DIOPHANTINE APPROXIMATION BY PRIME NUMBERS
OF A SPECIAL FORM

Tatiana L. Todorova

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Abstract. We show that whenever $\delta > 0$, $\eta$ are reals and constants $\lambda_i$ subject to certain assumptions, there are infinitely many prime triples $p_1, p_2, p_3$ satisfying the inequality $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-1/18+\delta}$ and such that, for each $i \in \{1, 2, 3\}$, $p_i + 2$ has at most 7 prime factors. The proof uses Davenport–Heilbronn adaption of the circle method together with a vector sieve method.

1. Introduction. The famous prime twins conjecture states that there exist infinitely many primes $p$ such that $p + 2$ is a prime too. This hypothesis is still not proved but there are established many approximations to this result. Let $P_r$ be an integer with no more than $r$ prime factors, counted with their multiplicities. In 1973 Chen [3] showed that there are infinitely many primes $p$ with $p + 2 = P_2$.

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Suppose that we have a problem including primes (equation, inequality, . . .) and let $r \geq 2$. We may consider the same problem with primes $p$, such that $p + 2 = p_r$. Up to now many hybrid theorems were proved. We mention some of them.

In 1937, Vinogradov [23] proved that every sufficiently large odd $n$ can be represented as a sum

\begin{equation}
    n = p_1 + p_2 + p_3
\end{equation}

of primes $p_1$, $p_2$, $p_3$. In 2000 Peneva [15] and Tolev [20] looked for representations with primes $p_i$, subject to $p_i + 2 = p_{r_i}$ for some $r_i \geq 2$. It was established in [20] that if $n$ is sufficiently large and $n \equiv 3 \pmod{6}$ then (1) has a solution in primes $p_1$, $p_2$, $p_3$ with

\begin{align*}
p_1 + 2 &= P_2, \quad p_2 + 2 = P_5, \quad p_3 + 2 = P_7.
\end{align*}

The last theorem was subsequently sharpened and finally K. Matomäki and Shao [13] proved (1) with

\begin{align*}
p_1 + 2 &= P_2, \quad p_2 + 2 = P_2, \quad p_3 + 2 = P_2.
\end{align*}

In 1947 Vinogradov [24] established that if $0 < \theta < 1/5$ then there are infinitely many primes $p$, subject to the inequality

\begin{equation}
    \|\alpha p + \beta\| < p^{-\theta}.
\end{equation}

In 2007 Todorova and Tolev [19] proved that if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\beta \in \mathbb{R}$ and $0 < \theta \leq 1/100$ then there are infinitely many primes $p$ with $p + 2 = P_4$, satisfying the inequality (2). Latter Matomäki [11] proved a Bombieri–Vinogradov type result for linear exponential sums over primes and showed that this actually holds with $p + 2 = P_2$ and $\theta = 1/1000$. The best result is due to Shi San-Ying [17] with $p + 2 = P_2$ and $\theta = 1.5/100$. The present paper is devoted to another popular problem for primes $p_i$, which is studied under the additional restrictions $p_i + 2 = p_{r_i}$ for some $r_i \geq 2$. According to R. C. Vaughan [21] there are infinitely many ordered triples of primes $p_1$, $p_2$, $p_3$ with

\begin{equation}
    |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-\xi + \delta}
\end{equation}

for $\xi = 1/10$, $\delta > 0$ and some constants $\lambda_j$, $j = 1, 2, 3$, $\eta$, subject to the following restrictions:

\begin{equation}
    \lambda_i \in \mathbb{R}, \quad \lambda_i \neq 0, \quad i = 1, 2, 3;
\end{equation}
Diophantine approximation by prime numbers of a special form

\( \lambda_1, \lambda_2, \lambda_3 \) not all of the same sign;

\( \lambda_1/\lambda_2 \in \mathbb{R} \setminus \mathbb{Q}; \)

\( \eta \in \mathbb{R}. \)

Latter the upper bound for \( \xi \) was improved by Baker and Harman [1] to \( \xi = 1/6 \), by Harman [7] to \( \xi = 1/5 \) and the strongest published result is due to K. Matomäki [12] with \( \xi = 2/9 \).

In 2015 Dimitrov, Todorova [5] proved that there are infinitely many ordered triples of primes \( p_1, p_2, p_3 \) such that the inequality (3) is fulfilled with right-hand side \( \max \{ \log p_j \}^{-A} \) and \( p_i + 2 = P_8, i = 1, 2, 3 \).

