental solution of the heat equation associated to $\mathcal{L}$, $\vartheta(x, u; t)$ defined by

$$(3.1) \quad \begin{cases} \frac{\partial}{\partial t} \vartheta(x, u; t) = \mathcal{L} \vartheta(x, u; t); & t > 0, (x, u) \in \mathbb{H}_O \\ \lim_{t \rightarrow 0^+} \vartheta(x, u; t) = f(x, u) \in C_0^\infty(\mathbb{H}_O) \end{cases}.$$

In this section, we deal with the Cauchy problem of heat type (3.1). For this we need some preparations.

Denoting by $\mathcal{F}_{(u)}$, the partial Fourier transform with respect to the variable $u \in \mathbb{R}^7$ is given by

$$(3.2) \quad \mathcal{F}_{(u)} \varphi(x, u) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_v^7} e^{i\langle u, v \rangle} \varphi(x, v) dm(v) = \lim_{\rho \rightarrow +\infty} \int_{|v| \leq \rho} e^{i\langle u, v \rangle} \varphi(x, v) dm(v),$$

considered as the limit in the $L^2(\mathbb{R}^7)$ -norm sense.

The partial Fourier transform, with respect to the variable $u \in \mathbb{R}^7$, of the Kohn Laplacian $\mathcal{L}$ defined in (2.5) associated to the octonionic Heisenberg group $\mathbb{H}_O$ is given by

$$(3.3) \quad \Delta_v := \mathcal{F}_{(u)} \circ \mathcal{L} = -\Delta_{\mathbb{R}_x^8} + i\langle J(v)x, \vec{\partial}_x \rangle + \frac{1}{4}|x|^2|v|^2,$$

where $v = (v_1, \dots, v_7)$ is the dual variable of $u \in \mathbb{R}^7$. It is a differential operator acting on $C^\infty(\mathbb{O}; \mathbb{C})$.

Definition 3.1. We will call Δ_v given in (3.3), the octonionic Landau operator.

We denoted by $h_v(x, y, t)$, the heat kernel associated to the octonionic Landau operator Δ_v which can be computed using the geometric variational method considered in [4]. Indeed, we have

$$(3.4) \quad h_v(x, y, t) = a_v(t) e^{-iS(x, y; it)},$$

where S is the classical action associated with the operator Δ_v and $a_v(t)$ is the volume element defined by

$$(3.5) \quad a_v(t) = \left(\frac{|v|}{\sinh(|v|t)} \right)^4.$$

Introducing the dual variable $\xi_j = i \frac{\partial}{\partial x_j}$, $j = 0, \dots, 7$, we get the Hamilton function H associated to Δ_v

$$(3.6) \quad H(x, \xi) = |\xi|^2 + \langle J(v)x, \xi \rangle + \frac{1}{4}|x|^2|v|^2; \quad \xi = (\xi_0, \dots, \xi_7).$$

The corresponding Lagrangian function reads as

$$(3.7) \quad L(x, \rho) = \max_{\xi \in \mathbb{R}^8} (\langle \rho, \xi \rangle - H(x, \xi)), \quad \rho \in \mathbb{R}^8.$$

Now, let $x(s)$ be a smooth curve joining $x(0) = y$ and $x(t) = x$. Therefore, the action along $x(t)$ is given by the functional

$$(3.8) \quad S(x, y; t) := S(x(t)) = \int_0^t L(x(s), \dot{x}(s)) ds.$$

The following assertion gives the explicit expression of $S(x, y; t)$.

Proposition 3.2. *The functional $S(x, y; t)$ is given by*

$$(3.9) \quad S(x, y; t) = 2 \langle J(v)x, y \rangle + 2|v| \cot(4|v|t) |x - y|^2.$$

Proof. Recall that the classical action $S(x(t))$ is obtained by integrating the Lagrangian along the classical path $x(s)$ which is a solution of the Euler-Lagrange equations given by

$$(3.10) \quad \frac{d}{ds} \left[\frac{\partial}{\partial \dot{x}} L(x(s), \dot{x}(s)) \right] = \frac{\partial}{\partial x} L(x, \dot{x}).$$

The necessary condition for reaching the maximum is vanishing of partial derivatives of the function $\xi \mapsto \langle \rho, \xi \rangle - H(x, \xi)$. This condition can be written as

$$\xi^* = \frac{1}{2}\rho - \frac{1}{2}J(v)x.$$

So the Lagrangian along the solution is

$$L(x, \rho) = \langle \rho, \xi^* \rangle - H(x, \xi^*) = \langle \rho, \xi^* \rangle - |\xi^*|^2 - \langle J(v)x, \xi^* \rangle - \frac{1}{4}|v|^2|x|^2$$

$$\begin{aligned}
&= \left\langle \rho, \frac{1}{2}\rho - \frac{1}{2}J(v)x \right\rangle - \frac{1}{2}|\rho - J(v)x|^2 \\
&\quad - \left\langle J(v)x, \frac{1}{2}v - \frac{1}{2}J(v)x \right\rangle - \frac{1}{4}|v|^2|x|^2.
\end{aligned}$$

Using the fact that $\langle J(v)x, J(v)x \rangle = |v|^2|x|^2$, for any $x \in \mathbb{R}^8$, we deduce that

$$L(x, \rho) = \frac{1}{4}|\rho|^2 - \frac{1}{2}\langle J(v)x, \rho \rangle.$$

Hence,

$$\frac{\partial}{\partial \dot{x}} L(x, \dot{x}) = \frac{1}{2}\dot{x} - \frac{1}{2}J(v)x, \quad \frac{\partial}{\partial x} L(x, \dot{x}) = \frac{1}{2}J(v)\dot{x}$$

and

$$\frac{d}{ds} \left[\frac{\partial}{\partial \dot{x}} L(x(s), \dot{x}(s)) \right] = \frac{1}{2}\ddot{x}(s) - \frac{1}{2}J(v)\dot{x}(s).$$

