Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Serdica
Mathematical Journal
Сердика
Математическо списание

The attached copy is furnished for non-commercial research and education use only.
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **47** (2021), 179–190

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON THE LCM-SUM FUNCTION OVER ARBITRARY SETS OF INTEGERS

P. Anantha Reddy, M. Ganeshwar Rao, V. Siva Rama Prasad

Communicated by V. Drensky

Abstract. Let $\mathbb{N}$ denote the set of all positive integers. For $j, n \in \mathbb{N}$, let (j, n) and $[j, n]$ respectively denote their gcd and lcm. If $S \subseteq \mathbb{N}$ and α is a real number then define $L_{S,\alpha}(n)$ to be the sum of $[j, n]^\alpha$, where $j \in \{1, 2, 3, \ldots, n\}$ for which $(j, n) \in S$. In this paper we obtain asymptotic formulae for the summatory functions of $L_{S,a}(n)$ and $L_{S,-a}(n)$, where $a \in \mathbb{N}$ and $a \geq 2$. Apart from deducing some results proved earlier for $S = \mathbb{N}$ by Ikeda and Matsuoka, certain new asymptotic formulae are obtained here.

1. Introduction. Let $\mathbb{N}$ denote the set of all positive integers. For $j, n \in \mathbb{N}$, let (j, n) and $[j, n]$ be their greatest common divisor and least common multiple respectively.

If $S \subseteq \mathbb{N}$ and α is a real number then define, for any $n \in \mathbb{N}$,

$$L_{S,\alpha}(n) = \sum_{\substack{j=1 \\ (j,n)\in S}}^{n} [j, n]^\alpha, \tag{1.1}$$

2020 *Mathematics Subject Classification:* Primary 11A25; Secondary 11N37.

Key words: Zeta-function of S, unitary divisor, r-free integer, semi-r-free integer, unitary r-free integer, (k, r)-integer.

where the sum is over those $j \in \{1, 2, 3, \ldots, n\}$ for which $(j, n) \in S$.

Observe that $L_{\mathbb{N},\alpha}(n) = L_\alpha(n)$, the arithmetic function first considered by Alladi [1] who proved:

Lemma 1.1. *If $x \geq 1$ and $\alpha \geq 1$ then*

$$\sum_{n \leq x} L_\alpha(n) = C(\alpha).x^{2\alpha+2} + O\left(x^{2\alpha+1+\varepsilon}\right),$$

for any $\varepsilon > 0$, where

$$C(\alpha) = \frac{\zeta(\alpha+2)}{2(\alpha+1)^2\zeta(2)}, \tag{1.2}$$

in which $\zeta(t)$ is the Riemann-zeta function.

The order term in Lemma 1.1 has been improved in the case $\alpha = 1$ by Bordellès [3], who also studied the case $\alpha = -1$ in the same paper. Recently Ikeda and Matsuoka [6] have considered the cases of Lemma 1.1 for $\alpha = a$ and $\alpha = -a$ where $a \in \mathbb{N}$ and $a \geq 2$.

Let $S \subseteq \mathbb{N}$ be arbitrary. In this paper, an asymptotic formula for $\sum_{n \leq x} L_{S,a}(n)$, when $a \in \mathbb{N}$, is obtained in Theorem 3.1. Also finding the sum $\sum_{n=1}^{\infty} L_{S,-\alpha}(n)$ for any $\alpha > 1$; we give an estimate for $\sum_{n \leq x} L_{S,-a}(n)$, when $a \in \mathbb{N}$ with $a \geq 2$, in Theorem 3.4. Noting the results proved in [3] and [6] are deducible from our theorems, several new asymptotic formulae are given for certain special subsets of $\mathbb{N}$ in Section 4.

2. Notation and preliminaries. For any $S \subseteq \mathbb{N}$, let $\chi_S(n)$ be its *characteristic function.* (That is, $\chi_S(n)$=1 or 0 according as $n \in S$ *or* $n \notin S$).