Result of this type were obtained by Dimitrov [4] with \( \xi = 1/12 \) in the inequality (3) and \( p_i + 2 = P_{28}, i = 1, 2, 3 \).

We prove the following

**Theorem 1.** If conditions (4), (5), (6), (7) are fulfilled then there are infinitely many ordered triples of primes \( p_1, p_2, p_3 \) with

\[
|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-1/18}
\]

and

\[ p_1 + 2 = P'_7, \quad p_2 + 2 = P''_7, \quad p_3 + 2 = P'''_7. \]

**2. Notations.** By \( p \) and \( q \) we always denote primes. As usual, \( \varphi(n) \) and \( \mu(n) \) denote respectively, Euler’s function and Möbius’ function. We denote by \( \tau_k(n) \) the number of solutions of the equation \( m_1 m_2 \cdots m_k = n \) in natural numbers \( m_1, \ldots, m_k \) and \( \tau_2(n) = \tau(n) \). Let \( (m_1, m_2) \) and \([m_1, m_2]\) be the greatest common divisor and the least common multiple of \( m_1, m_2 \). Instead of \( m \equiv n \pmod{k} \) we write for simplicity \( m \equiv n(k) \). As usual, \([y]\) denotes the integer part of \( y \), \( e(y) = e^{2\pi i y} \).

Let \( q_0 \) be an arbitrary positive integer and \( X \) be such that

\[
q_0^2 = X^{7/9};
\]

\[
\varepsilon = X^{-1/18+\delta}, \quad \delta > 0 \text{ is arbitrary small};
\]

\[
\Delta = X^{-7/9} \log X;
\]

\[
H = \frac{\log^2 X}{\varepsilon};
\]

\[
D = X^{7/18};
\]
\[ z = X^\beta, \quad 0 < \beta < 7/36; \]
\[ P(z) = \prod_{2 < p \leq z} p; \]
\[ S_d(\alpha) = \sum_{\lambda_0 X < p \leq X} e(p\alpha) \log p, \quad 0 < \lambda_0 < 1; \]
\[ I(\alpha) = \int_{\lambda_0 X}^X e(\alpha y) dy; \]
\[ E(x, q, a) = \sum_{p \leq x} \log p - \frac{x}{\varphi(q)}. \]

The restrictions on \( \lambda_0 \) and the value of \( \beta \) will be specified later. We will write \( k \sim K \) when \( K/2 < k \leq K \). The letter \( \theta \) denotes an arbitrary small positive number, not the same in all appearances. For example this convention allows us to write \( x^\theta \log x \ll x^\theta \).

3. Outline of the proof. We notice that if \((p+2, P(z)) = 1\) then \( p+2 = P_{1/\beta} \). Our aim is to prove that for a specific (as large as possible) value of \( \beta \) there exists a sequence \( X_1, X_2, \ldots \to \infty \) and primes \( p_i \in (\lambda_0 X_j, X_j], i = 1, 2, 3 \) with \( |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \varepsilon \) and \( p_i + 2 = P_{1/\beta}, i = 1, 2, 3 \). In such a way, we get an infinite sequence of triples of primes \( p_1, p_2, p_3 \) with the desired properties.

Our method goes back to Vaughan [21] but we also use the Davenport–Heilbronn adaptation of the circle method (see[22], ch. 11) combined with a vector sieve similar to that one from [20].

We take a function \( v \) such that

\[ v(x) = 1 \quad \text{for} \ |x| \leq 3\varepsilon/4; \]
\[ 0 < v(x) < 1 \quad \text{for} \ 3\varepsilon/4 < |x| < \varepsilon; \]
\[ v(x) = 0 \quad \text{for} \ |x| \geq \varepsilon \]

and it has derivatives of sufficiently large order.

So if

\[ \Gamma_0(X) = \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} v(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \log p_1 \log p_2 \log p_3 > 0 \]
then the number of solutions of (8) in primes $p_i \in (\lambda_0 X, X]$, $p_i + 2 = P_{[1/\beta]}$, $i = 1, 2, 3$ is positive.

Let $\lambda^\pm(d)$ be the lower and upper bounds Rosser’s weights of level $D$, hence if $d = p_1 \cdots p_r$, $p_1 > \cdots > p_r$, $r \geq 1$ then

$$(21) \quad \lambda^+(d) = \begin{cases} (-1)^r, & \text{if } p_1 \cdots p_{2s}p_{2s+1}^3 < D \text{ for all } 0 \leq s \leq (r - 1)/2; \\ 0, & \text{otherwise}; \end{cases}$$

$$(22) \quad \lambda^-(d) = \begin{cases} (-1)^r, & \text{if } p_1 \cdots p_{2s-1}p_{2s}^3 < D \text{ for all } 0 \leq s \leq r/2; \\ 0, & \text{otherwise}; \end{cases}$$

For further properties of Rosser’s weights we refer to [8], [9].