Then, the Euler-Lagrange equations (3.10) becomes

$$(3.11) \quad \ddot{x}(s) - 2J(v)\dot{x}(s) = 0.$$

With the substitution $\dot{x}(s) = \rho(s)$ equation (3.11) becomes

$$\dot{\rho}(s) - 2J(v)\rho(s) = 0.$$

Integrating this equation yields

$$\rho(s) = e^{2J(v)s} C_1.$$

It follows that the classical path $x(s)$ is then given by

$$x(s) = \frac{1}{2}J(v)^{-1}e^{2J(v)s} C_1 + C_2,$$

where C_1 and C_2 are some constants to determinate. The Lagrangian $L(x(s), \dot{x}(s))$ evaluated along the previous solution is

$$L(x(s), \dot{x}(s)) = -\frac{1}{2} \left\langle e^{2J(v)s} C_1, J(v) C_2 \right\rangle.$$

Integrating the Lagrangian along the solution yields the following classical action

$$S(x, y; t) = -\frac{1}{4} \left\langle J(v)^{-1} \left(e^{2J(v)t} - I_8 \right) C_1, J(v) C_2 \right\rangle.$$

Using the initial conditions $x(0) = y$ and $x(t) = x$, we get

$$C_1 = 2J(v) \left(e^{2J(v)t} - I_8 \right)^{-1} (x - y)$$

and

$$C_2 = y - \left(e^{2J(v)t} - I_8 \right)^{-1} (x - y).$$

This infers

$$\begin{aligned} S(x, y; t) &= -\frac{1}{2} \langle x - y, J(v) y \rangle + \frac{1}{2} \left\langle x - y, J(v) \left(e^{2J(v)t} - I_8 \right)^{-1} (x - y) \right\rangle \\ (3.12) \quad &= \frac{1}{2} \langle J(v) x, y \rangle - \frac{1}{2} \left\langle J(v) (x - y), \left(e^{2J(v)t} - I_8 \right)^{-1} (x - y) \right\rangle, \end{aligned}$$

where we have used the fact that $\langle J(v) x, x \rangle = 0$ for $J(v)$ being antisymmetric.

As a step toward computing the inverse matrix $\left(e^{2J(v)t} - I_8 \right)^{-1}$, let's first attempt to compute the exponential of the matrix $2J(v)t$. For all $v \in \mathbb{R}^7$, the matrix $J(v)$ satisfies the following rules

$$\begin{aligned} (3.13) \quad J(u)^2 &= -|u|^2 I_8, \quad J(u)^3 = -|u|^2 J(u), \\ J(u)^4 &= |u|^4 I_8, \quad J(u)^5 = |u|^4 J(v), \quad \dots \end{aligned}$$

where I_8 is the 8×8 identity matrix.

By (3.13), we observe that

$$\begin{aligned} e^{2J(v)t} &= \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} J(v)^n = I_8 \sum_{k=0}^{\infty} \frac{(2t|v|)^{4k}}{(4k)!} + \frac{J(v)}{|v|} \sum_{k=0}^{\infty} \frac{(2t|v|)^{4k+1}}{(4k+1)!} \\ &\quad - I_8 \sum_{k=0}^{\infty} \frac{(2t|v|)^{4k+2}}{(4k+2)!} - \frac{J(v)}{|v|} \sum_{k=0}^{\infty} \frac{(2t|v|)^{4k+3}}{(4k+3)!}. \end{aligned}$$

We conclude that the matrices $J(v)$ and $e^{2J(v)t}$ commute. Note that

$$\sum_{k=0}^{\infty} \frac{(2t|v|)^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{(2t|v|)^{4k+2}}{(4k+2)!} = \cos(2t|v|)$$

and

$$\sum_{k=0}^{\infty} \frac{(2t|v|)^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{(2t|v|)^{4k+3}}{(4k+3)!} = \sin(2t|v|).$$

It follows that, the matrix $e^{2J(v)t}$ has the form

$$e^{2J(v)t} = \cos(2t|v|)I_8 + \frac{\sin(2t|v|)}{|v|}J(v).$$

To this end, we claim

$$\left(e^{2J(v)t} - I_8\right)^{-1} = -\frac{1}{2}I_8 - \frac{\cot(|v|t)}{2|v|}J(v).$$

Inserting this in (3.12) gives rise to

$$S(x, y; t) = \frac{1}{2} \langle J(v)x, y \rangle + \frac{|v|}{4} \cot(|v|t) |x - y|^2$$

by straightforward computation. This completes the proof. $\square$

We summarize the computations and the formula as a Lemma:

Lemma 3.3. *The heat kernel associated to Δ_v the octonionic Landau operator is given by*

$$(3.14) \quad h_v(x, y; t) = \left(\frac{|v|}{\sinh(t|v|)}\right)^4 e^{-\frac{i}{2}\langle J(v)x, y \rangle} e^{-\frac{|v|}{4} \coth(t|v|) |x - y|^2}.$$

With these preparations, we assert the following

Theorem 3.4. *The solution of the Cauchy problem (3.1) is given by*

$$(3.15) \quad \vartheta(x, u; t) = \int_{\mathbb{H}_O} k_t(x, u; y, v) f(y, v) dm(y, v),$$

where $k_t(x, u; y, v)$ is the heat kernel on the octonionic Heisenberg group $\mathbb{H}_O$ associated to $\mathcal{L}$ given by

$$(3.16) \quad k_t(x, u; y, v) = \int_{\mathbb{R}^7} e^{-i\langle u - v, \lambda \rangle} e^{-\frac{i}{2}\langle J(\lambda)x, y \rangle} \left(\frac{|\lambda|}{\sinh(t|\lambda|)}\right)^4 e^{-\frac{|\lambda|}{4} \coth(t|\lambda|) |x - y|^2} dm(\lambda),$$

where $x, y \in \mathbb{R}^8$ and $u, v \in \mathbb{R}^7$.