Cohen [4] defined *the zeta-function of* S , $\zeta_S(t)$, by

$$\zeta_S(t) = \sum_{n=1}^{\infty} \frac{\chi_S(n)}{n^t} \text{ for } t > 1. \tag{2.1}$$

Cleary

$$\zeta_{\mathbb{N}}(t) = \sum_{n=1}^{\infty} \frac{1}{n^t} = \zeta(t) \text{ for } t > 1. \tag{2.2}$$

and this has Euler product representation [2, Theorem 11.6]

$$\zeta(t) = \prod_p \left(1 - \frac{1}{p^t}\right)^{-1} \quad \text{for } t > 1, \tag{2.3}$$

the product being over all primes p.

Also if $\mathbb{N}_r = \{1^r, 2^r, 3^r, \dots\}$, where $r \in \mathbb{N}$, then

$$\zeta_{\mathbb{N}_r}(t) = \zeta(rt) = \prod_p \left(1 - \frac{1}{p^{rt}}\right)^{-1}. \tag{2.4}$$

First we prove

Lemma 2.1. *For $n \in \mathbb{N}, S \subseteq \mathbb{N}$ and any real number α, we have*

$$L_{S,\alpha}(n) = \sum_{d\delta=n} \chi_S(d) d^\alpha . \phi_\alpha(\delta)\delta^\alpha,$$

where

$$\phi_\alpha(n) = \sum_{\substack{u=1 \\ (u,n)=1}}^{n} u^\alpha. \tag{2.5}$$

P r o o f. By definition, we have

$$L_{S,\alpha}(n) = \sum_{\substack{0<j\le n \\ (j,n)=d}} \chi_S(d) \left(\frac{jn}{d}\right)^\alpha = \sum_{\substack{d|n, d|j \\ 0<j\le n \\ (\frac{j}{d},\frac{n}{d})=1}} \chi_S(d) \left(\frac{j}{d}\right)^\alpha . n^\alpha$$

$$= \sum_{d\delta=n} \chi_S(d)(d\delta)^\alpha . \left(\sum_{\substack{du=j\le d\delta \\ (u,\delta)=1}} u^\alpha \right)$$

$$= \sum_{d\delta=n} \chi_S(d) d^\alpha . \delta^\alpha \phi_\alpha(\delta),$$

proving the lemma. □

Under a slightly different notation,Ikeda and Matsuoka ([6], page 6) have proved:

Lemma 2.2. *If $a \in \mathbb{N}$ then*

$$\phi_a(n) = n^a \left\{ \frac{1}{a+1}\phi(n) + \frac{1}{2}\varepsilon_0(n) + \frac{1}{a+1}\sum_{k=1}^{a-1} \binom{a+1}{k+1} B_{k+1}\Psi_k(n) \right\},$$

where $\phi(n) = \phi_0(n)$ is the totient function of Euler; $\varepsilon_0(n) = 1$ or 0 according as $n = 1$ or $n > 1$;

$$\Psi_k(n) = \sum_{d\delta=n} \frac{\mu(d)}{\delta^k}, \tag{2.6}$$

in which $\mu(n)$ is the well-known Mobius function and B_k are the Bernoulli numbers defined by the relation $\dfrac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \dfrac{z^k}{k!}$.

Also they proved:

Lemma 2.3 ([6, Lemma 6])**.** *Let $r \in \mathbb{N}$ and $x > e$. Then*

$$\sum_{n\le x} n^r.\phi(n) = \frac{x^{r+2}}{(r+2)\zeta(2)} + O_r\left(x^{r+1}\Delta(x)\right), as\ x \to \infty$$

where $\Delta(x) = (\log x)^{2/3}(\log\log x)^{4/3}$; and the implied constant depends on r.

For this they used the best estimate of Walfisz [10, Satz1, p. 144]

$$\sum_{n\le x} \phi(n) = \frac{x^2}{2\zeta(2)} + O\left(x.\Delta(x)\right), whenever\ x > e;$$

and the Abel's identity (cite[Theorem 4.2].