Let $\Lambda_i = \sum_{\lambda(d) | (p_i + 2, P(z))} \mu(d)$ be the characteristic function of primes $p_i$, such that $(p_i + 2, P(z)) = 1$ for $i = 1, 2, 3$. Then from (20) we obtain the condition

$$(23) \quad \Gamma_0(X) = \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} v(\lambda_1p_1 + \lambda_2p_2 + \lambda_3p_3 + \eta)\Lambda_1\Lambda_2\Lambda_3 \log p_1 \log p_2 \log p_3 > 0.$$

To set up a vector sieve, we use lower and upper bounds

$$(24) \quad \Lambda^\pm_i = \sum_{d | (p_i + 2, P(z))} \lambda^\pm(d), \quad i = 1, 2, 3.$$ 

From the linear sieve we know that $\Lambda^-_i \leq \Lambda_i \leq \Lambda^+_i$ (see [2, Lemma 10]). Then we have a simple inequality

$$(25) \quad \Lambda_1\Lambda_2\Lambda_3 \geq \Lambda^-_1\Lambda^+_2\Lambda^+_3 + \Lambda^+_1\Lambda^-_2\Lambda^+_3 + \Lambda^+_1\Lambda^+_2\Lambda^-_3 - 2\Lambda^+_1\Lambda^-_2\Lambda^-_3.$$

analogous to this one in [2, Lemma 13]. Using (23) and (25) we get

$$(26) \quad \Gamma_0(X) \geq \Gamma(X) = \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} v(\lambda_1p_1 + \lambda_2p_2 + \lambda_3p_3 + \eta)$$

$$(27) \quad \times (\Lambda^-_1\Lambda^+_2\Lambda^+_3 + \Lambda^+_1\Lambda^-_2\Lambda^+_3 + \Lambda^+_1\Lambda^+_2\Lambda^-_3 - 2\Lambda^+_1\Lambda^-_2\Lambda^-_3) \log p_1 \log p_2 \log p_3 > 0.$$ 

Let $\Upsilon(x) = \int_{-\infty}^{\infty} v(t)e(-xt)dt$ be the Fourier transform of the function $v$ defined by (19). Then

$$(28) \quad |\Upsilon(x)| \leq \min\left(\frac{7\varepsilon}{4}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left(\frac{k}{2\pi|x|\varepsilon/8}\right)^{k}\right),$$
for all \( k \in \mathbb{N} \) – see [16].

We substitute the function \( v(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \) in (26) with its inverse Fourier transform and get

\[
\sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} (\Lambda_1^{-\lambda_2^+ \Lambda_3} + \Lambda_1^{-\lambda_2^- \Lambda_3} + \Lambda_1^{-\lambda_2^+ \Lambda_3} - 2\Lambda_1^{-\lambda_2^+ \Lambda_3}) \log p_1 \log p_2 \log p_3
\]

(28)

\[
\times \int_{-\infty}^{\infty} \Upsilon(t) e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t) \, dt.
\]

Hence our argument is based on the following consequence of (28).

**Lemma 1.** If the integral

\[
\Gamma(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t) \log p_1 \log p_2 \log p_3
\]

(29)

\[
\times (\Lambda_1^{-\lambda_2^+ \Lambda_3} + \Lambda_1^{-\lambda_2^+ \Lambda_3} + \Lambda_1^{-\lambda_2^+ \Lambda_3} - 2\Lambda_1^{-\lambda_2^+ \Lambda_3}) \, dt
\]

is positive then the number of solutions of (8) in primes \( p_i \in (\lambda_0 X, X] \), \( p_i + 2 = P_{1/\beta} \), \( i = 1, 2, 3 \) is positive. Here

\[
\Gamma_1(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t)
\]

\[
\times \Lambda_1^{-\lambda_2^+ \Lambda_3} \log p_1 \log p_2 \log p_3 \, dt;
\]

\[
\Gamma_2(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t)
\]

\[
\times \Lambda_1^{-\lambda_2^- \Lambda_3} \log p_1 \log p_2 \log p_3 \, dt;
\]

\[
\Gamma_3(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t)
\]

\[
\times \Lambda_1^{-\lambda_2^+ \Lambda_3} \log p_1 \log p_2 \log p_3 \, dt;
\]
\[ \Gamma_4(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} e\left((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\right) \times \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \log p_1 \log p_2 \log p_3 dt. \]