Proof. Let ϑ be the solution of the Cauchy problem of heat type:

$$(3.17) \quad \begin{cases} \frac{\partial}{\partial t} \vartheta(x, u; t) = \mathcal{L} \vartheta(x, u; t); & t > 0, (x, u) \in \mathbb{H}_O \\ \lim_{t \rightarrow 0^+} \vartheta(x, u; t) = f(x, u) \end{cases}.$$

Applying the Fourier transform in the u -variable of both sides of this equation, then ϑ satisfies

$$(3.18) \quad \begin{cases} \frac{\partial}{\partial t} \mathcal{F}_{(u)} \vartheta(x, v; t) = \Delta_v \mathcal{F}_{(u)} \vartheta(x, v; t); \\ \lim_{t \rightarrow 0^+} \mathcal{F}_{(u)} \vartheta(x, v; t) = \mathcal{F}_{(u)} f(x, v) \end{cases},$$

where v is the dual variable of $u \in \mathbb{R}^7$ and Δ_v is the octonionic Landau operator given in (3.3).

According to Lemma 3.3, we get

$$(3.19) \quad \mathcal{F}_{(u)} \vartheta(x, v; t) = \int_{\mathbb{R}_y^8} h_v(x, y; t) \mathcal{F}_{(u)} f(y, v) dm(y),$$

where $h_v(x, y; t)$ is the heat kernel associated to Δ_v given by

$$(3.20) \quad h_v(x, y; t) = \left(\frac{|v|}{\sinh(t|v|)} \right)^4 e^{-\frac{i}{2} \langle J(v) x, y \rangle} e^{-\frac{|v|}{4} \coth(t|v|) |x-y|^2}.$$

Now, by the Parseval-Plancherel theorem [23, p.39], we get from (3.19) the integral representation

$$(3.21) \quad \begin{aligned} \vartheta(x, u; t) &= \int_{\mathbb{R}_v^7} e^{-i \langle v, u \rangle} \mathcal{F}_{(u)} \vartheta(x, v; t) dm(v) \\ &= \int_{\mathbb{R}_v^7} e^{-i \langle v, u \rangle} \left[\int_{\mathbb{R}_y^8} h_v(x, y; t) \mathcal{F}_{(u)} f(y, v) dm(y) \right] dm(v) \\ &= \int_{\mathbb{R}_y^8} \left[\int_{\mathbb{R}_v^7} e^{-i \langle v, u \rangle} h_v(x, y; t) \mathcal{F}_{(u)} f(y, v) dm(v) \right] dm(y). \end{aligned}$$

On the other hand, Corollary 2 in [23, p.49] enables us to write

$$(3.22) \quad \begin{aligned} &\int_{\mathbb{R}_v^7} e^{-i \langle v, u \rangle} h_v(x, y; t) \mathcal{F}_{(u)} f(y, v) dm(v) \\ &= \int_{\mathbb{R}_v^7} \left(\int_{\mathbb{R}_\lambda^7} e^{-i \langle \lambda, u-v \rangle} h_\lambda(x, y; t) dm(\lambda) \right) f(y, v) dm(v), \end{aligned}$$

where we have used the square integrability of the function $v \mapsto e^{-i \langle v, u \rangle} h_v(x, y; t)$ which follows by applying the following inequality

$$\frac{s}{\sinh s} \leq e^s \quad \text{for all } s > 0.$$

Substituting in (3.21) yields

$$\vartheta(x, u; t) = \int_{\mathbb{R}_y^8 \times \mathbb{R}_v^7} k_t(x, u; y, v) f(y, v) dm(y, v),$$

where $k_t(x, u; y, v)$ is given by

$$k_t(x, u; y, v) = \int_{\mathbb{R}_\lambda^7} e^{-i\langle \lambda, u-v \rangle} h_\lambda(x, y; t) dm(\lambda).$$

More explicitly

$$k_t(x, u; y, v) = \int_{\mathbb{R}_\lambda^7} e^{-i\langle u-v, \lambda \rangle} e^{-\frac{i}{2}\langle J(\lambda)x, y \rangle} \left(\frac{|\lambda|}{\sinh(t|\lambda|)} \right)^4 e^{-\frac{|\lambda|}{4} \coth(t|\lambda|)|x-y|^2} dm(\lambda).$$

This completes the proof. $\square$

Using polar coordinates, we establish a new integral representation of the heat kernel (3.16).

Proposition 3.5. *The heat kernel in (3.16) can be written as*

$$(3.23) \quad k_t(x, u; y, v) = \frac{(2\pi)^{\frac{7}{2}}}{|u-v + \Im m(y\bar{x})/2|^{\frac{5}{2}}} \int_0^\infty \frac{e^{-\frac{r}{4} \coth(tr)|x-y|^2}}{\sinh^4(tr)} J_{\frac{5}{2}}(|u-v + \Im m(y\bar{x})/2|r) r^{\frac{15}{2}} dr,$$

where J_ν is the Bessel functions of the first kind.

