Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Serdica
Mathematical Journal
Сердика
Математическо списание

The attached copy is furnished for non-commercial research and education use only.
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **47** (2021), 255–272

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

DIOPHANTINE APPROXIMATION BY PRIME NUMBERS OF A SPECIAL FORM

Tatiana L. Todorova

Communicated by V. Drensky

Abstract. We show that whenever $\delta > 0$, η are reals and constants λ_i subject to certain assumptions, there are infinitely many prime triples p_1, p_2, p_3 satisfying the inequality $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-1/18+\delta}$ and such that, for each $i \in \{1, 2, 3\}$, $p_i + 2$ has at most 7 prime factors. The proof uses Davenport–Heilbronn adaption of the circle method together with a vector sieve method.

1. Introduction. The famous prime twins conjecture states that there exist infinitely many primes p such that $p + 2$ is a prime too. This hypothesis is still not proved but there are established many approximations to this result. Let P_r be an integer with no more than r prime factors, counted with their multiplicities. In 1973 Chen [3] showed that there are infinitely many primes p with $p + 2 = P_2$.

2020 *Mathematics Subject Classification:* 11D75, 11N36, 11P32.

Key words: Rosser's weights, vector sieve, circle method, almost primes, diophantine inequality.

The author was supported, in part, by a the Science Foundation of Sofia University under contract 80-10-52/10.05.2022.

Suppose that we have a problem including primes (equation, inequality, ...) and let $r \geq 2$. We may consider the same problem with primes p, such that $p+2 = P_r$. Up to now many hybrid theorems were proved. We mention some of them.

In 1937, Vinogradov [23] proved that every sufficiently large odd n can be represented as a sum

$$n = p_1 + p_2 + p_3 \tag{1}$$

of primes p_1, p_2, p_3. In 2000 Peneva [15] and Tolev [20] looked for representations with primes p_i, subject to $p_i + 2 = P_{r_i}$ for some $r_i \geq 2$. It was established in [20] that if n is sufficiently large and $n \equiv 3 \pmod 6$ then (1) has a solution in primes p_1, p_2, p_3 with

$$p_1 + 2 = P_2, \quad p_2 + 2 = P_5, \quad p_3 + 2 = P_7.$$

The last theorem was subsequently sharpened and finally K. Matomäki and Shao [13] proved (1) with

$$p_1 + 2 = P_2, \quad p_2 + 2 = P_2, \quad p_3 + 2 = P_2.$$

In 1947 Vinogradov [24] established that if $0 < \theta < 1/5$ then there are infinitely many primes p, subject to the inequality

$$\|\alpha p + \beta\| < p^{-\theta}. \tag{2}$$

In 2007 Todorova and Tolev [19] proved that if $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\beta \in \mathbb{R}$ and $0 < \theta \leq 1/100$ then there are infinitely many primes p with $p + 2 = P_4$, satisfying the inequality (2). Latter Matomäki [11] proved a Bombieri–Vinogradov type result for linear exponential sums over primes and showed that this actually holds with $p + 2 = P_2$ and $\theta = 1/1000$. The best result is due to Shi San-Ying [17] with $p + 2 = P_2$ and $\theta = 1.5/100$. The present paper is devoted to another popular problem for primes p_i, which is studied under the additional restrictions $p_i + 2 = P_{r_i}$ for some $r_i \geq 2$. According to R. C. Vaughan [21] there are infinitely many ordered triples of primes p_1, p_2, p_3 with

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-\xi+\delta} \tag{3}$$

for $\xi = 1/10$, $\delta > 0$ and some constants λ_j, $j = 1, 2, 3$, η, subject to the following restrictions:

$$\lambda_i \in \mathbb{R}, \quad \lambda_i \neq 0, \quad i = 1, 2, 3; \tag{4}$$

(5) $\lambda_1, \lambda_2, \lambda_3$ not all of the same sign;

(6) $\lambda_1/\lambda_2 \in \mathbb{R} \setminus \mathbb{Q}$;

(7) $\eta \in \mathbb{R}$.

Latter the upper bound for ξ was improved by Baker and Harman [1] to $\xi = 1/6$, by Harman [7] to $\xi = 1/5$ and the strongest published result is due to K. Matomäki [12] with $\xi = 2/9$.

In 2015 Dimitrov, Todorova [5] proved that there are infinitely many ordered triples of primes p_1, p_2, p_3 such that the inequality (3) is fulfilled with righ-hand side $\max\left(\log p_j\right)^{-A}$ and $p_i + 2 = P_8$, $i = 1, 2, 3$.

Result of this type were obtained by Dimitrov [4] with $\xi = 1/12$ in the inequality (3) and $p_i + 2 = P_{28}$, $i = 1, 2, 3$.

We prove the following

Theorem 1. *If conditions* (4), (5), (6), (7) *are fulfilled then there are infinitely many ordered triples of primes* p_1, p_2, p_3 *with*

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-1/18} \tag{8}$$

and

$$p_1 + 2 = P_7', \quad p_2 + 2 = P_7'', \quad p_3 + 2 = P_7'''.$$

2. Notations. By p and q we always denote primes. As usual, $\varphi(n)$ and $\mu(n)$ denote respectively, Euler's function and Möbius' function. We denote by $\tau_k(n)$ the number of solutions of the equation $m_1 m_2 \cdots m_k = n$ in natural numbers $m_1, \ldots, m_k$ and $\tau_2(n) = \tau(n)$. Let (m_1, m_2) and $[m_1, m_2]$ be the greatest common divisor and the least common multiple of m_1, m_2. Instead of $m \equiv n \pmod k$ we write for simplicity $m \equiv n(k)$. As usual, $[y]$ denotes the integer part of y, $e(y) = e^{2\pi i y}$.