We are going to estimate \( \Gamma_1(X) \). The integrals \( \Gamma_2(X), \Gamma_3(X), \Gamma_4(X) \) can be treated in a similar way. Changing the order of summation and bearing in mind (24) we obtain

\[ \Gamma_1(X) = \int_{-\infty}^{\infty} \Upsilon(t) e(\eta t) L^- (\lambda_1 t, X) L^+ (\lambda_2 t, X) L^+ (\lambda_3 t, X) dt, \]

where

\[ L^\pm (t, X) = \sum_{d \mid P(z)} \lambda^\pm (d) \sum_{\lambda_0 X < p \leq X \atop p+2 \equiv 0 (d)} e(pt) \log p, \]

Let us subdivide \( \Gamma_1(X) \) into three integrals

\[ \Gamma_1(X) = \Gamma_1^{(1)}(X) + \Gamma_1^{(2)}(X) + \Gamma_1^{(3)}(X), \]

where

\[ \Gamma_1^{(1)}(X) = \int_{|t| \leq \Delta} \Upsilon(t) e(\eta t) L^- (\lambda_1 t, X) L^+ (\lambda_2 t, X) L^+ (\lambda_3 t, X) dt, \]

\[ \Gamma_1^{(2)}(X) = \int_{\Delta < |t| < H} \Upsilon(t) e(\eta t) L^- (\lambda_1 t, X) L^+ (\lambda_2 t, X) L^+ (\lambda_3 t, X) dt, \]

\[ \Gamma_1^{(3)}(X) = \int_{|t| \geq H} \Upsilon(t) e(\eta t) L^- (\lambda_1 t, X) L^+ (\lambda_2 t, X) L^+ (\lambda_3 t, X) dt. \]

Here the functions \( \Delta = \Delta(X) \) and \( H = H(X) \) are defined with (11) and (12).

We shall estimate \( \Gamma_1^{(3)}(X), \Gamma_1^{(1)}(X), \Gamma_1^{(2)}(X), \) respectively, in the Sections 4, 5, 6. In Section 7 we shall complete the proof of the Theorem.

4. Upper bound for \( \Gamma_1^{(3)}(X) \).

Lemma 2. For the integral \( \Gamma_1^{(3)}(X) \), defined by (34), we have

\[ \Gamma_1^{(3)}(X) \ll 1. \]

Proof. See [5] Lemma 2. \( \square \)
5. Asymptotic formula for $\Gamma_1^{(1)}(X)$. The main term of the integral $\Gamma_1(X)$ we will derive from $\Gamma_1^{(1)}(X)$. Working as in §5 of [4], but with $C = A$, we get

$$\Gamma_1^{(1)}(X) = B(X) \left( \sum_{d \mid P(z)} \frac{\lambda^-(d)}{\varphi(d)} \right) \left( \sum_{d \mid P(z)} \frac{\lambda^+(d)}{\varphi(d)} \right)^2 + O \left( \frac{\varepsilon X^2}{(\log X)^{A-5}} \right).$$

where

$$B(X) = \int_{-\infty}^{\infty} \Upsilon(t) e(\eta t) \left( \int_X X X X e(t(\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3)) dy_1 dy_2 dy_3 \right) dt.$$

According to ([5], Lemma 4), we have following

**Lemma 3.** If there hold (4), (5) and

$$\lambda_0 < \min \left( \frac{\lambda_1}{4|\lambda_3|}, \frac{\lambda_2}{4|\lambda_3|}, \frac{1}{16} \right),$$

then $B(X)$, defined by (37) satisfies

$$B(X) \gg \varepsilon X^2,$$

with a constant implied by the $\gg$-symbol depending only on $\lambda_1$, $\lambda_2$ and $\lambda_3$.

Let

$$G^\pm = \sum_{d \mid P(z)} \frac{\lambda^\pm(d)}{\varphi(d)}.$$

Then from (36) and (38) it follows

$$\Gamma_1^{(1)}(X) = B(X)G^- (G^+)^2 + O \left( \frac{\varepsilon X^2}{(\log X)^{A-5}} \right).$$

6. Upper bound for $\Gamma_1^{(2)}(X)$. To find estimate for $\Gamma_1^{(2)}(X)$ we need some statements.

We will say that a squarefree number $d \in \mathbb{N}$, $d < D$ is well-separated if for any $D_1 \geq 1$ and $D_2 \geq 1$ such that $D = D_1 D_2$ one can write $d = d_1 d_2$ with $d_1 \leq D_1$ and $d_2 \leq D_2$ (see [6, ch. 6, §6.1.2]).
Lemma 4. If $d | P(z)$, $z < D^{1/2}$, $\lambda^\pm$ are Rosser's weight and either 
$\lambda^+(d) \neq 0$ or $\lambda^-(d) \neq 0$ then $d$ is a well-separated number.