Proof. We introduce the polar coordinates for the λ -variable such that $\lambda = r\omega$, where $r = |\lambda|$ and $\omega = (\omega_1, \dots, \omega_7)$ is a point in the unit sphere $\mathbb{S}^6$ in $\mathbb{R}^7$ with center at the origin. Then $dm(\lambda) = r^6 dr d\sigma(\omega)$, where $d\sigma$ is the surface measure on $\mathbb{S}^6$. By Theorem 3.4,

$$(3.24) \quad \begin{aligned} k_t(x, u; y, v) &= \int_0^\infty \int_{\mathbb{S}^6} e^{-ir\langle u-v, \omega \rangle} e^{-\frac{i}{2}\langle J(r\omega)x, y \rangle} \left(\frac{r}{\sinh(tr)} \right)^4 e^{-\frac{r}{4} \coth(tr)|x-y|^2} r^6 dr d\sigma(\omega) \\ &= \int_0^\infty \frac{r^{10}}{\sinh^4(tr)} e^{-\frac{r}{4} \coth(tr)|x-y|^2} \mathcal{I}_{x,y,u,v}(r) dr, \end{aligned}$$

where

$$(3.25) \quad \mathcal{I}_{x,y,u,v}(r) = \int_{\mathbb{S}^6} e^{-ir\langle u-v, \omega \rangle} e^{-\frac{i}{2}\langle J(r\omega)x, y \rangle} d\sigma(\omega).$$

Let $x = (x_0, \dots, x_7)^t$, $y = (y_0, \dots, y_7)^t \in \mathbb{R}^8$. We have

$$\langle J(r\omega) x, y \rangle = r \langle \tau, \omega \rangle,$$

where

$$\tau := (-x_0 y_1, \dots, x_6 y_7),$$

which can be identified with the pure octonion $\Im m(y\bar{x})$, by using the obvious identification

$$\mathbb{R}^7 \ni (z_1, \dots, z_7) \longleftrightarrow \sum_{k=1}^7 z_k e_k \in \Im m \mathbb{O}.$$

Hence, we can rewrite (3.25) as

$$(3.26) \quad \mathcal{I}_{x,y,u,v}(r) = \int_{\mathbb{S}^6} e^{-ir \langle u-v+\tau/2, \omega \rangle} d\sigma(\omega).$$

Using the identity [26, p.347]

$$\int_{\mathbb{S}^{n-1}} e^{i \langle a, \omega \rangle} d\sigma(\omega) = (2\pi)^{\nu+1} |a|^{-\nu} J_{\nu}(|a|), \quad \nu = \frac{n}{2} - 1,$$

for $n = 7$ and $a = r(u - v + \tau/2)$, we get

$$(3.27) \quad \mathcal{I}_{x,y,u,v}(r) = (2\pi)^{\frac{7}{2}} r^{-\frac{5}{2}} |u - v + \tau/2|^{-\frac{5}{2}} J_{\frac{5}{2}}(|u - v + \tau/2|r).$$

Substituting (3.27) into the expression of the heat kernel in (3.24), we finally obtain

$$\begin{aligned} & k_t(x, u; y, v) \\ &= \frac{(2\pi)^{\frac{7}{2}}}{|u - v + \Im m(y\bar{x})/2|^{\frac{5}{2}}} \int_0^\infty \frac{e^{-\frac{r}{4} \coth(tr) |x-y|^2}}{\sinh^4(tr)} J_{\frac{5}{2}}(|u - v + \Im m(y\bar{x})/2|r) r^{\frac{15}{2}} dr, \end{aligned}$$

as required. $\square$

4. Resolvent kernel on the octonionic Heisenberg group. By $W_{\kappa, \mu}(z)$ we denote the Whittaker function given by

$$(4.1) \quad W_{\kappa, \mu}(z) = e^{-z/2} z^{\mu+\frac{1}{2}} U\left(\mu - \kappa + \frac{1}{2}, 1 + 2\mu; z\right),$$

where $U(a, b; z)$ is the Kummer's Function of the second kind [1, p.505].

Theorem 4.1. *Let $\zeta \in \mathbb{C}$ such that $\Re \zeta > 0$. Then, the resolvent operator $(\zeta - \mathcal{L})^{-1}$ is given by*

$$(4.2) \quad (\zeta - \mathcal{L})^{-1} f(x, u) = \int_{\mathbb{H}_O} R(\zeta; (x, u), (y, v)) f(y, v) dm(y, v),$$

where

$$(4.3) \quad \begin{aligned} \mathcal{R}(\zeta; (x, u), (y, v)) = & \frac{4(2\pi)^{\frac{7}{2}} |x - y|^{-2}}{|u - v + \Im m(y\bar{x})/2|^{\frac{5}{2}}} \int_0^\infty \Gamma\left(\frac{\zeta}{2r} + 2\right) J_{\frac{5}{2}}(|u - v + \Im m(y\bar{x})/2|r) \\ & \times W_{-\frac{\zeta}{2r}, \frac{3}{2}}\left(\frac{|x - y|^2}{2}r\right) r^{\frac{11}{2}} dr, \end{aligned}$$

and where $\Gamma(\cdot)$ is Euler's Gamma-function.