Now we prove two lemmas:

Lemma 2.4. *If $\Psi_k(n)$ is the function defined in* (2.6) *then for $r > k$,*

$$\sum_{n\le x} n^r.\Psi_k(n) = O(x^{r+1})\ whenever\ x \ge 1.$$

P r o o f.

$$\sum_{n\le x} n^r.\Psi_k(n) = \sum_{d\le x} \mu(d)d^r \left(\sum_{\delta\le \frac{x}{d}} \delta^{r-k} \right)$$

$$= O\left(\sum_{d\le x} |\mu(d)| d^r \left(\frac{x}{d}\right)^{r-k+1}\right)$$
$$= O\left(x^{r-k+1} \sum_{d\le x} \frac{1}{d^{1-k}}\right)$$
$$= O\left(x^{r-k+1}.x^k\right) = O\left(x^{r+1}\right). \qquad \square$$

Lemma 2.5. *For $a \in \mathbb{N}$ and $x > e$, we have*

$$\sum_{n\le x} n^a.\phi_a(n) = \frac{1}{2(a+1)^2\zeta(2)}.x^{2a+2} + O_a\left(x^{2a+1}.\Delta(x)\right),$$

where the implied constant depends on a.

P r o o f. By Lemma 2.2, we have

$$\begin{aligned}\sum_{n\le x} n^a\phi_a(n) &= \frac{1}{a+1}\sum_{n\le x} n^{2a}.\phi(n) + \frac{1}{2}\sum_{n\le x} n^{2a}.\varepsilon_0(n)\\ &\quad + \frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1} B_{k+1}\left(\sum_{n\le x} n^{2a}\Psi_k(n)\right),\end{aligned}$$

wherein using Lemma 2.3 and Lemma 2.4, we get

$$\sum_{n\le x} n^a\phi_a(n) = \frac{1}{a+1}\left\{\frac{x^{2a+2}}{(2a+2)\zeta(2)} + O_a\left(x^{2a+1}.\Delta(x)\right)\right\} + \frac{1}{2} + O_a\left(x^{2a+1}\right),$$

from which the lemma follows. $\square$

3. Main results. Now we prove:

Theorem 3.1. *For any $S \subseteq \mathbb{N}$, $a \in \mathbb{N}$ and $x > e$, we have*

$$\sum_{n\le x} L_{S,a}(n) = C_S(a)\cdot x^{2a+2} + O_a\left(x^{2a+1}\cdot\Delta(x)\right),$$

where

$$C_S(\alpha) = \frac{\zeta_S(\alpha+2)}{2(\alpha+1)^2\zeta(2)} \text{ for } \alpha \ge 1; \tag{3.1}$$

and the implied constant depends on a.

P r o o f. By Lemma 2.1 and Lemma 2.5, we get

$$\begin{aligned}\sum_{n\le x} L_{S,a}(n) &= \sum_{d\le x} \chi_S(d) d^a \left(\sum_{\delta \le \frac{x}{d}} \delta^a \phi_a(\delta) \right) \\ &= \sum_{d\le x} \chi_S(d) d^a \left\{ \frac{1}{2(a+1)^2\zeta(2)} \left(\frac{x}{d}\right)^{2a+2} + O_a\left(\left(\frac{x}{d}\right)^{2a+1} \cdot \Delta\left(\frac{x}{d}\right)\right)\right\} \\ &= \frac{x^{2a+2}}{2(a+1)^2\zeta(2)} \cdot \sum_{d\le x} \frac{\chi_S(d)}{d^{a+2}} + O_a\left(x^{2a+1} \cdot \Delta(x) \cdot \sum_{d\le x} \frac{\chi_S(d)}{d^{a+1}} \right) \\ &= \frac{x^{2a+2}}{2(a+1)^2\zeta(2)} \left\{ \zeta_S(a+2) + O\left(\frac{1}{x^{a+1}}\right)\right\} + O_a\left(x^{2a+1}\Delta(x)\right),\end{aligned}$$

since $\sum_{n\le x} \frac{\chi_S(n)}{n^t} = \zeta_S(t) + O\left(\frac{1}{x^{t-1}}\right)$ for $t > 1$ and $\sum_{n\le x} \frac{\chi_S(n)}{n^{a+1}} = O(1)$, from which the theorem follows. □