Proof. See remark 2, [9] □

We shall use following

Lemma 5. Let $k, m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and

(40) $\left| \alpha - \frac{a}{q} \right| < \frac{1}{q^2}$, $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a, q) = 1$, $q \geq 1$.

Then for every arbitrary small $\theta > 0$ the inequality

$$
\sum_{m \sim M} \tau_\mu(m) \sum_{j \sim J} \tau_\zeta(j) \min \left\{ \frac{x}{m^2 j}, \frac{1}{\| \alpha m^2 j \|} \right\} \ll x^\theta \left( MJ + \frac{x}{M^{3/2}} + \frac{x}{Mq^{1/2}} + \frac{x^{1/2}q^{1/2}}{M} \right)
$$

is fulfilled.


Lemma 6. Suppose $\alpha \in \mathbb{R}$ and $\alpha$ satisfy conditions (40). Let $\xi(d)$ be complex numbers defined for $d \leq D$ and $\xi(d) \ll 1$. If

(41) $\mathcal{L}(X) = \sum_{d \sim D} \xi(d) \sum_{\substack{p \sim Z \setminus \xi(d) \equiv 0 (d)}} e(p\alpha) \log p.$

then for any arbitrary small $\theta > 0$ we have

$$
\mathcal{L}(X) \ll X^\theta \left( \frac{X}{q^{1/2}} + X^{1/2}q^{1/2} + X^{2/3}D + X^{5/6} \right).
$$

Proof. This is Lemma 1 from [19]. □

Above theorem give good bound for $D \leq x^{1/3-\vartheta}$ with $\vartheta > \theta$. To prove upper bound for $\mathcal{L}(X)$ with bigger $D$ we will use following version of Theorem 1, [11]:

Lemma 7. Suppose $\alpha \in \mathbb{R}$ and $\alpha$ satisfy conditions (40), $\xi(d)$ are complex numbers defined for $d \leq D$, $D \leq x^{1/2-\vartheta}$,

(42) $\xi(d) \ll 1$ and $\xi(d) \neq 0$ if and only if $d$ is a well-separated number.
Then for any arbitrary small $\theta > 0$ for sum (41) we have

\[
\mathfrak{L}(X) \ll X^\theta \left( \frac{X}{q^{1/4}} + X^{3/4}q^{1/4} + X^{3/4}D^{1/2} + \frac{X}{D^{1/4}} + X^{17/18} \right).
\]

**Proof.** The proof is similarly to proof of Theorem 1, [11]. □

**Corollary 1.** Suppose $\alpha \in \mathbb{R}$ and $\alpha$ satisfy conditions (40), $\xi(d)$ are complex numbers defined for $d \leq D$, $D \leq x^{17/18}$ and satisfying conditions (42). Then for any arbitrary small $\theta > 0$ for sum

\[
W = \sum_{d \leq D} \xi(d) \sum_{p \sim X \atop p+2\equiv 0 (d)} e(p\alpha) \log p
\]

we have

\[
W \ll X^\theta \left( \frac{X}{q^{1/4}} + X^{3/4}q^{1/4} + X^{17/18} \right).
\]

**Proof.** It is obvious that we can represent the sum $W$ as $O(\log x)$ sums of the type (41). For $D \leq x^{5/18}$ from Lemma 6 we get (44). When $x^{5/18} \leq D \leq x^{7/18}$ from Lemma 7 we obtain (44). □

Let us consider any sum $L^\pm(\alpha, X)$ denoted by (30) and let

\[
L(\alpha, Y) = \sum_{d \leq D} \xi(d) \sum_{p \sim Y \atop p+2\equiv 0 (d)} e(p\alpha) \log p,
\]

where

\[
\xi(d) = \begin{cases} 
\lambda^\pm(d), & \text{if } d \mid P(z), \\
0, & \text{otherwise}.
\end{cases}
\]

It is obvious that

\[
L^\pm(\alpha, X) \ll \max_{\lambda_0 X \leq Y \leq X} |L(\alpha, Y)|.
\]