Let q_0 be an arbitrary positive integer and X be such that

(9) $q_0^2 = X^{7/9}$;

(10) $\varepsilon = X^{-1/18+\delta}$, $\quad \delta > 0$ is arbitrary small;

(11) $\Delta = X^{-7/9} \log X$;

(12) $H = \dfrac{\log^2 X}{\varepsilon}$;

(13) $D = X^{7/18}$;

$$z = X^{\beta}, \quad 0 < \beta < 7/36; \tag{14}$$

$$P(z) = \prod_{2<p\le z} p; \tag{15}$$

$$S_d(\alpha) = \sum_{\substack{\lambda_0 X < p \le X \\ p+2 \equiv 0\ (d)}} e(p\alpha)\log p, \quad 0 < \lambda_0 < 1; \tag{16}$$

$$I(\alpha) = \int_{\lambda_0 X}^{X} e(\alpha y)dy; \tag{17}$$

$$E(x,q,a) = \sum_{\substack{p\le x \\ p\equiv a\ (q)}} \log p - \frac{x}{\varphi(q)}. \tag{18}$$

The restrictions on λ_0 and the value of β will be specified latter. We will write $k \sim K$ when $K/2 < k \le K$. The letter θ denotes an arbitrary small positive number, not the same in all appearances. For example this convention allows us to write $x^{\theta}\log x \ll x^{\theta}$.

3. Outline of the proof. We notice that if $(p+2,\, P(z)) = 1$ then $p + 2 = P_{[1/\beta]}$. Our aim is to prove that for a specific (as large as possible) value of β there exists a sequence $X_1,\, X_2, \ldots \to \infty$ and primes $p_i \in (\lambda_0 X_j,\, X_j]$, $i = 1,2,3$ with $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \varepsilon$ and $p_i + 2 = P_{[1/\beta]}$, $i = 1,2,3$. In such a way, we get an infinite sequence of triples of primes $p_1,\, p_2,\, p_3$ with the desired properties.

Our method goes back to Vaughan [21] but we also use the Davenport–Heilbronn adaptation of the circle method (see[[22], ch. 11]) combined with a vector sieve similar to that one from [20].

We take a function υ such that

$$\begin{aligned} &\upsilon(x) = 1 && \text{for } |x| \le 3\varepsilon/4; \\ &0 < \upsilon(x) < 1 && \text{for } 3\varepsilon/4 < |x| < \varepsilon; \\ &\upsilon(x) = 0 && \text{for } |x| \ge \varepsilon \end{aligned} \tag{19}$$

and it has derivatives of sufficiently large order.

So if

$$\Gamma_0(X) = \sum_{\substack{\lambda_0 X < p_1,p_2,p_3 \le X \\ (p_i+2,P(z))=1, i=1,2,3}} \upsilon(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)\log p_1 \log p_2 \log p_3 > 0 \tag{20}$$

then the number of solutions of (8) in primes $p_i \in (\lambda_0 X,\, X]$, $p_i + 2 = P_{[1/\beta]}$, $i = 1,\, 2,\, 3$ is positive.

Let $\lambda^{\pm}(d)$ be the lower and upper bounds Rosser's weights of level D, hence if $d = p_1 \cdots p_r$, $p_1 > \cdots > p_r$, $r \geq 1$ then

$$\lambda^{+}(d) = \begin{cases} (-1)^r, & \text{if} \quad p_1 \cdots p_{2s} p_{2s+1}^3 < D \quad \text{for all} \quad 0 \leq s \leq (r-1)/2; \\ 0, & \text{otherwise;} \end{cases} \tag{21}$$

$$\lambda^{-}(d) = \begin{cases} (-1)^r, & \text{if} \quad p_1 \cdots p_{2s-1} p_{2s}^3 < D \quad \text{for all} \quad 0 \leq s \leq r/2; \\ 0, & \text{otherwise;} \end{cases} \tag{22}$$

For further properties of Rosser's weights we refer to [8], [9].

Let $\Lambda_i = \sum_{d|(p_i+2,P(z))} \mu(d)$ be the characteristic function of primes p_i, such that $(p_i + 2,\, P(z)) = 1$ for $i = 1,\, 2,\, 3$. Then from (20) we obtain the condition

$$\begin{aligned} &\Gamma_0(X) \\ &= \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \upsilon(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \Lambda_1 \Lambda_2 \Lambda_3 \log p_1 \log p_2 \log p_3 > 0. \end{aligned} \tag{23}$$

To set up a vector sieve, we use lower and upper bounds

$$\Lambda_i^{\pm} = \sum_{d|(p_i+2,P(z))} \lambda^{\pm}(d), \quad i = 1, 2, 3. \tag{24}$$