Proof. We use the well known formula connecting the resolvent and the heat kernels

$$(4.4) \quad \mathcal{R}(\zeta; (x, u), (y, v)) = \int_0^\infty e^{-\zeta t} k_t(x, u; y, v) dt; \quad \Re \zeta < 0,$$

as well as the explicit formula for the heat kernel in (3.23), to get

$$\begin{aligned} \mathcal{R}(\zeta; (x, u), (y, v)) \\ = \frac{(2\pi)^{\frac{7}{2}}}{|u - v + \Im m(y\bar{x})/2|^{\frac{5}{2}}} \int_0^\infty J_{\frac{5}{2}}(|u - v + \Im m(y\bar{x})/2|r) \mathcal{J}_{x,y,\zeta}(r) r^{\frac{15}{2}} dr, \end{aligned}$$

where

$$(4.5) \quad \mathcal{J}_{x,y,\zeta}(r) = \int_0^\infty e^{-\zeta t} e^{-\frac{r}{4} \coth(tr) |x-y|^2} \sinh^{-4}(rt) dt.$$

The change $s = rt$ yields

$$\mathcal{J}_{x,y,\zeta}(r) = \frac{1}{r} \int_0^\infty e^{-\frac{\zeta}{r}s} e^{-\frac{r|x-y|^2}{4} \coth(s)} \sinh^{-4}(s) ds.$$

Next, using the integral representation

$$(4.6) \quad \int_0^{+\infty} e^{-2\mu s} e^{-2\beta \coth(s)} (\sinh(s))^{2\nu} ds = \frac{1}{2} \beta^\nu \Gamma(\mu - \nu) W_{-\mu, \nu - \frac{1}{2}}(4\beta),$$

where $\Re(\beta) > 0$ and $\Re(\mu) > \Re(\nu)$ [9, p.384], we can write the integral $\mathcal{J}_{x,y,\zeta}(r)$ in terms of the Whittaker function $W_{\kappa,\mu}(z)$ given in (4.1). Hence for

$$\mu = \frac{\zeta}{2r}, \beta = \frac{r|x-y|^2}{8} \text{ and } \nu = -2,$$

we get

$$\begin{aligned} \mathcal{J}_{x,y,\zeta}(r) &= \frac{32 \Gamma\left(\frac{\zeta}{2r} + 2\right)}{|x-y|r^2} W_{-\frac{\zeta}{2r}, -\frac{5}{2}}(r|x-y|^2/2) \\ &= \frac{32 \Gamma\left(\frac{\zeta}{2r} + 2\right)}{|x-y|r^2} W_{-\frac{\zeta}{2r}, \frac{5}{2}}(r|x-y|^2/2), \end{aligned}$$

where we have used the following identities on the Whittaker function [20, p.299]

$$(4.7) \quad W_{\kappa,\mu}(z) = W_{\kappa,-\mu}(z).$$

Hence, the resolvent kernel can be expressed as

$$\begin{aligned} \mathcal{R}(\zeta; (x, u), (y, v)) &= \frac{32(2\pi)^{\frac{7}{2}}|x-y|^{-1}}{|u-v + \Im m(y\bar{x})/2|^{\frac{5}{2}}} \int_0^\infty \Gamma\left(\frac{\zeta}{2r} + 2\right) J_{\frac{5}{2}}(|u-v + \Im m(y\bar{x})/2|r) \\ &\quad \times W_{-\frac{\zeta}{2r}, \frac{5}{2}}(r|x-y|^2/2) r^{\frac{11}{2}} dr. \quad \square \end{aligned}$$

5. Construction of the fundamental solution. We introduce on $\mathbb{H}_O$ the group $\{\delta_r : 0 < r < \infty\}$ of dilations, which is defined by

$$\delta_r(x, u) = (rx, r^2u).$$

These dilations satisfy the distributive law

$$\delta_r((x, u) \cdot (y, v)) = (\delta_r(x, u)) \cdot (\delta_r(y, v)).$$

We also define the norm function on $\mathbb{H}_O$ by

$$\rho(x, u) = (|x|^4 + 16|u|^2)^{\frac{1}{4}} = \left(\left(\sum_{i=0}^7 x_i^2 \right)^2 + 16 \sum_{j=1}^7 u_j^2 \right)^{\frac{1}{4}},$$

which satisfies

$$\rho(\delta_r(x, u)) = r\rho(x, u).$$

By analogy with the fact that $|x|^{2-m}$ is (a constant multiple of) the fundamental solution for the Laplacian on $\mathbb{R}^m$ with source at 0, and the results in [10], we now prove

Theorem 5.1. $c_Q\rho^{2-Q}$ is a fundamental solution for $\mathcal{L}$ with source at e , where

$$(5.1) \quad c_Q^{-1} = (Q^2 - 4) \int_{\mathbb{H}_O} |x|^2 (\rho(x, u)^4 + 1)^{-(Q+6)/4} d(x, u),$$

$e = (0, 0)$ is the identity element of $\mathbb{H}_O$ and where $Q = 22$ is the homogeneous dimension of $\mathbb{H}_O$.