If

\[
q \in \left[ X^{2/9}, X^{7/9} \right],
\]

then from Corollary 1 for the sum $L(\alpha, Y)$ we get

\[
L(\alpha, Y) \ll Y^{17/18+\theta}.
\]
Therefore
\[ L^{±}(α, X) \ll \max_{λ_0 X \leq Y \leq X} Y^{17/18+θ} \ll X^{17/18+θ}. \]

Let
\[ V(t, X) = \min \{|L^{±}(λ_1 t, X)|, |L^{±}(λ_2 t, X)|\}. \]

We shall prove the following

**Lemma 8.** Let \( t, X, λ_1, λ_2 \in \mathbb{R} \),
\[ |t| \in (\Delta, H), \]
where \( \Delta \) and \( H \) are denoted by (11) and (12), \( λ_1, λ_2 \) satisfy (6) and \( V(t, X) \) is defined by (47). Then there exists a sequence of real numbers \( X_1, X_2, \ldots \to ∞ \) such that
\[ V(t, X_j) \ll X_j^{17/18+θ}, \quad j = 1, 2, \ldots. \]

**Proof.** Our aim is to prove that there exists a sequence \( X_1, X_2, \ldots \to ∞ \) such that for each \( j = 1, 2, \ldots \) at least one of the numbers \( λ_1 t \) and \( λ_2 t \) with \( t \), subject to (48) can be approximated by rational numbers with denominators, satisfying (45). Then the proof follows from (46) and (47).

Since \( \frac{λ_1}{λ_2} \in \mathbb{R}/\mathbb{Q} \) then by [18, Corollary 1B], there exist infinitely many fractions \( \frac{a_0}{q_0} \) with arbitrary large denominators such that
\[ \left| \frac{λ_1}{λ_2} - \frac{a_0}{q_0} \right| < \frac{1}{q_0^2}, \quad (a_0, q_0) = 1. \]

Let \( q_0 \) be sufficiently large and \( X \) be such that \( q_0^2 = X^{7/9} \) (see (9)). Let us notice that there exist \( a_1, q_1 \in \mathbb{Z} \), such that
\[ \left| λ_1 t - \frac{a_1}{q_1} \right| < \frac{1}{q_1 q_0^2}, \quad (a_1, q_1) = 1, \quad 1 \leq q_1 \leq q_0^2, \quad a_1 \neq 0. \]

From Dirichlet’s Theorem (see [10, ch.10, §1]) it follows the existence of integers \( a_1 \) and \( q_1 \), satisfying the first three conditions. If \( a_1 = 0 \) then \( |λ_1 t| < \frac{1}{q_1 q_0^2} \) and from (48) it follows
\[ λ_1 \Delta < λ_1 |t| < \frac{1}{q_0^2}, \quad q_0^2 < \frac{1}{λ_1 \Delta}. \]
From the last inequality, (9) and (11) we obtain

\[ X^{7/9} < \frac{X^{7/9}}{\lambda_1 \log X}, \]

which is impossible for large \( q_0 \), respectively, for a large \( X \). So \( a_1 \neq 0 \). By analogy there exist \( a_2, q_2 \in \mathbb{Z} \), such that

\[
\left| \lambda_2 t - \frac{a_2}{q_2} \right| < \frac{1}{q_2 q_0^2}, \quad (a_2, q_2) = 1, \quad 1 \leq q_2 \leq q_0^2, \quad a_2 \neq 0.
\]

If \( q_i \in \left[ X^{2/9}, X^{7/9} \right] \) for \( i = 1 \) or \( i = 2 \), then the proof is completed. From (9), (51) and (52) we have

\[ q_i \leq X^{7/9} = q_0^2, \quad i = 1, 2. \]

Thus it remains to prove that the case

\[
q_i < X^{2/9}, \quad i = 1, 2
\]

is impossible. Let \( q_i < X^{2/9}, i = 1, 2 \). From (48), (51)–(53) it follows

\[ 1 \leq |a_i| < \frac{1}{q_0^2} + q_i \lambda_i |t| < \frac{1}{q_0^2} + q_i \lambda_i H, \]

\[
1 \leq |a_i| < \frac{1}{q_0^2} + \lambda_i X^{1/18-\delta} \log^2 X, \quad i = 1, 2.
\]

