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ON THE LCM-SUM FUNCTION OVER ARBITRARY SETS OF INTEGERS

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Abstract. Let \( \mathbb{N} \) denote the set of all positive integers. For \( j, n \in \mathbb{N} \), let \((j, n)\) and \([j, n]\) respectively denote their gcd and lcm. If \( S \subseteq \mathbb{N} \) and \( \alpha \) is a real number then define \( L_{S, \alpha}(n) \) to be the sum of \([j, n]^\alpha\), where \( j \in \{1, 2, 3, \ldots, n\} \) for which \((j, n) \in S\). In this paper we obtain asymptotic formulae for the summatory functions of \( L_{S, \alpha}(n) \) and \( L_{S, -\alpha}(n) \), where \( a \in \mathbb{N} \) and \( a \geq 2 \). Apart from deducing some results proved earlier for \( S = \mathbb{N} \) by Ikeda and Matsuoka, certain new asymptotic formulae are obtained here.

1. Introduction. Let \( \mathbb{N} \) denote the set of all positive integers. For \( j, n \in \mathbb{N} \), let \((j, n)\) and \([j, n]\) be their greatest common divisor and least common multiple respectively.

If \( S \subseteq \mathbb{N} \) and \( \alpha \) is a real number then define, for any \( n \in \mathbb{N} \),

\[
L_{S, \alpha}(n) = \sum_{j=1}^{n} [j, n]^\alpha, \quad (j, n) \in S
\]
where the sum is over those \( j \in \{1, 2, 3, \ldots, n\} \) for which \((j, n) \in S\).

Observe that \( L_{N, \alpha}(n) = L_\alpha(n) \), the arithmetic function first considered by Alladi [1] who proved:

**Lemma 1.1.** If \( x \geq 1 \) and \( \alpha \geq 1 \) then

\[
\sum_{n \leq x} L_\alpha(n) = C(\alpha).x^{2\alpha+2} + O \left( x^{2\alpha+1+\varepsilon} \right),
\]

for any \( \varepsilon > 0 \), where

\[
(1.2) \quad C(\alpha) = \frac{\zeta(\alpha + 2)}{2(\alpha + 1)^2\zeta(2)},
\]

in which \( \zeta(t) \) is the Riemann-zeta function.

The order term in Lemma 1.1 has been improved in the case \( \alpha = 1 \) by Bordellès [3], who also studied the case \( \alpha = -1 \) in the same paper. Recently Ikeda and Matsuoka [6] have considered the cases of Lemma 1.1 for \( \alpha = a \) and \( \alpha = -a \) where \( a \in \mathbb{N} \) and \( a \geq 2 \).

Let \( S \subseteq \mathbb{N} \) be arbitrary. In this paper, an asymptotic formula for

\[
\sum_{n \leq x} L_{S, a}(n), \text{ when } a \in \mathbb{N},
\]

is obtained in Theorem 3.1. Also finding the sum

\[
\sum_{n=1}^\infty L_{S, -\alpha}(n) \text{ for any } \alpha > 1;
\]

we give an estimate for \( \sum_{n \leq x} L_{S, -a}(n) \), when \( a \in \mathbb{N} \) with \( a \geq 2 \), in Theorem 3.4. Noting the results proved in [3] and [6] are deducible from our theorems, several new asymptotic formulae are given for certain special subsets of \( \mathbb{N} \) in Section 4.

**2. Notation and preliminaries.** For any \( S \subseteq \mathbb{N} \), let \( \chi_S(n) \) be its characteristic function. (That is, \( \chi_S(n) = 1 \) or 0 according as \( n \in S \) or \( n \notin S \)).

Cohen [4] defined the zeta-function of \( S \), \( \zeta(t) \), by

\[
(2.1) \quad \zeta_S(t) = \sum_{n=1}^\infty \frac{\chi_S(n)}{n^t} \text{ for } t > 1.
\]

Cleary

\[
(2.2) \quad \zeta_{\mathbb{N}}(t) = \sum_{n=1}^\infty \frac{1}{n^t} = \zeta(t) \text{ for } t > 1.
\]
and this has Euler product representation [2, Theorem 11.6]

\[(2.3) \quad \zeta(t) = \prod_p \left(1 - \frac{1}{p^t}\right)^{-1} \text{ for } t > 1,\]

the product being over all primes \(p\).