Proof. For any $f \in C_0^\infty(\mathbb{H}_O)$, we will prove that

$$\langle \mathcal{L}f, c_Q\rho^{2-Q} \rangle = f(e).$$

Given any $\varepsilon > 0$, let $\rho_\varepsilon = (\rho^4 + \varepsilon^4)^{\frac{1}{4}}$, we have

$$\langle \mathcal{L}f, c_Q\rho^{2-Q} \rangle = \lim_{\varepsilon \rightarrow 0^+} \langle \mathcal{L}f, c_Q\rho_\varepsilon^{2-Q} \rangle = \lim_{\varepsilon \rightarrow 0^+} \langle f, \mathcal{L} c_Q\rho_\varepsilon^{2-Q} \rangle.$$

It is easy to see that:

$$\left\{ \sum_{l=1}^7 \left(x_0 \frac{\partial^2}{\partial x_l \partial u_l} - x_l \frac{\partial^2}{\partial x_0 \partial u_l} \right) + \sum_{(i,j,k)} x_i \left(\frac{\partial^2}{\partial x_j \partial u_k} - \frac{\partial^2}{\partial x_k \partial u_j} \right) \right\} c_Q\rho_\varepsilon^{2-Q} = 0.$$

Hence

$$\mathcal{L} c_Q\rho_\varepsilon^{2-Q} = -c_Q \left\{ \sum_{k=0}^7 \frac{\partial^2}{\partial x_k^2} + \frac{|x|^2}{4} \left(\sum_{l=1}^7 \frac{\partial^2}{\partial u_l^2} \right) \right\} \rho_\varepsilon^{2-Q}.$$

We have

$$\frac{\partial}{\partial x_k} \rho_\varepsilon^{2-Q} = (2-Q) x_k |x|^2 \rho_\varepsilon^{-(Q+2)},$$

and

$$\frac{\partial^2}{\partial x_k^2} \rho_\varepsilon^{2-Q} = (2-Q) \left\{ (|x|^2 + 2x_k^2) \rho_\varepsilon^{-(Q+2)} - (Q+2) x_k^2 |x|^4 \rho_\varepsilon^{-(Q+6)} \right\}.$$

Then,

$$\left(\sum_{k=0}^7 \frac{\partial^2}{\partial x_k^2} \right) c_Q \rho_\varepsilon^{2-Q} = c_Q \left\{ 10(2-Q) |x|^2 \rho_\varepsilon^{-(Q+2)} + (Q^2-4) |x|^6 \rho_\varepsilon^{-(Q+6)} \right\}.$$

In other hand, we have

$$\frac{\partial}{\partial u_l} \rho_\varepsilon^{2-Q} = 8(2-Q) u_l \rho_\varepsilon^{-(Q+2)},$$

and

$$\frac{\partial^2}{\partial u_l^2} \rho_\varepsilon^{2-Q} = 8(2-Q) \left\{ \rho_\varepsilon^{-(Q+2)} - 8(Q+2) u_l^2 \rho_\varepsilon^{-(Q+6)} \right\}.$$

Hence

$$\begin{aligned} \frac{|x|^2}{4} \left(\sum_{l=1}^7 \frac{\partial^2}{\partial u_l^2} \right) c_Q \rho_\varepsilon^{2-Q} \\ = 2(2-Q) c_Q \left\{ 7|x|^2 \rho_\varepsilon^{-(Q+2)} - 8(Q+2) |x|^2 |u|^2 \rho_\varepsilon^{-(Q+6)} \right\}. \end{aligned}$$

From these we get that

$$\mathcal{L} c_Q \rho_\varepsilon^{2-Q} = (Q^2-4) c_Q \varepsilon^4 |x|^2 \rho_\varepsilon^{-(Q+6)}.$$

It follows that

$$\begin{aligned} \langle f, \mathcal{L} c_Q \rho_\varepsilon^{2-Q} \rangle &= (Q^2-4) c_Q \varepsilon^4 \int_{\mathbb{H}_O} f(x, u) |x|^2 \rho_\varepsilon^{-(Q+6)} d(x, u) \\ (5.2) \quad &= (Q^2-4) c_Q \int_{\mathbb{H}_O} f(\varepsilon y, \varepsilon^2 v) \frac{|y|^2}{(|y|^4 + 16|v|^2 + 1)^{(Q+6)/4}} d(y, v), \end{aligned}$$

in the above integral we let $(y, v) = \delta_{\frac{1}{\varepsilon}}(x, u)$.

Now letting $\varepsilon \rightarrow 0$, we see that for any $f \in C_0^\infty(\mathbb{H}_O)$,

$$\langle \mathcal{L} f, c_Q \rho_\varepsilon^{2-Q} \rangle = \lim_{\varepsilon \rightarrow 0^+} \langle \mathcal{L} f, c_Q \rho_\varepsilon^{2-Q} \rangle = \lim_{\varepsilon \rightarrow 0^+} \langle f, \mathcal{L} c_Q \rho_\varepsilon^{2-Q} \rangle = f(e),$$

by (5.2) and the formula (5.1) for c_Q^{-1} . The theorem is proved. $\square$

Here the integral in (5.1) is convergent, but it is not computed explicitly.

We will give a description of polar coordinates on the octonionic Heisenberg group $\mathbb{H}_O$, and then compute explicitly c_Q . Then, we have

Lemma 5.2. $c_Q \rho^{2-Q}$ being defined in Theorem 5.1, we have

$$(5.3) \quad c_Q = \frac{24}{\pi^8} = \frac{Q+2}{\pi^{\frac{Q+2}{3}}}.$$

Proof. Let S be the unit sphere in the octonionic Heisenberg group $\mathbb{H}_O$, i.e.,

$$S = \left\{ (x, u) \in \mathbb{H}_O \mid x = (x_0, \dots, x_7) \in \mathbb{O}, u = (u_1, \dots, u_7) \in \mathbb{R}^7, \right. \\ \left. \left(\sum_{i=0}^7 x_i^2 \right)^2 + 16 \sum_{j=1}^7 u_j^2 = 1 \right\}.$$