From the linear sieve we know that $\Lambda_i^{-} \leq \Lambda_i \leq \Lambda_i^{+}$ (see [2, Lemma 10]). Then we have a simple inequality

$$\Lambda_1 \Lambda_2 \Lambda_3 \geq \Lambda_1^{-} \Lambda_2^{+} \Lambda_3^{+} + \Lambda_1^{+} \Lambda_2^{-} \Lambda_3^{+} + \Lambda_1^{+} \Lambda_2^{+} \Lambda_3^{-} - 2\Lambda_1^{+} \Lambda_2^{+} \Lambda_3^{+}. \tag{25}$$

analogous to this one in [2, Lemma 13]. Using (23) and (25) we get

$$\begin{aligned} &\Gamma_0(X) \geq \Gamma(X) \\ &= \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \upsilon(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \\ &\times \left(\Lambda_1^{-} \Lambda_2^{+} \Lambda_3^{+} + \Lambda_1^{+} \Lambda_2^{-} \Lambda_3^{+} + \Lambda_1^{+} \Lambda_2^{+} \Lambda_3^{-} - 2\Lambda_1^{+} \Lambda_2^{+} \Lambda_3^{+} \right) \log p_1 \log p_2 \log p_3 > 0. \end{aligned} \tag{26}$$

Let $\Upsilon(x) = \int_{-\infty}^{\infty} \upsilon(t) e(-xt) dt$ be the Fourier transform of the function υ defined by (19). Then

$$|\Upsilon(x)| \leq \min \left(\frac{7\varepsilon}{4}, \frac{1}{\pi |x|}, \frac{1}{\pi |x|} \left(\frac{k}{2\pi |x| \varepsilon / 8} \right)^k \right), \tag{27}$$

for all $k \in \mathbb{N}$ – see [16].

We substitute the function $v(\lambda_1 p_1+\lambda_2 p_2+\lambda_3 p_3+\eta)$ in (26) with its invers Fourier transform and get

$$\sum_{\lambda_0 X<p_1,p_2,p_3\le X} (\Lambda_1^-\Lambda_2^+\Lambda_3^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^- - 2\Lambda_1^+\Lambda_2^+\Lambda_3^+) \log p_1 \log p_2 \log p_3$$

(28)

$$\times \int_{-\infty}^{\infty} \Upsilon(t) e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)\, dt.$$

Hence our argument is based on the following consequence of (28).

Lemma 1. *If the integral*

$$\Gamma(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X<p_1,p_2,p_3\le X} e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big) \log p_1 \log p_2 \log p_3$$
$$\times (\Lambda_1^-\Lambda_2^+\Lambda_3^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^- - 2\Lambda_1^+\Lambda_2^+\Lambda_3^+)\, dt$$

(29) $$= \Gamma_1(X) + \Gamma_2(X) + \Gamma_3(X) - 2\Gamma_4(X) > 0.$$

is positive then the number of solutions of (8) *in primes* $p_i \in (\lambda_0 X, X]$, $p_i + 2 = P_{[1/\beta]}$, $i = 1, 2, 3$ *is positive. Here*

$$\Gamma_1(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X<p_1,p_2,p_3\le X} e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)$$
$$\times \Lambda_1^-\Lambda_2^+\Lambda_3^+ \log p_1 \log p_2 \log p_3\, dt;$$

$$\Gamma_2(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X<p_1,p_2,p_3\le X} e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)$$
$$\times \Lambda_1^+\Lambda_2^-\Lambda_3^+ \log p_1 \log p_2 \log p_3\, dt;$$

$$\Gamma_3(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X<p_1,p_2,p_3\le X} e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)$$
$$\times \Lambda_1^+\Lambda_2^+\Lambda_3^- \log p_1 \log p_2 \log p_3\, dt;$$

$$\Gamma_4(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1,p_2,p_3 \le X} e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big) \times \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \log p_1 \log p_2 \log p_3 \, dt.$$

We are going to estimate $\Gamma_1(X)$. The integrals $\Gamma_2(X)$, $\Gamma_3(X)$, $\Gamma_4(X)$ can be treated in a similar way. Changing the order of summation and bearing in mind (24) we obtain

$$\Gamma_1(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) e(\eta t) L^-(\lambda_1 t, X) L^+(\lambda_2 t, X) L^+(\lambda_3 t, X) dt,$$

where

$$L^{\pm}(t, X) = \sum_{d \mid P(z)} \lambda^{\pm}(d) \sum_{\substack{\lambda_0 X < p \le X \\ p+2 \equiv 0 \ (d)}} e(pt) \log p, \tag{30}$$

Let us subdivide $\Gamma_1(X)$ into three integrals

$$\Gamma_1(X) = \Gamma_1^{(1)}(X) + \Gamma_1^{(2)}(X) + \Gamma_1^{(3)}(X), \tag{31}$$

where

$$\Gamma_1^{(1)}(X) = \int\limits_{|t| \le \Delta} \Upsilon(t) e(\eta t) L^-(\lambda_1 t, X) L^+(\lambda_2 t, X) L^+(\lambda_3 t, X) dt, \tag{32}$$

$$\Gamma_1^{(2)}(X) = \int\limits_{\Delta < |t| < H} \Upsilon(t) e(\eta t) L^-(\lambda_1 t, X) L^+(\lambda_2 t, X) L^+(\lambda_3 t, X) dt, \tag{33}$$

$$\Gamma_1^{(3)}(X) = \int\limits_{|t| \ge H} \Upsilon(t) e(\eta t) L^-(\lambda_1 t, X) L^+(\lambda_2 t, X) L^+(\lambda_3 t, X) dt. \tag{34}$$

Here the functions $\Delta = \Delta(X)$ and $H = H(X)$ are defined with (11) and (12).