We have

\[
\frac{\lambda_1}{\lambda_2} = \frac{\lambda_1 t}{\lambda_2 t} = \frac{a_1}{q_1} + \left( \frac{\lambda_1 t - a_1}{q_1} \right) = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + \Xi_1}{1 + \Xi_2},
\]

where \( \Xi_i = \frac{q_i}{a_i} \left( \lambda_i t - \frac{a_i}{q_i} \right), \ i = 1, 2 \). According to (51), (52) and (55) we obtain

\[ |\Xi_i| < \frac{q_i}{|a_i|} \cdot \frac{1}{q_i q_0^2} \leq \frac{1}{q_0^2}, \quad i = 1, 2, \]

\[ \frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + O \left( \frac{1}{q_0^2} \right)}{1 + O \left( \frac{1}{q_0^2} \right)} = \frac{a_1 q_2}{a_2 q_1} \left( 1 + O \left( \frac{1}{q_0^2} \right) \right). \]
Thus \( \frac{a_1 q_2}{a_2 q_1} = \mathcal{O}(1) \) and

\[
\frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} + \mathcal{O} \left( \frac{1}{q_0} \right).
\]

(56)

Therefore, both fractions \( \frac{a_0}{q_0} \) and \( \frac{a_1 q_2}{a_2 q_1} \) approximate \( \frac{\lambda_1}{\lambda_2} \). Using (10), (51), (53) and inequality (54) with \( i = 2 \) we obtain

\[
|a_2| q_1 < 1 + \lambda_2 X^{5/18 - \delta} \log^2 X < \frac{q_0}{\log X}
\]

(57)

so \( |a_2| q_1 \neq q_0 \) and the fractions \( \frac{a_0}{q_0} \) and \( \frac{a_1 q_2}{a_2 q_1} \) are different. Then using (57) it follows

\[
\frac{|a_0|}{q_0} - \frac{a_1 q_2}{a_2 q_1} = \frac{|a_0 a_2 q_1 - a_1 q_2 q_0|}{|a_2 q_1 q_0|} \geq \frac{1}{|a_2 q_1 q_0|} \gg \log X \frac{q_0}{q_0^2}.
\]

(58)

On the other hand, from (50) and (56) we have

\[
\left| \frac{a_0}{q_0} - \frac{a_1 q_2}{a_2 q_1} \right| \leq \left| \frac{a_0}{q_0} - \frac{\lambda_1}{\lambda_2} \right| + \left| \frac{\lambda_1}{\lambda_2} - \frac{a_1 q_2}{a_2 q_1} \right| \ll \frac{1}{q_0^2},
\]

which contradicts (58). This rejects the assumption (53). Let \( q_0^{(1)}, q_0^{(2)}, \ldots \) be an infinite sequence of values of \( q_0 \), satisfying (50). Then using (9) one gets an infinite sequence \( X_1, X_2, \ldots \) of values of \( X \), such that at least one of the numbers \( \lambda_1 t \) and \( \lambda_2 t \) can be approximated by rational numbers with denominators, satisfying (45). Hence, the proof is completed. \( \square \)

Let us estimate the integral \( \Gamma^{(2)}_1(X_j) \), denoted by (33). Using \( |\Upsilon(t)| \leq \frac{7 \varepsilon}{4} \) (see (27)), (47) and estimate (49) we find

\[
\Gamma^{(2)}_1(X_j)
\]

\[
\ll \varepsilon \int_{\Delta < |t| < H} V(t, X_j) \left[ \left| L^- \left( \lambda_1 t, X_j \right) L^+ \left( \lambda_3 t, X_j \right) \right| + \left| L^+ \left( \lambda_2 t, X_j \right) L^+ \left( \lambda_3 t, X_j \right) \right| \right] dt
\]

\[
\ll \varepsilon \int_{\Delta < |t| < H} V(t, X_j) \left( \left| L^- \left( \lambda_1 t, X_j \right) \right|^2 + \left| L^+ \left( \lambda_2 t, X_j \right) \right|^2 + \left| L^+ \left( \lambda_3 t, X_j \right) \right|^2 \right) dt
\]

(59)

\[
\ll \varepsilon X_j^{17/18 + \theta} \max_{1 \leq k \leq 3} \mathcal{I}_k,
\]
where
\[ I_k = \int_{\Delta < |t| < H} |L^\pm(\lambda_k t, X_j)|^2 \, dt. \]

Arguing as in [5] we obtain
\[ I_k \ll X_j^{19/18-\delta} \log^7 X_j. \]

Using (59) and (60) and choosing \( \theta < \delta \) we get
\[ \Gamma_{1\,2}(X_j) \ll \varepsilon X_j^{17/18+\theta} X_j^{19/18-\delta} \log^7 X_j \ll \frac{\varepsilon X_j^2}{(\log X_j)^{4-5}}. \]