Corollary 3.2 ([6, Theorem 2])**.** *If $x > e$ and $a \in \mathbb{N}$ then*

$$\sum_{n\le x} L_a(n) = \frac{\zeta(a+2)}{2(a+1)^2\zeta(2)} x^{2a+2} + O_a\left(x^{2a+1} \cdot \Delta(x)\right) \text{ as } x \to \infty,$$

where the implied constant depends on a

P r o o f. Taking $S = \mathbb{N}$ in Theorem 3.1, we get the corollary in view of (2.2) and the fact $C_{\mathbb{N}}(a) = C(a)$, given in (1.2). □

Corollary 3.3. *If $x > e$, and $a, r \in \mathbb{N}$ then*

$$\sum_{n\le x} L_{\mathbb{N}_r,a}(n) = \frac{\zeta(ar+2r)}{2(a+1)^2\zeta(2)} x^{2a+2} + O_a\left(x^{2a+1} \cdot \Delta(x)\right),$$

where the implied constant depends on a

P r o o f. Taking $S = \mathbb{N}_r$ in Theorem 3.1, the corollary follows in view of (2.4) and (3.1). □

Here the formula in Corollary 3.3 is new. Also the case $r = 1$ of the Corollary 3.3 gives Corollary 3.2.

Some more special cases of S will be discussed in Section 4.

Before proving our next result, we recall that a divisor d of $n \in \mathbb{N}$ is said to be *unitary* if $\left(d.\frac{n}{d}\right) = 1$. If $\tau^*(n)$ is the number of unitary divisors of n

(That is, $\tau^*(n) = \sum_{\substack{d\delta=n \\ (d,\delta)=1}} 1$), then it is easy to see that $\tau^*(n) = 2^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of n. Also its Dirichlet series is given by

$$\sum_{n=1}^{\infty} \frac{\tau^*(n)}{n^t} = \frac{\zeta^2(t)}{\zeta(2t)} \text{ for } t > 1 \tag{3.2}$$

(see [2, Exercise 7, page 247]).

Now we prove the theorem given below:

Theorem 3.4. *Suppose $S \subseteq \mathbb{N}$. Then*
(i) for any $\alpha > 1$,

$$\sum_{n=1}^{\infty} L_{S,-\alpha}(n) = B_S(\alpha),$$

where

$$B_S(\alpha) = \frac{1}{2}\zeta_S(\alpha)\left\{1 + \frac{\zeta^2(\alpha)}{\zeta(2\alpha)}\right\}. \tag{3.3}$$

(ii) for $x \geq 1$ and $a \in \mathbb{N}$ with $a \geq 2$, we have

$$\sum_{n\leq x} L_{S,-a}(n) = B_S(a) + O_a\left(\frac{\log x}{x^{a-1}}\right), \text{ as } x \to \infty,$$

where the implied constant depends on a.

Proof. (*i*) For $\alpha > 1$, we have, by definition

$$L_{S,-\alpha}(n) = \sum_{\substack{0<j\leq n \\ (j,n)=d}} \chi_S(d)\left(\frac{jn}{d}\right)^{-\alpha} = \frac{1}{n^\alpha} \sum_{\substack{d\delta=n \\ du=j \\ 0<u\leq\delta \\ (u,\delta)=1}} \chi_S(d)\cdot u^{-\alpha}$$

$$= \frac{1}{n^\alpha} \sum_{d\delta=n} \chi_S(d) \left(\sum_{\substack{0<u\leq\delta \\ (u,\delta)=1}} \frac{1}{u^\alpha} \right)$$

so that

$$\sum_{n=1}^{\infty} L_{S,-\alpha}(n) = \sum_{d\delta=1}^{\infty} \frac{\chi_S(d)}{d^\alpha \delta^\alpha} \left(\sum_{\substack{0<u\le\delta \\ (u,\delta)=1}} \frac{1}{u^\alpha} \right) \tag{3.4}$$

$$= \left\{ \sum_{d=1}^{\infty} \frac{\chi_S(d)}{d^\alpha} \right\} \left\{ \sum_{\delta=1}^{\infty} \frac{1}{\delta^\alpha} \left(\sum_{\substack{0<u\le\delta \\ (u,\delta)=1}} \frac{1}{u^\alpha} \right) \right\}$$

$$= \zeta_S(\alpha) \cdot \sum_{m=1}^{\infty} \frac{1}{m^\alpha} \left(\sum_{\substack{u\delta=m,(u,\delta)=1 \\ 0<u\le\delta}} 1 \right)$$