Also if \(\mathbb{N}_r = \{1^r, 2^r, 3^r, \ldots\}\), where \(r \in \mathbb{N}\), then

\[(2.4) \quad \zeta_{\mathbb{N}_r}(t) = \zeta(rt) = \prod_p \left(1 - \frac{1}{p^{rt}}\right)^{-1}.\]

First we prove

**Lemma 2.1.** For \(n \in \mathbb{N}, S \subseteq \mathbb{N}\) and any real number \(\alpha\), we have

\[L_{S,\alpha}(n) = \sum_{d \delta = n} \chi_S(d) d^\alpha \phi_\alpha(\delta) \delta^\alpha,\]

where

\[(2.5) \quad \phi_\alpha(n) = \sum_{u=1}^{n} \frac{u^\alpha}{(u,n) = 1} \cdot (u,n).\]

**Proof.** By definition, we have

\[L_{S,\alpha}(n) = \sum_{0 < j \leq n \atop (j,n) = d} \chi_S(d) \left(\frac{jn}{d}\right)^\alpha = \sum_{d | n, d | j \atop 0 < j \leq n \atop (\frac{j}{d}, \frac{n}{d}) = 1} \chi_S(d) \left(\frac{j}{d}\right)^\alpha \cdot \frac{1}{n^\alpha}\]

\[= \sum_{d \delta = n} \chi_S(d) (d \delta)^\alpha \cdot \sum_{du = j \leq d \delta \atop (u,\delta) = 1} u^\alpha \]

\[= \sum_{d \delta = n} \chi_S(d) d^\alpha \delta^\alpha \phi_\alpha(\delta),\]

proving the lemma. \(\square\)

Under a slightly different notation, Ikeda and Matsuoka ([6], page 6) have proved:
**Lemma 2.2.** If \( a \in \mathbb{N} \) then

\[
\phi_a(n) = n^a \left\{ \frac{1}{a+1} \phi(n) + \frac{1}{2} \varepsilon_0(n) + \frac{1}{a+1} \sum_{k=1}^{a-1} \left( \frac{a+1}{k+1} \right) B_{k+1} \Psi_k(n) \right\},
\]

where \( \phi(n) = \phi_0(n) \) is the totient function of Euler; \( \varepsilon_0(n) = 1 \) or 0 according as \( n = 1 \) or \( n > 1 \);

\[
(2.6) \quad \Psi_k(n) = \sum_{d \delta = n} \frac{\mu(d)}{\delta^k},
\]

in which \( \mu(n) \) is the well-known Mobius function and \( B_k \) are the Bernoulli numbers defined by the relation \( \frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \).

Also they proved:

**Lemma 2.3** ([6, Lemma 6]). Let \( r \in \mathbb{N} \) and \( x > e \). Then

\[
\sum_{n \leq x} n^r \phi(n) = \frac{x^{r+2}}{(r+2)\zeta(2)} + O_r \left( x^{r+1} \Delta(x) \right), \text{as } x \to \infty
\]

where \( \Delta(x) = (\log x)^{2/3} (\log \log x)^{4/3} \); and the implied constant depends on \( r \).

For this they used the best estimate of Walfisz [10, Satz1, p. 144]

\[
\sum_{n \leq x} \phi(n) = \frac{x^2}{2\zeta(2)} + O(x.\Delta(x)) \text{, whenever } x > e;
\]

and the Abel’s identity (cite[Theorem 4.2]).

Now we prove two lemmas:

**Lemma 2.4.** If \( \Psi_k(n) \) is the function defined in (2.6) then for \( r > k \),

\[
\sum_{n \leq x} n^r \Psi_k(n) = O(x^{r+1}) \text{ whenever } x \geq 1.
\]

**Proof.**

\[
\sum_{n \leq x} n^r \Psi_k(n) = \sum_{d \leq x} \mu(d) d^r \left( \sum_{\delta \leq \frac{x}{d}} \delta^{r-k} \right)
\]
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\[= O\left(\sum_{d \leq x} |\mu(d)|d^{r} \left(\frac{x}{d}\right)^{r-k+1}\right)\]

\[= O\left(x^{r-k+1} \sum_{d \leq x} \frac{1}{d^{1-k}}\right)\]

\[= O\left(x^{r-k+1} \cdot x^{k}\right) = O\left(x^{r+1}\right). \quad \square\]

**Lemma 2.5.** For \(a \in \mathbb{N}\) and \(x > e\), we have

\[\sum_{n \leq x} n^{a} \cdot \phi_{a}(n) = \frac{1}{2(a+1)^{2} \zeta(2)} \cdot x^{2a+2} + O_{a}\left(x^{2a+1} \cdot \Delta(x)\right),\]

where the implied constant depends on \(a\).