We shall estimate $\Gamma_1^{(3)}(X)$, $\Gamma_1^{(1)}(X)$, $\Gamma_1^{(2)}(X)$, respectively, in the Sections 4, 5, 6. In Section 7 we shall complete the proof of the Theorem.

4. Upper bound for $\Gamma_1^{(3)}(X)$.

Lemma 2. *For the integral* $\Gamma_1^{(3)}(X)$*, defined by* (34)*, we have*

$$\Gamma_1^{(3)}(X) \ll 1. \tag{35}$$

P r o o f. See [5] Lemma 2. □

5. Asymptotic formula for $\Gamma_1^{(1)}(X)$. The main term of the integral $\Gamma_1(X)$ we will derive from $\Gamma_1^{(1)}(X)$. Working as in §5 of [4], but with $C = A$, we get

$$\Gamma_1^{(1)}(X) = B(X)\left(\sum_{d|P(z)}\frac{\lambda^-(d)}{\varphi(d)}\right)\left(\sum_{d|P(z)}\frac{\lambda^+(d)}{\varphi(d)}\right)^2 + \mathcal{O}\Big(\frac{\varepsilon X^2}{(\log X)^{A-5}}\Big). \tag{36}$$

where

$$B(X) = \int\limits_{-\infty}^{\infty}\Upsilon(t)e(\eta t)\left(\int\limits_{\lambda_0 X}^{X}\int\limits_{\lambda_0 X}^{X}\int\limits_{\lambda_0 X}^{X} e(t(\lambda_1 y_1+\lambda_2 y_2+\lambda_3 y_3))dy_1dy_2dy_3\right)dt. \tag{37}$$

According to ([5], Lemma 4), we have following

Lemma 3. *If there hold* (4), (5) *and*

$$\lambda_0 < \min\left(\frac{\lambda_1}{4|\lambda_3|}, \frac{\lambda_2}{4|\lambda_3|}, \frac{1}{16}\right),$$

then $B(X)$*, defined by* (37) *satisfies*

$$B(X) \gg \varepsilon X^2,$$

with a constant implied by the $\gg$*-symbol depending only on* λ_1, λ_2 *and* λ_3.

Let

$$G^{\pm} = \sum_{d|P(z)}\frac{\lambda^{\pm}(d)}{\varphi(d)}. \tag{38}$$

Then from (36) and (38) it follows

$$\Gamma_1^{(1)}(X) = B(X)G^-\left(G^+\right)^2 + \mathcal{O}\Big(\frac{\varepsilon X^2}{(\log X)^{A-5}}\Big). \tag{39}$$

6. Upper bound for $\Gamma_1^{(2)}(X)$. To find estimate for $\Gamma_1^{(2)}(X)$ we need some statements.

We will say that a squarefree number $d \in \mathbb{N}$, $d < D$ is well-separated if for any $D_1 \geq 1$ and $D_2 \geq 1$ such that $D = D_1D_2$ one can write $d = d_1d_2$ with $d_1 \leq D_1$ and $d_2 \leq D_2$ (see [6, ch. 6, §6.1.2]).

Lemma 4. *If $d|P(z)$, $z < D^{1/2}$, $\lambda^{\pm}$ are Rosser's weigts and either $\lambda^{+}(d) \neq 0$ or $\lambda^{-}(d) \neq 0$ then d is a well-separated number.*

P r o o f. See remark 2, [9] □

We shall use following

Lemma 5. *Let k, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and*

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}, \quad a \in \mathbb{Z},; \; q \in \mathbb{N}, \quad (a,q) = 1, \quad q \geq 1. \tag{40}$$

Then for every arbitrary small $\theta > 0$ the inequality

$$\sum_{m\sim M} \tau_{\mu}(m) \sum_{j\sim J} \tau_{\zeta}(j) \min\left\{\frac{x}{m^2 j}, \frac{1}{||\alpha m^2 j||}\right\} \ll x^{\theta}\left(MJ + \frac{x}{M^{3/2}} + \frac{x}{Mq^{1/2}} + \frac{x^{1/2}q^{1/2}}{M}\right)$$

is fulfilled.