But

$$\sum_{m=1}^{\infty} \frac{1}{m^\alpha} \left(\sum_{\substack{u\delta=m,(u,\delta)=1 \\ 0<u\le\delta}} 1 \right) = 1 + \frac{1}{2} \sum_{m=2}^{\infty} \frac{1}{m^\alpha} \left(\sum_{\substack{u\delta=m \\ (u,\delta)=1}} 1 \right) \tag{3.5}$$

$$= 1 + \frac{1}{2} \sum_{m=2}^{\infty} \frac{\tau^*(m)}{m^\alpha} = 1 + \frac{1}{2} \left\{ \frac{\zeta^2(\alpha)}{\zeta(2\alpha)} - 1 \right\},$$

in view of (3.2)

Now (3.4) and (3.5) prove part (i) of Theorem 3.4.

Again if $a \in \mathbb{N}$ and $a \ge 2$ then for $x \ge 1$, we have

$$\sum_{n\le x} L_{S,-a}(n) = \sum_{n=1}^{\infty} L_{S,-a}(n) - \sum_{n>x} L_{S,-a}(n) \tag{3.6}$$

$$= B_S(a) - \sum_{n>x} L_{S,-a}(n).$$

But for any $S \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, it is clear that $L_{S,-a}(n) \leq L_{-a}(n)$, so that $\sum_{n>x} L_{S,-a}(n) \leq \sum_{n>x} L_{-a}(n)$. It has been proved in [6, pages 9 and 10] that $\sum_{n>x} L_{-a}(n) = O_a\left(\frac{\log x}{x^{a-1}}\right)$ for $x \geq 1$. This together with (3.6) prove part (ii) of Theorem 3.4. □

Remark 3.5. Taking $S = \mathbb{N}$, in view of (2.2), part (i) of the Theorem 3.4 gives the equation (2) of Theorem 2 in [6].

4. Some new asymptotic formulae. In this section we consider certain special subsets of $\mathbb{N}$ to derive new asymptotic formulae from our theorems. For this a brief study of M-free integers introduced by Rieger [7] is useful.

Let M be a set of positive integer with its minimal element r is such that $r > 1$. A positive integer of the form $n = \prod_{i=1}^{l} p_i^{\beta_i}$, where p_i are distinct primes and β_i are non-negative integers is said to be *M-free* if $\beta_i \notin M$ for $i = 1, 2, \ldots, l$. Denote the set of all *M-free integers* by Q_M.

Clearly $1 \in Q_M$ for every $M \subseteq \mathbb{N}$. Also χ_{Q_M} is a multiplicative function (That is, $\chi_{Q_M}(ab) = \chi_{Q_M}(a) \cdot \chi_{Q_M}(b)$ whenever $(a, b) = 1$). Further the series $\sum_{n=1}^{\infty} \frac{\chi_{Q_M}(n)}{n^t} = \zeta_{Q_M}(t)$ converges absolutely for $t > 1$ so that it has Euler product representation [2, Theorem 11.6] given by

$$\zeta_{Q_M}(t) = \prod_p \left\{ \sum_{\beta=0}^{\infty} \frac{\chi_{Q_M}(p^\beta)}{p^{\beta t}} \right\} = \prod_p \left\{ \sum_{\substack{\beta=0 \\ \beta \notin M}}^{\infty} \frac{1}{p^{\beta t}} \right\} \tag{4.1}$$

$$= \prod_p \left\{ \sum_{\beta=0}^{\infty} \frac{1}{p^{\beta t}} - \sum_{\substack{\beta=0 \\ \beta \in M}}^{\infty} \frac{1}{p^{\beta t}} \right\} = \prod_p \left\{ \frac{1}{1 - \frac{1}{p^t}} - \sum_{\substack{\beta=r \\ \beta \in M}}^{\infty} \frac{1}{p^{\beta t}} \right\}$$

If the elements of M are explicitly known then the sum on the right of (4.1) and thereby $\zeta_{Q_M}(t)$ can be expressed in compact form. For example, we consider the sets given below:

$A = \{r, r+1, r+2, \dots\}$; $B = \{r\}$; $C = \{r, 2r, 3r, \dots\}$ and $D = \{\beta : \beta \geq r, \beta \equiv j \pmod{k}$ for some j with $r \leq j \leq k-1\}$, in which $k \in \mathbb{N}$ is such that $2 \leq r < k$.