Summarizing (31), (35), (39) and (61) we find
\[ \Gamma_1(X_j) = B(X_j) G^- (G^+)^2 + O\left(\frac{\varepsilon X_j^2}{(\log X_j)^{4-5}}\right). \]

### 7. Proof of the Theorem.

Since the sums \( \Gamma_2(X_j), \Gamma_3(X_j) \) and \( \Gamma_4(X_j) \) are estimated in the same way then from (29) and (62) we obtain
\[ \Gamma(X_j) = B(X_j) W(X_j) + O\left(\frac{\varepsilon X_j^2}{(\log X_j)^{4-5}}\right), \]

where
\[ W(X_j) = 3 (G^+)^2 \left(G^- - \frac{2}{3} G^+\right). \]

Let \( f(s) \) and \( F(s) \) are the lower and the upper functions of the linear sieve. We know that if
\[ s = \frac{\log D}{\log z}, \quad 2 \leq s \leq 3 \]

then
\[ f(s) = \frac{2e^\gamma \log(s-1)}{s}, \quad F(s) = \frac{2e^\gamma}{s}. \]
where $\gamma = 0.577\ldots$ is the Euler constant (see [2, Lemma 10]). Using (38) and Lemma 10 from [2] we get

$$\mathcal{F}(z)\left( f(s) + \mathcal{O}\left( (\log X)^{-1/3} \right) \right) \leq G^- \leq \mathcal{F}(z) \leq G^+$$

$$\leq \mathcal{F}(z) \left( F(s) + \mathcal{O}\left( (\log X)^{-1/3} \right) \right).$$

Here

$$\mathcal{F}(z) = \prod_{2 < p \leq z} \left( 1 - \frac{1}{p-1} \right) \approx \frac{1}{\log X};$$

see Mertens formula [14, ch.9, §9.1, Theorem 9.1.3] and (14). To estimate $W(X_j)$ from below we shall use the inequalities (see (67)):

$$G^- - \frac{2}{3}G^+ \geq \mathcal{F}(z) \left( f(s) - \frac{2}{3} F(s) + \mathcal{O}\left( (\log X)^{-1/3} \right) \right)$$

$$G^+ \geq \mathcal{F}(z).$$

Let $X = X_j$. Then from (64) and (69) it follows

$$W(X_j) \geq 3\mathcal{F}^3(z) \left( f(s) - \frac{2}{3} F(s) + \mathcal{O}\left( (\log X_j)^{-1/3} \right) \right)$$

We choose $s = 2.949$. Then

$$f(s) - \frac{2}{3} F(s) \geq 0,0001$$

and from (65) for sufficiently large $X$ we get $\frac{1}{\beta} = 7.583$.

We choose $A \geq 10$. Then from (10), (63), (68), (70), (71) and Lemma 3 we obtain

$$\Gamma(X_j) \gg \frac{X_j^{35/18+\delta}}{(\log X_j)^3}. $$

The last inequality implies that $\Gamma(X_j) \to \infty$ as $X_j \to \infty$.

By the definition (26) of $\Gamma(X)$ and the inequality (72) we conclude that for some constant $c_0$ there are at least $c_0 \frac{X_j^{35/18+\delta}}{(\log X_j)^3}$ triples of primes $p_1, p_2, p_3$. 

satisfying $p_i \in (\lambda_0 X_j, X_j)$, $i = 1, 2, 3$, $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \varepsilon$ and such that for every prime factor $p$ of $p_i + 2$, $i = 1, 2, 3$ we have $p \geq X^{0.1318}$.

That completes the proof of Theorem.

Using the same manner of the above proof but with parameters

$$q_0^2 = X^{6/7}, \quad \varepsilon = X^{-1/28+\delta}, \quad \Delta = X^{-6/7} \log X,$$

$$H = \frac{\log^2 X}{\varepsilon}, \quad D = X^{3/7}, \quad z = X^\beta, \quad \beta = 0.1429;$$

we get

**Theorem 2.** If conditions (4), (5), (6), (7) are fulfilled then there are infinitely many ordered triples of primes $p_1, p_2, p_3$ with

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-1/28}$$

and

$$p_1 + 2 = P_5^\ell, \quad p_2 + 2 = P_5^\eta, \quad p_3 + 2 = P_5^{\eta\eta}.$$"
Diophantine approximation by prime numbers of a special form


*Faculty of Mathematics and Informatics*

"St. Kl. Ohridski" University of Sofia

5, J. Bourchier Blvd

1164 Sofia, Bulgaria

e-mail: tlt@fmi.uni-sofia.bg

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