P r o o f. See Lemma 8, [11]. □

Lemma 6. *Suppose $\alpha \in \mathbb{R}$ and α satisfy conditions* (40). *Let $\xi(d)$ be complex numbers defined for $d \leq D$ and $\xi(d) \ll 1$. If*

$$\mathfrak{L}(X) = \sum_{d\sim D} \xi(d) \sum_{\substack{p\sim X \\ p+2\equiv 0\,(d)}} e(p\alpha)\log p. \tag{41}$$

then for any arbitrary small $\theta > 0$ we have

$$\mathfrak{L}(X) \ll X^{\theta}\left(\frac{X}{q^{1/2}} + X^{1/2}q^{1/2} + X^{2/3}D + X^{5/6}\right).$$

P r o o f. This is Lemma 1 from [19]. □

Above theorem give good bound for $D \leq x^{1/3-\vartheta}$ with $\vartheta > \theta$. To prove upper bound for $\mathfrak{L}(X)$ with bigger D we will use following version of Theorem 1, [11]:

Lemma 7. *Suppose $\alpha \in \mathbb{R}$ and α satisfy conditions* (40), *$\xi(d)$ are complex numbers defined for $d \leq D$, $D \leq x^{1/2-\vartheta}$,*

$$\xi(d) \ll 1 \quad \textit{and} \quad \xi(d) \neq 0 \quad \textit{if and only if } d \textit{ is a well-separated number.} \tag{42}$$

Then for any arbitrary small $\theta > 0$ for sum (41) *we have*

$$\mathfrak{L}(X) \ll X^{\theta}\left(\frac{X}{q^{1/4}} + X^{3/4}q^{1/4} + X^{3/4}D^{1/2} + \frac{X}{D^{1/4}} + X^{17/18}\right). \tag{43}$$

P r o o f. The proof is similarly to proof of Theorem 1, [11]. □

Corollary 1. *Suppose $\alpha \in \mathbb{R}$ and α satisfy conditions* (40)*, $\xi(d)$ are complex numbers defined for $d \le D$, $D \le x^{17/18}$ and satisfying conditions* (42)*. Then for any arbitrary small $\theta > 0$ for sum*

$$W = \sum_{d \le D} \xi(d) \sum_{\substack{p \sim X \\ p+2 \equiv 0\,(d)}} e(p\alpha)\log p$$

we have

$$W \ll X^{\theta}\left(\frac{X}{q^{1/4}} + X^{3/4}q^{1/4} + X^{17/18}\right). \tag{44}$$

P r o o f. It is obvious that we can represent the sum W as $O(\log x)$ sums of the type (41). For $D \le x^{5/18}$ from Lemma 6 we get (44). When $x^{5/18} \le D \le x^{7/18}$ from Lemma 7 we obtain (44). □

Let us consider any sum $L^{\pm}(\alpha,\, X)$ denoted by (30) and let

$$L(\alpha,\, Y) = \sum_{d \le D} \xi(d) \sum_{\substack{p \sim Y \\ p+2 \equiv 0\,(d)}} e(p\alpha)\log p,$$

where

$$\xi(d) = \begin{cases} \lambda^{\pm}(d), & \text{if} \quad d|\, P(z), \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that

$$L^{\pm}(\alpha,\, X) \ll \max_{\lambda_0 X \le Y \le X} |L(\alpha,\, Y)|.$$

If

$$q \in \left[X^{2/9},\, X^{7/9}\right], \tag{45}$$

then from Corollary 1 for the sum $L(\alpha,\, Y)$ we get

$$L(\alpha,\, Y) \ll Y^{17/18+\theta}. \tag{46}$$

Therefore

$$L^{\pm}(\alpha,\, X) \ll \max_{\lambda_0 X \le Y \le X} Y^{17/18+\theta} \ll X^{17/18+\theta}.$$

Let

$$V(t,\, X) = \min\left\{\left|L^{\pm}(\lambda_1 t,\, X)\right|, \left|L^{\pm}(\lambda_2 t,\, X)\right|\right\}. \tag{47}$$

We shall prove the following

Lemma 8. *Let* $t,\, X,\, \lambda_1,\, \lambda_2 \in \mathbb{R}$,

$$|t| \in (\Delta,\, H), \tag{48}$$

where Δ *and* H *are denoted by* (11) *and* (12), $\lambda_1,\, \lambda_2$ *satisfy* (6) *and* $V(t,\, X)$ *is defined by* (47). *Then there exists a sequence of real numbers* $X_1,\, X_2, \ldots \to \infty$ *such that*

$$V(t, X_j) \ll X_j^{17/18+\theta}, \quad j = 1, 2, \ldots. \tag{49}$$

P r o o f. Our aim is to prove that there exists a sequence $X_1,\, X_2, \cdots \to \infty$ such that for each $j = 1, 2, \ldots$ at least one of the numbers $\lambda_1 t$ and $\lambda_2 t$ with t, subject to (48) can be approximated by rational numbers with denominators, satisfying (45). Then the proof follows from (46) and (47).

Since $\dfrac{\lambda_1}{\lambda_2} \in \mathbb{R}/\mathbb{Q}$ then by [18, Corollary 1B], there exist infinitely many fractions $\dfrac{a_0}{q_0}$ with arbitrary large denominators such that

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a_0}{q_0}\right| < \frac{1}{q_0^2}, \qquad (a_0,\, q_0) = 1. \tag{50}$$

Let q_0 be sufficiently large and X be such that $q_0^2 = X^{7/9}$ (see (9)). Let us notice that there exist $a_1,\, q_1 \in \mathbb{Z}$, such that

$$\left|\lambda_1 t - \frac{a_1}{q_1}\right| < \frac{1}{q_1 q_0^2}, \qquad (a_1,\, q_1) = 1, \qquad 1 \le q_1 \le q_0^2, \qquad a_1 \ne 0. \tag{51}$$