Taking polar coordinates, i.e., setting $(x, u) = \delta_r(y, v)$ with $(y, v) \in S$, we get

$$\begin{aligned} c_Q^{-1} &= (Q^2 - 4) \int_{\mathbb{H}_O} |x|^2 (\rho(x, u)^4 + 1)^{-(Q+6)/4} d(x, u) \\ &= (Q^2 - 4) \int_S \int_0^\infty |ry|^2 (\rho(\delta_r(y, v))^4 + 1)^{-(Q+6)/4} r^{Q-1} dr d(y, v); (y, v) \in S, \\ &= (Q^2 - 4) \int_S \int_0^\infty |y|^2 (r^4 \rho(y, v)^4 + 1)^{-(Q+6)/4} r^{Q+1} dr d(y, v) \\ &= (Q^2 - 4) \int_S \int_0^\infty |y|^2 (r^4 + 1)^{-(Q+6)/4} r^{Q+1} dr d(y, v) \\ &= (Q^2 - 4) \left(\int_0^\infty r^{Q+1} (r^4 + 1)^{-(Q+6)/4} dr \right) \left(\int_S |y|^2 d(y, v) \right). \end{aligned}$$

Let's compute the two integral separately. The first integral is

$$\begin{aligned} I_1 &= \int_0^\infty r^{Q+1} (r^4 + 1)^{-(Q+6)/4} dr \\ &= \frac{1}{4} \int_0^\infty s^{\frac{Q-2}{4}} (1+s)^{-\frac{Q+6}{4}} ds; \quad s = r^4, \\ &= \frac{\Gamma(\frac{Q+2}{4})}{4\Gamma(\frac{Q+6}{4})}, \end{aligned}$$

$$\text{since } \int_0^\infty \frac{s^{y-1}}{(1+s)^{x+y}} ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

To compute the second integral, we use the Lemma 2.9 in [17, p.911]:

$$\begin{aligned} I_2 &= \int_S |y|^2 d(y, v) \\ &= \frac{\pi^{\frac{15}{2}} \Gamma(\frac{5}{2})}{3\Gamma(6)}. \end{aligned}$$

Then, for $Q = 22$, we have

$$\begin{aligned} c_Q^{-1} &= (Q^2 - 4) \times \frac{\Gamma(\frac{Q+2}{4})}{4\Gamma(\frac{Q+6}{4})} \times \frac{\pi^{\frac{15}{2}} \Gamma(\frac{5}{2})}{3\Gamma(6)}, \\ &= 480 \times \frac{\Gamma(6)}{4\Gamma(7)} \times \frac{\pi^{\frac{15}{2}} \Gamma(\frac{5}{2})}{3\Gamma(6)} \\ &= \frac{40\pi^{\frac{15}{2}} \Gamma(\frac{5}{2})}{\Gamma(7)} \\ &= \frac{\pi^8}{24} \\ &= \frac{\pi^{\frac{Q+2}{3}}}{Q+2}. \end{aligned}$$

The proof is complete. $\square$

6. The Green functions of the octonionic Heisenberg group.

We compute in this section the kernel $\mathcal{G}$ of the integral operator $\mathcal{L}^{-1}$, which is also known as the Green function of $\mathcal{L}$. It is related to the heat kernel k_t , $t > 0$, by the formula

$$\mathcal{G}(h) = \int_0^\infty k_t(e, h) dt, \quad h \in \mathbb{H}_O.$$

The explicit formula for $\mathcal{G}$ is given in the following Proposition.

Proposition 6.1. *The Green function of the octonionic Heisenberg group $\mathbb{H}_O$ is given by*

$$(6.1) \quad \mathcal{G}(x, u) = 3 \times 4^{16} \pi^3 \rho^{2-Q}.$$

Proof. Using Proposition 3.5, the Green function of the octonionic Heisenberg group can be written as

$$\mathcal{G}(x, u) = \int_0^\infty k_t(e, (x, u)) dt$$

$$= \frac{(2\pi)^{\frac{7}{2}}}{|u|^{\frac{5}{2}}} \int_0^\infty \left(\int_0^\infty \frac{e^{-\frac{|x|^2 r}{4} \coth(tr)}}{\sinh^4(tr)} J_{\frac{5}{2}}(|u|r) r^{\frac{15}{2}} dr \right) dt.$$

It is possible to define the Bessel functions of the first kind J_ν by its series expansion as follows [1, p.360]

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left(\frac{x}{2}\right)^{2m+\nu},$$

where $\Gamma(z)$ is the Gamma function. Then it follows that

$$\begin{aligned} \mathcal{G}(x, u) &= \frac{(2\pi)^{\frac{7}{2}}}{|u|^{\frac{5}{2}}} \sum_{m=0}^{\infty} \frac{(-1)^m |u|^{2m+\frac{5}{2}}}{m! \Gamma(m + \frac{5}{2} + 1) 2^{2m+\frac{5}{2}}} \int_0^\infty \left(\int_0^\infty \frac{e^{-\frac{|x|^2 r}{4} \coth(tr)}}{\sinh^4(tr)} r^{2m+10} dr \right) dt. \end{aligned}$$

By performing the change of variables $tr \rightarrow \rho$, we get

$$\begin{aligned} \mathcal{G}(x, u) &= 2\pi^{\frac{7}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m |u|^{2m}}{m! \Gamma(m + \frac{7}{2}) 2^{2m}} \int_0^\infty \left(\int_0^\infty \frac{e^{-\frac{|x|^2 \rho}{4t} \coth(\rho)} \rho^{2m+10}}{t^{2m+11} \sinh^4(\rho)} d\rho \right) dt \\ &= 2\pi^{\frac{7}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m |u|^{2m}}{m! \Gamma(m + \frac{7}{2}) 2^{2m}} \int_0^\infty \frac{\rho^{2m+10}}{\sinh^4(\rho)} \left(\int_0^\infty \frac{e^{-\frac{|x|^2 \rho \coth(\rho)}{4t}}}{t^{2m+11}} dt \right) d\rho. \end{aligned}$$