Now (4.1) together with (2.3) and (2.4) give, for $t > 1$, that

$$\zeta_{Q_A}(t) = \prod_p \left\{ \frac{1}{1-\frac{1}{p^t}} - \sum_{\beta=r}^{\infty} \frac{1}{p^{\beta t}} \right\} = \prod_p \left\{ \frac{1-\frac{1}{p^{rt}}}{1-\frac{1}{p^t}} \right\} = \frac{\zeta(t)}{\zeta(rt)}; \tag{4.2}$$

$$\zeta_{Q_B}(t) = \prod_p \left\{ \frac{1}{1-\frac{1}{p^t}} - \frac{1}{p^{rt}} \right\} = \zeta(t) \cdot \prod_p \left\{ 1 - \frac{1}{p^{rt}} + \frac{1}{p^{(r+1)t}} \right\}; \tag{4.3}$$

$$\zeta_{Q_C}(t) = \prod_p \left\{ \frac{1}{1-\frac{1}{p^t}} - \sum_{k=1}^{\infty} \frac{1}{p^{krt}} \right\} = \zeta(t)\zeta(rt) \cdot \prod_p \left\{ 1 - \frac{2}{p^{rt}} + \frac{1}{p^{(r+1)t}} \right\}; \tag{4.4}$$

and

$$\begin{aligned} \zeta_{Q_D}(t) &= \prod_p \left\{ \frac{1}{1-\frac{1}{p^t}} - \sum_{u=0}^{\infty} \sum_{j=r}^{k-1} \frac{1}{p^{(ku+j)t}} \right\} \\ &= \prod_p \left\{ \frac{(1-\frac{1}{p^{rt}})}{(1-\frac{1}{p^t})(1-\frac{1}{p^{kt}})} \right\} = \frac{\zeta(t)\zeta(kt)}{\zeta(rt)}. \end{aligned} \tag{4.5}$$

Let $r, n \in \mathbb{N}$ and $r > 1$.

n is said to be *r-free* if it is not divisible by p^r for any prime p and the set of all *r-free integer* is denoted by Q_r.

Suryanarayana [9] called n as *semi-r-free* if p^r is not a unitary divisor of it for any prime p. We denote the set of all *semi-r-free integers* by S_r.

Cohen [5] termed n as *unitarily-r-free* if p^{kr} is not a unitary divisor of it for any prime p and any integer $k \geq 1$. Let U_r denote the set of all unitarily-r-free integers.

If $2 \leq r < k$ and if n is of the form $n = a^k \cdot b$, where $a \in \mathbb{N}$ and b is r-free (that is, $b \in Q_r$) then n is called a (k,r)-*integer* by Cohen [5]; and independently Subbarao and Harris [8] studied these numbers (under a different notation). Let $Q_{k,r}$ denote the set of all (k,r)-integers. It may be noted that $Q_{\infty,r} = Q_r$, showing (k,r)-integer is a generalized r-free integer.