From Dirichlet's Theorem (see [10, ch.10, §1]) it follows the existence of integers a_1 and q_1, satisfying the first three conditions. If $a_1 = 0$ then $|\lambda_1 t| < \dfrac{1}{q_1 q_0^2}$ and from (48) it follows

$$\lambda_1 \Delta < \lambda_1 |t| < \frac{1}{q_0^2}, \qquad q_0^2 < \frac{1}{\lambda_1 \Delta}.$$

From the last inequality, (9) and (11) we obtain

$$X^{7/9} < \frac{X^{7/9}}{\lambda_1 \log X},$$

which is impossible for large q_0, respectively, for a large X. So $a_1 \neq 0$. By analogy there exist a_2, $q_2 \in \mathbb{Z}$, such that

$$\left|\lambda_2 t - \frac{a_2}{q_2}\right| < \frac{1}{q_2 q_0^2}, \qquad (a_2,\, q_2) = 1, \qquad 1 \leq q_2 \leq q_0^2, \qquad a_2 \neq 0. \tag{52}$$

If $q_i \in \left[X^{2/9},\, X^{7/9}\right]$ for $i = 1$ or $i = 2$, then the proof is completed. From (9), (51) and (52) we have

$$q_i \leq X^{7/9} = q_0^2, \quad i = 1, 2.$$

Thus it remains to prove that the case

$$q_i < X^{2/9}, \quad i = 1, 2 \tag{53}$$

is impossible. Let $q_i < X^{2/9}$, $i = 1, 2$. From (48), (51)–(53) it follows

$$1 \leq |a_i| < \frac{1}{q_0^2} + q_i \lambda_i |t| < \frac{1}{q_0^2} + q_i \lambda_i H,$$

$$1 \leq |a_i| < \frac{1}{q_0^2} + \lambda_i X^{1/18-\delta} \log^2 X, \quad i = 1,\, 2. \tag{54}$$

We have

$$\frac{\lambda_1}{\lambda_2} = \frac{\lambda_1 t}{\lambda_2 t} = \frac{\dfrac{a_1}{q_1} + \left(\lambda_1 t - \dfrac{a_1}{q_1}\right)}{\dfrac{a_2}{q_2} + \left(\lambda_2 t - \dfrac{a_2}{q_2}\right)} = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + \mathfrak{T}_1}{1 + \mathfrak{T}_2}, \tag{55}$$

where $\mathfrak{T}_i = \dfrac{q_i}{a_i}\left(\lambda_i t - \dfrac{a_i}{q_i}\right)$, $i = 1,\, 2$. According to (51), (52) and (55) we obtain

$$|\mathfrak{T}_i| < \frac{q_i}{|a_i|} \cdot \frac{1}{q_i q_0^2} = \frac{1}{|a_i| q_0^2} \leq \frac{1}{q_0^2}, \quad i = 1, 2,$$

$$\frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + \mathcal{O}\left(\dfrac{1}{q_0^2}\right)}{1 + \mathcal{O}\left(\dfrac{1}{q_0^2}\right)} = \frac{a_1 q_2}{a_2 q_1}\left(1 + \mathcal{O}\left(\frac{1}{q_0^2}\right)\right).$$

Thus $\dfrac{a_1q_2}{a_2q_1} = \mathcal{O}(1)$ and

$$\frac{\lambda_1}{\lambda_2} = \frac{a_1q_2}{a_2q_1} + \mathcal{O}\left(\frac{1}{q_0^2}\right). \tag{56}$$

Therefore, both fractions $\dfrac{a_0}{q_0}$ and $\dfrac{a_1q_2}{a_2q_1}$ approximate $\dfrac{\lambda_1}{\lambda_2}$. Using (10), (51), (53) and inequality (54) with $i = 2$ we obtain

$$|a_2|q_1 < 1 + \lambda_2 X^{5/18-\delta}\log^2 X < \frac{q_0}{\log X} \tag{57}$$

so $|a_2|q_1 \neq q_0$ and the fractions $\dfrac{a_0}{q_0}$ and $\dfrac{a_1q_2}{a_2q_1}$ are different. Then using (57) it follows

$$\left|\frac{a_0}{q_0} - \frac{a_1q_2}{a_2q_1}\right| = \frac{|a_0a_2q_1 - a_1q_2q_0|}{|a_2|q_1q_0} \geq \frac{1}{|a_2|q_1q_0} \gg \frac{\log X}{q_0^2}. \tag{58}$$

On the other hand, from (50) and (56) we have

$$\left|\frac{a_0}{q_0} - \frac{a_1q_2}{a_2q_1}\right| \leq \left|\frac{a_0}{q_0} - \frac{\lambda_1}{\lambda_2}\right| + \left|\frac{\lambda_1}{\lambda_2} - \frac{a_1q_2}{a_2q_1}\right| \ll \frac{1}{q_0^2},$$

which contradicts (58). This rejects the assumption (53). Let $q_0^{(1)}$, $q_0^{(2)}$, ... be an infinite sequence of values of q_0, satisfying (50). Then using (9) one gets an infinite sequence X_1, X_2, ... of values of X, such that at least one of the numbers $\lambda_1 t$ and $\lambda_2 t$ can be approximated by rational numbers with denominators, satisfying (45). Hence, the proof is completed. □