Making the change variable $s = \frac{|x|^2 \rho \coth(\rho)}{4t}$, and using the integral representation of the Gamma function $\Gamma(\gamma) = \int_0^\infty s^{\gamma-1} e^{-s} ds$, we arrive at

$$\begin{aligned} \mathcal{G}(x, u) &= 2\pi^{\frac{7}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m |u|^{2m}}{m! \Gamma(m + \frac{7}{2}) 2^{2m}} \\ &\quad \times \int_0^\infty \frac{\rho^{2m+10}}{\sinh^4(\rho)} \left(\frac{4}{|x|^2 \rho \coth(\rho)} \right)^{2m+10} \left(\int_0^\infty s^{2m+9} e^{-s} ds \right) d\rho \\ (6.2) \quad &= \frac{4^{11} \pi^{\frac{7}{2}}}{2|x|^{20}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(2m+10) (4|u|)^{2m}}{m! \Gamma(m + \frac{7}{2}) 2^{2m} |x|^{4m}} \int_0^\infty \frac{d\rho}{\sinh^4(\rho) \coth^{2m+10}(\rho)}. \end{aligned}$$

On the other hand, the integral $I_m = \int_0^\infty \frac{d\rho}{\sinh^4(\rho) \coth^{2m+10}(\rho)} d\rho$ is calculated very well by expressing the integrand function with the function $\tanh$. As

$I_m = \int_0^\infty \frac{d\rho}{\sinh^4(\rho) \coth^{2m+10}(\rho)} d\rho = \int_0^\infty \tanh^{2m+6}(\rho) (1 - \tanh^2(\rho))^2 d\rho$ and $\tanh'(\rho) = 1 - \tanh^2(\rho)$, we get

$$I_m = \frac{2}{(2m+7)(2m+9)}.$$

Then,

$$\begin{aligned} \mathcal{G}(x, u) &= \frac{4^{11} \pi^{\frac{7}{2}}}{2|x|^{20}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(2m+10) (4|u|)^{2m}}{m! \Gamma(m + \frac{7}{2}) 2^{2m} |x|^{4m}} \frac{2}{(2m+7)(2m+9)} \\ &= \frac{4^{11} \pi^{\frac{7}{2}}}{|x|^{20}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(2m+10)}{m! \Gamma(m + \frac{7}{2}) (2m+7)(2m+9) 2^{2m}} \left(\frac{4|u|}{|x|^2} \right)^{2m} \\ &= \frac{4^{11} \pi^{\frac{7}{2}}}{|x|^{20}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(2m+10)}{\Gamma(m+1) \Gamma(m+1+\frac{1}{2}) (m+\frac{5}{2}) (m+\frac{3}{2}) (2m+7)(2m+9) 2^{2m}} \left(\frac{4|u|}{|x|^2} \right)^{2m}, \\ &\quad \Gamma(z+1) = z\Gamma(z), \\ &= \frac{4^{11} \pi^{\frac{7}{2}}}{|x|^{20}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(2m+10)}{\sqrt{\pi} 2^{-2m-1} \Gamma(2m+2) (m+\frac{5}{2}) (m+\frac{3}{2}) (2m+7)(2m+9) 2^{2m}} \left(\frac{4|u|}{|x|^2} \right)^{2m}. \end{aligned}$$

The last equality follows using Legendre's duplication formula [4, p.896]:

$$\Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \sqrt{\pi} \Gamma(2z).$$

It follows that

$$\mathcal{G}(x, u) = \frac{4^{13} \pi^3}{2|x|^{20}} \sum_{m=0}^{\infty} (-1)^m (2m+2)(2m+4)(2m+6)(2m+8) \left(\frac{4|u|}{|x|^2} \right)^{2m}.$$

Using the usual methods of series expansion, we obtain

$$\begin{aligned} \mathcal{G}(x, u) &= \frac{4^{13} \pi^3}{2|x|^{20}} \times \frac{384}{\left(1 + \left(\frac{4|u|}{|x|^2}\right)^2\right)^5} \\ &= \frac{3 \times 4^{16} \pi^3}{|x|^{20}} \times \frac{|x|^{20}}{(|x|^4 + 16|u|^2)^5} \end{aligned}$$

$$\begin{aligned}
&= \frac{3 \times 4^{16} \pi^3}{(|x|^4 + 16|u|^2)^5} \\
&= \frac{3 \times 4^{16} \pi^3}{(\rho^4)^5} = \frac{3 \times 4^{16} \pi^3}{\rho^{20}} \\
&= 3 \times 4^{16} \pi^3 \rho^{-20} = 3 \times 4^{16} \pi^3 \rho^{2-Q},
\end{aligned}$$

where $Q = 22$ is the homogeneous dimension of $\mathbb{H}_O$ and where ρ is the norm function on $\mathbb{H}_O$ given by

$$\rho(x, u) = (|x|^4 + 16|u|^2)^{\frac{1}{4}}.$$

It follows that, the Green function of the octonionic Heisenberg group $\mathbb{H}_O$ is given by

$$\mathcal{G}(x, u) = 3 \times 4^{16} \pi^3 \rho^{2-Q}.$$

The asserted formula is established. $\square$

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