**Proof.** By Lemma 2.2, we have

\[\sum_{n \leq x} n^{a} \cdot \phi_{a}(n) = \frac{1}{a+1} \sum_{n \leq x} n^{2a} \cdot \phi(n) + \frac{1}{2} \sum_{n \leq x} n^{2a} \cdot \varepsilon_{0}(n)\]

\[+ \frac{1}{a+1} \sum_{k=1}^{a-1} \binom{a+1}{k+1} B_{k+1} \left(\sum_{n \leq x} n^{2a} \Psi_{k}(n)\right),\]

wherein using Lemma 2.3 and Lemma 2.4, we get

\[\sum_{n \leq x} n^{a} \cdot \phi_{a}(n) = \frac{1}{a+1} \left\{ \frac{x^{2a+2}}{(2a+2) \zeta(2)} + O_{a}\left(x^{2a+1} \cdot \Delta(x)\right) \right\} + \frac{1}{2} + O_{a}\left(x^{2a+1}\right),\]

from which the lemma follows. \(\square\)

**3. Main results.** Now we prove:

**Theorem 3.1.** For any \(S \subseteq \mathbb{N}\), \(a \in \mathbb{N}\) and \(x > e\), we have

\[\sum_{n \leq x} L_{S,a}(n) = C_{S}(a) \cdot x^{2a+2} + O_{a}\left(x^{2a+1} \cdot \Delta(x)\right),\]

where

\[(3.1) \quad C_{S}(\alpha) = \frac{\zeta_{S}(\alpha+2)}{2(\alpha+1)^{2} \zeta(2)} \text{ for } \alpha \geq 1;\]

and the implied constant depends on \(a\).
Proof. By Lemma 2.1 and Lemma 2.5, we get
\[
\sum_{n \leq x} L_{S,a}(n) = \sum_{d \leq x} \chi_S(d) d^a \left( \sum_{\delta \leq \frac{x}{d}} \delta^a \phi_a(\delta) \right) \\
= \sum_{d \leq x} \chi_S(d) d^a \left\{ \frac{1}{2(a+1)^2 \zeta(2)} \left( \frac{x}{d} \right)^{2a+2} + O_a \left( \left( \frac{x}{d} \right)^{2a+1} \cdot \Delta \left( \frac{x}{d} \right) \right) \right\} \\
= \frac{x^{2a+2}}{2(a+1)^2 \zeta(2)} \cdot \sum_{d \leq x} \frac{\chi_S(d)}{d^{a+2}} + O_a \left( \frac{x^{2a+1} \cdot \Delta(x)}{d^{a+1}} \right) \\
= \frac{x^{2a+2}}{2(a+1)^2 \zeta(2)} \left\{ \zeta_S(a+2) + O \left( \frac{1}{x^{a+1}} \right) \right\} + O_a \left( x^{2a+1} \Delta(x) \right),
\]
since \( \sum_{n \leq x} \frac{\chi_S(n)}{n^t} = \zeta_S(t) + O \left( \frac{1}{x^{t-1}} \right) \) for \( t > 1 \) and \( \sum_{n \leq x} \frac{\chi_S(n)}{n^{a+1}} = O(1) \), from which the theorem follows. \( \square \)

**Corollary 3.2** ([6, Theorem 2]). If \( x > e \) and \( a \in \mathbb{N} \) then
\[
\sum_{n \leq x} L_a(n) = \frac{\zeta(a+2)}{2(a+1)^2 \zeta(2)} x^{2a+2} + O_a \left( x^{2a+1} \cdot \Delta(x) \right) \text{ as } x \to \infty,
\]
where the implied constant depends on \( a \)

Proof. Taking \( S = \mathbb{N} \) in Theorem 3.1, we get the corollary in view of (2.2) and the fact \( C_{\mathbb{N}}(a) = C(a) \), given in (1.2). \( \square \)

**Corollary 3.3.** If \( x > e \), and \( a, r \in \mathbb{N} \) then
\[
\sum_{n \leq x} L_{N_r,a}(n) = \frac{\zeta(ar+2r)}{2(a+1)^2 \zeta(2)} x^{2a+2} + O_a \left( x^{2a+1} \cdot \Delta(x) \right),
\]
where the implied constant depends on \( a \)

Proof. Taking \( S = \mathbb{N}_r \) in Theorem 3.1, the corollary follows in view of (2.4) and (3.1). \( \square \)

Here the formula in Corollary 3.3 is new. Also the case \( r = 1 \) of the Corollary 3.3 gives Corollary 3.2.