Now it is easy to see that $Q_A = Q_r$, $Q_B = S_r$, $Q_C = U_r$ and $Q_D = Q_{k,r}$. Therefore (4.2), (4.3), (4.4) and (4.5) respectively give $\zeta_{Q_r}(t)$, $\zeta_{S_r}(t)$, $\zeta_{U_r}(t)$ and

$\zeta_{Q_{k,r}}(t)$ in compact forms. Hence by taking $S = Q_r$, S_r, U_r and $Q_{k,r}$ in Theorem 3.1 and Theorem 3.4, we find new asymptotic formulae in each case. For example, if $S = Q_{k,r}$ then we get from Theorem 3.1 the following:

Corollary 4.1. *If $a \in \mathbb{N}$ and $x > e$ then*

$$\sum_{n\le x} L_{Q_{k,r},a}(n) = C_{Q_{k,r}}(a)\cdot x^{2a+2} + O_a\left(x^{2a+1}\cdot\Delta(x)\right),\ \text{as } x\to\infty$$

where

$$C_{Q_{k,r}}(a) = \frac{\zeta(a+2)\zeta(ak+2k)}{2(a+1)^2\zeta(2)\zeta(ar+2r)}$$

and the implied constant depends on a.

Also in this case Theorem 3.4 gives:

Corollary 4.2. (*i*) *For any $\alpha > 1$,*

$$\sum_{n=1}^{\infty} L_{Q_{k,r},-\alpha}(n) = \frac{\zeta(\alpha)\zeta(k\alpha)}{2\zeta(r\alpha)}\left\{1+\frac{\zeta^2(\alpha)}{\zeta(2\alpha)}\right\} \tag{4.6}$$

and
(*ii*) *for $a \in \mathbb{N}$ with $a \ge 2$, $x \ge 1$ we have*

$$\sum_{n\le x} L_{Q_{k,r},-a}(n) = \frac{\zeta(a)\zeta(ka)}{2\zeta(ra)}\left(1+\frac{\zeta^2(a)}{\zeta(2a)}\right) + O_a\left(\frac{\log x}{x^{a-1}}\right), \tag{4.7}$$

where the implied constant depends on a.

Similarly the formulae in each of the cases $S = Q_r$, S_r and U_r can be obtained.

Acknowledgement. The authors would like to thank the referee for making changes, for more clarity, in the REFERENCES part of the paper.

REFERENCES

[1] K. Alladi. On generalized Euler function and related totients. New concepts in Arithmetical functions. MATSCIENCE Report No. 83. Institute of Mathematical Sciences, Madras, 1975.

[2] T. M. Apostol. Introduction to Analytic Number Theory. Undergraduate Texts in Mathematics. New York-Heidelberg, Springer-Verlag, 1976; Springer International Student Edition, Narosa Publishing House, New Delhi, 1998.

[3] O. Bordellès. Mean values of generalized gcd-sum and lcm-sum functions, *J. Integer Seq.* **10**, *9* (2007), Article. 07.9.2, 13 pp.

[4] E. Cohen. Arithmetical functions associated with arbitrary sets of integers. *Acta Arith.* **5** (1959), 407–415, 435.

[5] E. Cohen. Some sets of integers related to the k-free integers. *Acta Sci. Math. (Szeged)*, **22** (1961), 223–233.

[6] S.Ikeda, K. Matsuoka. On the lcm-sum function. *J. Integer Seq.* **17**, *1* (2014), Article. 14.1.7, 11 pp.

[7] G. J. Rieger. Einige Verteilungsfragen mit K-leeren Zahlen, r-Zahlen und Primzahlen. *J. Reine. Angew. Math.* 262(263) (1973), 189–193 (in German).

[8] M. V. Subba Rao, V. C. Harris. A new generalization on Ramanujan's sum. *J. London. Math. Soc.*, **41** (1966), 595–604.

[9] D. Suryanarayana. Semi-k-free integers. *Elem. Math.* 26 (1971), 39–40.

[10] A. Walfisz. Weylsche Exponentialsummen in der neuren Zahlentheorie. Mathematische Forschungsberichte, XV. VEB Deutscher Verlag der Wissenschaften, Berlin, 1963 (in German).

P. Anantha Reddy
Government Polytechnic
Nizamabad, Telangana-503002, India
e-mail: ananth_palle@yahoo.co.in

M. Ganeshwar Rao
Chaitanya Bharathi Institute of Technology
Hyderabad, Telangana-500075, India
e-mail: ganehwararao_maths@cbit.ac.in

V. Siva Rama Prasad
Department of Mathematics
Osmania University
Hyderabad, Telangana-500007, India
e-mail: vangalasrp@yahoo.co.in

Received January 17, 2021