Let us estimate the integral $\Gamma_1^{(2)}(X_j)$, denoted by (33). Using $|\Upsilon(t)| \leq \dfrac{7\varepsilon}{4}$ (see (27)), (47) and estimate (49) we find

$$\begin{aligned}
&\Gamma_1^{(2)}(X_j)\\
&\quad\ll \varepsilon \int\limits_{\Delta<|t|<H} V(t,\, X_j)\left(\left|L^-(\lambda_1 t, X_j)L^+(\lambda_3 t, X_j)\right| + \left|L^+(\lambda_2 t,\, X_j)L^+(\lambda_3 t,\, X_j)\right|\right) dt\\
&\quad\ll \varepsilon \int\limits_{\Delta<|t|<H} V(t, X_j)\left(\left|L^-(\lambda_1 t,\, X_j)\right|^2 + \left|L^+(\lambda_2 t,\, X_j)\right|^2 + \left|L^+(\lambda_3 t,\, X_j)\right|^2\right) dt\\
&\quad\ll \varepsilon X_j^{17/18+\theta} \max_{1\leq k\leq 3} \mathcal{I}_k,
\end{aligned} \tag{59}$$

where

$$\mathcal{I}_k = \int\limits_{\Delta<|t|<H} \left|L^{\pm}(\lambda_k t,\, X_j)\right|^2\, dt.$$

Arguing as in [5] we obtain

$$\mathcal{I}_k \ll X_j^{19/18-\delta} \log^7 X_j. \tag{60}$$

Using (59) and (60) and choosing $\theta < \delta$ we get

$$\Gamma_1^{(2)}(X_j) \ll \varepsilon X_j^{17/18+\theta} X_j^{19/18-\delta} \log^7 X_j \ll \frac{\varepsilon X_j^2}{(\log X_j)^{A-5}}. \tag{61}$$

Summarizing (31), (35), (39) and (61) we find

$$\Gamma_1(X_j) = B(X_j) G^- \left(G^+\right)^2 + \mathcal{O}\left(\frac{\varepsilon X_j^2}{(\log X_j)^{A-5}}\right). \tag{62}$$

7. Proof of the Theorem. Since the sums $\Gamma_2(X_j)$, $\Gamma_3(X_j)$ and $\Gamma_4(X_j)$ are estimated in the same way then from (29) and (62) we obtain

$$\Gamma(X_j) = B(X_j) W(X_j) + \mathcal{O}\left(\frac{\varepsilon X_j^2}{(\log X_j)^{A-5}}\right), \tag{63}$$

where

$$W(X_j) = 3\left(G^+\right)^2 \left(G^- - \frac{2}{3} G^+\right). \tag{64}$$

Let $f(s)$ and $F(s)$ are the lower and the upper functions of the linear sieve. We know that if

$$s = \frac{\log D}{\log z}, \quad 2 \le s \le 3 \tag{65}$$

then

$$f(s) = \frac{2e^{\gamma} \log(s-1)}{s}, \qquad F(s) = \frac{2e^{\gamma}}{s} \tag{66}$$

where $\gamma = 0.577\ldots$ is the Euler constant (see [2, Lemma 10]). Using (38) and Lemma 10 from [2] we get

$$\begin{aligned}\mathcal{F}(z)\Big(f(s)+\mathcal{O}\big((\log X)^{-1/3}\big)\Big)&\\ \le G^- \le \mathcal{F}(z) \le G^+&\\ \le \mathcal{F}(z)\Big(F(s)+\mathcal{O}\Big((\log X)^{-1/3}\Big)\Big).&\end{aligned} \tag{67}$$

Here

$$\mathcal{F}(z) = \prod_{2<p\le z}\left(1-\frac{1}{p-1}\right) \asymp \frac{1}{\log X}, \tag{68}$$

see Mertens formula [14, ch.9, §9.1, Theorem 9.1.3] and (14). To estimate $W(X_j)$ from below we shall use the inequalities (see (67)):

$$\begin{aligned}G^- - \frac{2}{3}G^+ &\ge \mathcal{F}(z)\Big(f(s)-\frac{2}{3}F(s)+\mathcal{O}\big((\log X)^{-1/3}\big)\Big)\\ G^+ &\ge \mathcal{F}(z).\end{aligned} \tag{69}$$

Let $X = X_j$. Then from (64) and (69) it follows

$$W(X_j) \ge 3\mathcal{F}^3(z)\Big(f(s)-\frac{2}{3}F(s)+\mathcal{O}\big((\log X_j)^{-1/3}\big)\Big) \tag{70}$$

We choose $s = 2.949$. Then

$$f(s)-\frac{2}{3}F(s) \ge 0,0001 \tag{71}$$

and from (65) for sufficiently large X we get $\dfrac{1}{\beta} = 7.583$.

We choose $A \ge 10$. Then from (10), (63), (68), (70), (71) and Lemma 3 we obtain

$$\Gamma(X_j) \gg \frac{X_j^{35/18+\delta}}{(\log X_j)^3}. \tag{72}$$

The last inequality implies that $\Gamma(X_j) \to \infty$ as $X_j \to \infty$.

By the definition (26) of $\Gamma(X)$ and the inequality (72) we conclude that for some constant c_0 there are at least $c_0\dfrac{X_j^{35/18+\delta}}{(\log X_j)^3}$ triples of primes p_1, p_2, p_3

satisfying $p_i \in (\lambda_0 X_j,\, X_j]$, $i = 1,\, 2,\, 3$, $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \varepsilon$ and such that for every prime factor p of $p_i + 2$, $i = 1, 2, 3$ we have $p \geq X^{0.1318}$.