Some more special cases of \( S \) will be discussed in Section 4.

Before proving our next result, we recall that a divisor \( d \) of \( n \in \mathbb{N} \) is said to be *unitary* if \( \left( d, \frac{n}{d} \right) = 1 \). If \( \tau^*(n) \) is the number of unitary divisors of \( n \)
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(That is, \( \tau^*(n) = \sum_{d\delta=n, (d,\delta)=1} 1 \)), then it is easy to see that \( \tau^*(n) = 2^\omega(n) \), where \( \omega(n) \) is the number of distinct prime factors of \( n \). Also its Dirichlet series is given by

\[
\sum_{n=1}^{\infty} \frac{\tau^*(n)}{n^t} = \frac{\zeta^2(t)}{\zeta(2t)} \text{ for } t > 1
\]

(see [2, Exercise 7, page 247]).

Now we prove the theorem given below:

**Theorem 3.4.** Suppose \( S \subseteq \mathbb{N} \). Then

(i) for any \( \alpha > 1 \),

\[
\sum_{n=1}^{\infty} L_{S,-\alpha}(n) = B_S(\alpha),
\]

where

\[
B_S(\alpha) = \frac{1}{2} \zeta_S(\alpha) \left\{ 1 + \frac{\zeta^2(\alpha)}{\zeta(2\alpha)} \right\}.
\]

(ii) for \( x \geq 1 \) and \( a \in \mathbb{N} \) with \( a \geq 2 \), we have

\[
\sum_{n \leq x} L_{S,-a}(n) = B_S(a) + O_a \left( \frac{\log x}{x^{a-1}} \right), \text{ as } x \to \infty,
\]

where the implied constant depends on \( a \).

**Proof.** (i) For \( \alpha > 1 \), we have, by definition

\[
L_{S,-\alpha}(n) = \sum_{0<j\leq n} \chi_S(d) \left( \frac{jn}{d} \right)^{-\alpha} = \frac{1}{n^\alpha} \sum_{d\delta=n, du=j, 0<u\leq\delta, (u,\delta)=1} \chi_S(d) \cdot u^{-\alpha}
\]

\[
= \frac{1}{n^\alpha} \sum_{d\delta=n} \chi_S(d) \left( \sum_{0<u\leq\delta} \frac{1}{u^\alpha} \right)
\]
so that

\[
\sum_{n=1}^{\infty} L_{S,-\alpha}(n) = \sum_{d=1}^{\infty} \frac{\chi_S(d)}{d^\alpha \delta^\alpha} \left( \sum_{0<u\leq \delta} \frac{1}{u^\alpha} \right)
\]

\[
= \left\{ \sum_{d=1}^{\infty} \frac{\chi_S(d)}{d^\alpha} \right\} \left\{ \sum_{\delta=1}^{\infty} \frac{1}{\delta^\alpha} \left( \sum_{0<u\leq \delta} \frac{1}{u^\alpha} \right) \right\}
\]

\[
= \zeta_S(\alpha) \cdot \sum_{m=1}^{\infty} \frac{1}{m^\alpha} \left( \sum_{0<u\leq \delta} \frac{1}{u^\alpha} \right)
\]

But

\[
\sum_{m=1}^{\infty} \frac{1}{m^\alpha} \left( \sum_{u\delta=m, (u,\delta)=1} \frac{1}{u^\alpha} \right)
= 1 + \frac{1}{2} \sum_{m=2}^{\infty} \frac{1}{m^\alpha} \left( \sum_{u\delta=m, (u,\delta)=1} \frac{1}{u^\alpha} \right)
\]

\[
= 1 + \frac{1}{2} \sum_{m=2}^{\infty} \frac{\tau^*(m)}{m^\alpha} = 1 + \frac{1}{2} \left\{ \frac{2}{\zeta(2\alpha)} - 1 \right\}.
\]

in view of (3.2)

Now (3.4) and (3.5) prove part (i) of Theorem 3.4.

Again if \( a \in \mathbb{N} \) and \( a \geq 2 \) then for \( x \geq 1 \), we have

\[
\sum_{n \leq x} L_{S,-\alpha}(n) = \sum_{n=1}^{\infty} L_{S,-\alpha}(n) - \sum_{n>x} L_{S,-\alpha}(n)
\]

\[
= B_S(a) - \sum_{n>x} L_{S,-\alpha}(n).
\]
But for any $S \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, it is clear that $L_{S,-a}(n) \leq L_{-a}(n)$, so that
\[
\sum_{n>x} L_{S,-a}(n) \leq \sum_{n>x} L_{-a}(n).
\]
It has been proved in [6, pages 9 and 10] that
\[
\sum_{n>x} L_{-a}(n) = O\left(\frac{\log x}{x^{a-1}}\right) \quad \text{for } x \geq 1.
\]
This together with (3.6) prove part (ii) of Theorem 3.4.

Remark 3.5. Taking $S = \mathbb{N}$, in view of (2.2), part (i) of the Theorem 3.4 gives the equation (2) of Theorem 2 in [6].

4. Some new asymptotic formulae. In this section we consider certain special subsets of $\mathbb{N}$ to derive new asymptotic formulae from our theorems. For this a brief study of $M$-free integers introduced by Rieger [7] is useful.

Let $M$ be a set of positive integer with its minimal element $r$ is such that $r > 1$. A positive integer of the form $n = \prod_{i=1}^{l} p_i^{\beta_i}$, where $p_i$ are distinct primes and $\beta_i$ are non-negative integers is said to be $M$-free if $\beta_i / \notin M$ for $i = 1, 2, \ldots, l$. Denote the set of all $M$-free integers by $Q_M$.

Clearly $1 \in Q_M$ for every $M \subseteq \mathbb{N}$. Also $\chi_{Q_M}$ is a multiplicative function (That is, $\chi_{Q_M}(ab) = \chi_{Q_M}(a) \cdot \chi_{Q_M}(b)$ whenever $(a, b) = 1$). Further the series
\[
\sum_{n=1}^{\infty} \frac{\chi_{Q_M}(n)}{n^t} = \zeta_{Q_M}(t)
\]
converges absolutely for $t > 1$ so that it has Euler product representation [2, Theorem 11.6] given by

\[
\zeta_{Q_M}(t) = \prod_{p} \left\{ \sum_{\beta=0}^{\infty} \frac{\chi_{Q_M}(p^\beta)}{p^{\beta t}} \right\} = \prod_{p} \left\{ \sum_{\beta=0}^{\infty} \frac{1}{p^{\beta t}} \right\} \left\{ \sum_{\beta \notin M} \frac{1}{p^{\beta t}} \right\}
\]

\[
= \prod_{p} \left\{ \sum_{\beta=0}^{\infty} \frac{1}{p^{\beta t}} - \sum_{\beta=0}^{\infty} \frac{1}{p^{\beta t}} \right\} = \prod_{p} \left\{ \frac{1}{1 - \frac{1}{p^t}} - \sum_{\beta=r}^{\infty} \frac{1}{p^{\beta t}} \right\}
\]

If the elements of $M$ are explicitly known then the sum on the right of (4.1) and thereby $\zeta_{Q_M}(t)$ can be expressed in compact form. For example, we consider the sets given below:
\(A = \{r, r+1, r+2, \ldots\}; \quad B = \{r\}; \quad C = \{r, 2r, 3r, \ldots\} \) and \(D = \{\beta : \beta \geq r, \beta \equiv j \pmod{k}\) for some \(j\) with \(r \leq j \leq k-1\), in which \(k \in \mathbb{N}\) is such that \(2 \leq r < k\).

Now (4.1) together with (2.3) and (2.4) give, for \(t > 1\), that

\[
\zeta_{Q_A}(t) = \prod_p \left\{ \frac{1}{1 - \frac{1}{p^t}} - \sum_{\beta=r}^{\infty} \frac{1}{p^{\beta t}} \right\} = \prod_p \left\{ \frac{1 - \frac{1}{p^{rt}}}{1 - \frac{1}{p^t}} \right\} = \frac{\zeta(t)}{\zeta(rt)}; \tag{4.2}
\]

\[
\zeta_{Q_B}(t) = \prod_p \left\{ \frac{1}{1 - \frac{1}{p^t}} - \frac{1}{p^{rt}} \right\} = \zeta(t) \cdot \prod_p \left\{ \frac{1 - \frac{1}{p^{rt}} + \frac{1}{p^{(r+1)t}}}{1 - \frac{1}{p^{rt}}} \right\}; \tag{4.3}
\]