That completes the proof of Theorem.

Using the same manner of the above proof but with parameters

$$q_0^2 = X^{6/7}, \quad \varepsilon = X^{-1/28+\delta}, \quad \Delta = X^{-6/7} \log X,$$
$$H = \frac{\log^2 X}{\varepsilon}, \quad D = X^{3/7}, \quad z = X^{\beta}, \quad \beta = 0.1429;$$

we get

Theorem 2. *If conditions* (4), (5), (6), (7) *are fulfilled then there are infinitely many ordered triples of primes* p_1, p_2, p_3 *with*

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-1/28} \tag{73}$$

and

$$p_1 + 2 = P_5', \quad p_2 + 2 = P_5'', \quad p_3 + 2 = P_5'''.$$

REFERENCES

[1] R. Baker, G. Harman. Diophantine approximation by prime numbers. *J. London Math. Soc. (2)* **25**, *2* (1982), 201–215.

[2] J. Brüdern, É. Fouvry. Lagrange's four squares theorem with almost prime variables. *J. Reine Angew. Math.* **454**, (1994), 59–96.

[3] J. R. Chen. On the representation of a large even integer as the sum of a prime and the product of at most two primes. *Sci. Sinica* **16**, (1973), 157–176.

[4] S. I. Dimitrov. Diophantine approximation by special primes. AIP Conference Proceedings vol. **2048**, 050005, 2018, `https://aip.scitation.org/doi/abs/10.1063/1.5082104`

[5] S. I. Dimitrov, T. Todorova. Diophantine approximation by prime numbers of a special form. *Annuaire Univ. Sofia, Fac. Math. Inform.* **102** (2015), 71–90.

[6] G. Greaves. Results in Mathematics and Related Areas (3), vol. **43**. Berlin, Springer-Verlag, 2001.

[7] G. Harman. Diophantine approximation by prime numbers. *J. London Math. Soc. (2)* **44**, *2* (1991), 218–226.

[8] H. Iwaniec. Rosser's sieve. *Acta Arith.* **36**, *2* (1980), 171–202.

[9] H. Iwaniec. A new form of the error term in the linear sieve. *Acta Arith.* **37**, (1980), 307–320.

[10] A. Karatsuba. Principles of the Analytic Number Theory, 2nd ed. Moscow, Nauka, 1983 (in Russian).

[11] K. Matomäki. A Bombieri–Vinogradov type exponential sum result with applications. *J. Number Theory* **129**, *9* (2009), 2214–2225.

[12] K. Matomäki. Diophantine approximation by primes. *Glasg. Math. J.* **52**, *1* (2010), 87–106.

[13] K. Matomäki, X. Shao. Vinogradov's three primes theorem with almost twin primes. *Compos. Math.* **153**, *6* (2017), 1220–1256, DOI: 10.1112/S0010437X17007072.

[14] M. R. Murty. Problems in Analytic Number Theory, 2nd ed. Graduate Texts in Mathematics, vol. **206**. Readings in Mathematics. New York, Springer, 2008.

[15] T. P. Peneva. On the ternary Goldbach problem with primes p_i such that $p_i + 2$ are almost-prime. *Acta Math. Hungar.* **86**, *4* (2000), 305–318.

[16] B. Segal. On a theorem analogous to Waring's theorem. *C. R. (Dokl.) Akad. Sci. URSS (N. S.)* **2**, (1933), 47–49, (in Russian).

[17] S.-Y. Shi. On the distribution of *alphap* modulo one for primes p of a special form. *Osaka J. Math.* **49**, *4* (2012), 993–1004.

[18] W. M. Schmidt. Diophantine approximation. Lecture Notes in Mathematics, vol. **785**. Berlin-Heidelberg-New York, Springer-Verlag, 1980.

[19] T. L. Todorova, D. I. Tolev. On the distribution of αp modulo one for primes p of a special form., *Math. Slovaca* **60**, *6* (2010), 771–786.

[20] D. I. Tolev. Representations of large integers as sums of two primes of special type. In: Algebraic number theory and Diophantine analysis (Graz, 1998). Berlin, de Gruyter, 2000, 485–495.

[21] R. C. Vaughan. Diophantine approximation by prime numbers. I. *Proc. Lond. Math. Soc. (3)* **28** (1974), 373–384.

[22] R. C. Vaughan. The Hardy–Littlewood method, 2nd ed. Cambridge Tracts in Mathematics, vol. **125**. Cambridge, Cambridge University Press, 1997.

[23] I. M. Vinogradov. Representation of an odd number as the sum of three primes. *C. R. (Dokl.) Akad. Sci. URSS (N. S.)* **15**, (1937), 169–172, DOI:10.1007/978-3-642-15086-9_13 (in Russian).

[24] I. M. Vinogradov. The method of trigonometrical sums in the theory of numbers. *Trud. Math. Inst. Steklov* **23** (1947), 109 pp, (in Russian).

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier Blvd
1164 Sofia, Bulgaria
e-mail: tlt@fmi.uni-sofia.bg

Received July 04, 2022