\[
\zeta_{Q_C}(t) = \prod_p \left\{ \frac{1}{1 - \frac{1}{p^t}} - \sum_{k=1}^{\infty} \frac{1}{p^{krt}} \right\} = \zeta(t) \cdot \zeta(rt) \cdot \prod_p \left\{ \frac{1 - \frac{2}{p^{rt}} + \frac{1}{p^{(r+1)t}}}{1 - \frac{1}{p^{rt}}} \right\}; \tag{4.4}
\]

and

\[
\zeta_{Q_D}(t) = \prod_p \left\{ \frac{1}{1 - \frac{1}{p^t}} - \sum_{u=0}^{\infty} \frac{1}{p^{(ku+j)t}} \right\}
= \prod_p \left\{ \frac{1 - \frac{1}{p^{rt}}}{(1 - \frac{1}{p^{rt}})(1 - \frac{1}{p^t})} \right\} = \frac{\zeta(t) \zeta(kt)}{\zeta(rt)}. \tag{4.5}
\]

Let \(r, n \in \mathbb{N}\) and \(r > 1\).

\(n\) is said to be \(r\)-free if it is not divisible by \(p^r\) for any prime \(p\) and the set of all \(r\)-free integers is denoted by \(Q_r\).

Suryanarayana [9] called \(n\) as semi-\(r\)-free if \(p^r\) is not a unitary divisor of it for any prime \(p\). We denote the set of all semi-\(r\)-free integers by \(S_r\).

Cohen [5] termed \(n\) as unitarily-\(r\)-free if \(p^{kr}\) is not a unitary divisor of it for any prime \(p\) and any integer \(k \geq 1\). Let \(U_r\) denote the set of all unitarily-\(r\)-free integers.

If \(2 \leq r < k\) and if \(n\) is of the form \(n = a^k \cdot b\), where \(a \in \mathbb{N}\) and \(b\) is \(r\)-free (that is, \(b \in Q_r\)) then \(n\) is called a \((k, r)\)-integer by Cohen [5]; and independently Subbarao and Harris [8] studied these numbers (under a different notation). Let \(Q_{k,r}\) denote the set of all \((k, r)\)-integers. It may be noted that \(Q_{\infty,r} = Q_r\), showing \((k, r)\)-integer is a generalized \(r\)-free integer.

Now it is easy to see that \(Q_A = Q_r, Q_B = S_r, Q_C = U_r\) and \(Q_D = Q_{k,r}\).

Therefore (4.2), (4.3), (4.4) and (4.5) respectively give \(\zeta_{Q_r}(t), \zeta_{S_r}(t), \zeta_{U_r}(t)\) and
\( \zeta_{Q,r}(t) \) in compact forms. Hence by taking \( S = Q_r, S_r, U_r \) and \( Q_{k,r} \) in Theorem 3.1 and Theorem 3.4, we find new asymptotic formulae in each case. For example, if \( S = Q_{k,r} \) then we get from Theorem 3.1 the following:

**Corollary 4.1.** If \( a \in \mathbb{N} \) and \( x > e \) then

\[
\sum_{n \leq x} L_{Q_{k,r},a}(n) = C_{Q_{k,r}}(a) \cdot x^{2a+2} + O_a \left( x^{2a+1} \cdot \Delta(x) \right), \text{ as } x \to \infty
\]

where

\[
C_{Q_{k,r}}(a) = \frac{\zeta(a+2)\zeta(ak+2k)}{2(a+1)^2\zeta(2)\zeta(ar+2r)}
\]

and the implied constant depends on \( a \).

Also in this case Theorem 3.4 gives:

**Corollary 4.2.** (i) For any \( \alpha > 1 \),

\[
\sum_{n=1}^{\infty} L_{Q_{k,r},a}(n) = \frac{\zeta(\alpha)\zeta(k\alpha)}{2\zeta(r\alpha)} \left\{ 1 + \frac{\zeta^2(\alpha)}{\zeta(2\alpha)} \right\}
\]

and

(ii) for \( a \in \mathbb{N} \) with \( a \geq 2 \), \( x \geq 1 \) we have

\[
\sum_{n \leq x} L_{Q_{k,r},a}(n) = \frac{\zeta(a)\zeta(ka)}{2\zeta(ra)} \left( 1 + \frac{\zeta^2(a)}{\zeta(2a)} \right) + O_a \left( \frac{\log x}{x^a-1} \right),
\]

where the implied constant depends on \( a \).

Similarly the formulae in each of the cases \( S = Q_r, S_r \) and \( U_r \) can be obtained.

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