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UNIQUENESS OF DERIVATIVE OF AN ENTIRE FUNCTION THAT SHARES PAIR VALUES WITH DERIVATIVE OF ITS DIFFERENCE OPERATOR*

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Abstract. The main objective of this article is to examine the uniqueness of k^{th} derivative of transcendental entire function and i^{th} derivative of its difference operator that share pair values (α_1, α_2), 0 IM and zeros of k^{th} derivative of transcendental entire function contained in i^{th} derivative of its difference operator. Meanwhile all of which greatly generalizes the outcomes obtained by X. Huang [11] and L. Sheng [16].

1. Introduction and definitions. In this paper, we employ the general notations found in [1, 2, 3] for the Nevanlinna's value distribution theory. Readers should be familiar with the basics of Nevanlinna's value distribution theory for meromorphic functions in $\mathbb{C}$, particularly the first and second main theorems and counting functions $N(r,\infty;f)$ (Counting multiplicity), $\overline{N}(r,\infty;f)$

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(ignoring multiplicity), the proximity function $m(r,f)$ and $T(r,\infty;f)$ (characteristic function). In addition, $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set of finite logarithmic measure.

Let f and g be two non-constant meromorphic functions and α, β be small functions with respect to f and g. If the zeros of $f-\alpha$ and $g-\beta$ concur in locations and multiplicities, then we say that f and g share the pair of small functions (α,β) CM (Counting multiplicities) and if we do not consider the multiplicities, then f and g are said to share the pair of small functions (α,β) IM (Ignoring multiplicities). The counting function $N_{(q}\left(r,\dfrac{1}{g-a}\right)$ of g means those a-points of g are counted according to multiplicities whose multiplicities are not less than q, $N_{q)}\left(r,\dfrac{1}{g-a}\right)$ denotes the counting function of g whose a-points are counted with proper multiplicity where the multiplicities are less than or equal to q and the corresponding reduced counting function is given by $\overline{N}_{(q}\left(r,\dfrac{1}{g-a}\right)$, $\overline{N}_{q)}\left(r,\dfrac{1}{g-a}\right)$ where the multiplicities are ignored. Define

$$\rho(f)=\overline{\lim_{r\to\infty}}\frac{\log^{+}T(r,f)}{\log r}.$$

$$\rho_2(f)=\overline{\lim_{r\to\infty}}\frac{\log^{+}\log^{+}T(r,f)}{\log r}.$$

as the order and the hyper-order of f respectively. For a non-constant meromorphic function f and the non-zero finite complex number c, we define its shift by $f(z+c)$ and its difference operator by $\triangle f(z)=f(z+c)-f(z)$, where $f(z)$ is not a periodic function of period c.

Definition 1.1 ([8]). *For a non-constant meromorphic function f, the set of all small functions of f is denoted by $S(f)$, i.e., $S(f)=\{a\in M(\mathbb{C}): T(r,a)=S(r,f)$ as $r\to\infty\}$. Clearly $S(f)\subset M(\mathbb{C})$.*

Let us consider $f-a$ and $g-a$ share 0. Then by $N_{(m,n)}\left(r,\dfrac{1}{f-a}\right)$ $\left(\overline{N}_{(m,n)}\left(r,\dfrac{1}{f-a}\right)\right)$ we represent the counting (reduced counting) function refers to the number of zeros of $f-a$ and $g-a$, where these zeros have multiplicity m and n respectively.

Definition 1.2 ([2]). *For a complex constant a, we denote the set of all zeros of $f-a$ by $E(a,f)$, where each zero with multiplicity m is counted m times* *and all distinct zeros of $f - a$ by $\overline{E}(a,f)$. For two non-constant meromorphic functions f and g, we say f and g partially share the value a CM, if $E(a,f) \subset E(a,g)$. On the other hand, if $\overline{E}(a,f) \subset \overline{E}(a,g)$, we say f and g share the value a IM.*

Here we present some results published recently regarding uniqueness of f' and $f(z+c)$ when they share values. X. Qi and L. Yang [15], proved the following results:

Theorem A. *Assume that $a(\neq 0) \in \mathbb{C}$, and that f is a transcendental entire function of finite order. If $f'(z)$ and $f(z+c)$ share 0 CM and a IM, then $f' \equiv f(z+c)$.*

Theorem B. *Let f be a transcendental entire function of finite order, and let a, b be two distinct finite values. If $f'(z)$ and $f(z+c)$ share a, b IM, and $\overline{N}\left(r, \frac{1}{f'-a}\right) = S(r,f)$. Then $f' \equiv f(z+c)$.*

H. X. Huang and M. L. Fang [13] improved the above Theorems and established the following result:

Theorem C. *Suppose f is a transcendental entire function with $\rho_2 < 1$, c is a non-zero finite complex value, and a and b are two distinct finite values. If f' and $f(z+c)$ share a, b IM, then $f' \equiv f(z+c)$.*

The following Theorem was established by X. Huang, B. Deng, and M. Fang [12] through their investigation into the uniqueness of f and $f^{(k)}$ when they share pair values.

Theorem D. *Let f be a non-constant entire function, let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be four small functions of f such that $\alpha_1 \not\equiv \beta_1$ and $\alpha_2 \not\equiv \beta_2$, and each of them not equal to none of them identically equal to ∞. If $f(z)$ and $f^{(k)}(z)$ share (α_1, α_2) CM and (β_1, β_2) IM, then $(\alpha_2 - \beta_2)f - (\alpha_1 - \beta_1)f^{(k)} \equiv \alpha_2\beta_1 - \alpha_1\beta_2$.*

In 2021, X. Huang [11] investigated uniqueness of $f(z)$ and $\triangle f$ when they share the pair values and as a result, they proved the following Theorem.

Theorem E. *Let $f(z)$ be a non-constant entire function of $\rho_2(f) < 1$, η a non-zero complex number, and $i = 1, 2$. Let a_i and b_i be four finite complex numbers such that $a_i \neq b_i$, and let*

$$\varphi(z) = \frac{f'(z)((a_2 - b_2)f(z) - (a_1 - b_1) \triangle_\eta f(z) - a_2 b_1 + a_1 b_2)}{(f(z) - a_1)(f(z) - b_1)}. \tag{1.1}$$

If $f(z)$ and $\triangle_\eta f(z)$ share (a_1, a_2), (b_1, b_2) IM, then either

$$(a_2 - b_2)f(z) - (a_1 - b_1) \triangle_\eta f(z) \equiv a_2 b_1 - a_1 b_2, \tag{1.2}$$

or $\varphi(z)$ is a periodic entire function with period η.

L. Sheng, D. Mei and B. Chen [16] conducted the investigation into the uniqueness of entire function with respect to its difference operator in 2017 and established the following conclusion.

Theorem F. *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a non-constant entire function of finite order. If $f(z)$ and $\triangle_c^n(f(z))$ share two distinct complex constants a^* CM and a IM and if*

$$N\left(r, \frac{1}{f(z) - a^*}\right) = T(r, f) + S(r, f),$$

then $f(z) \equiv \triangle_c^n(f(z))$.

In this paper we prove the following result for the uniqueness of $f^{(k)}$ and $(\triangle f)^{(i)}$.

Theorem 1.1. *Let f be a transcendental entire function with $\rho_2(f) < 1$, let c be a non-zero finite complex value, and α_1, α_2 be two non-zero finite values. If $f^{(k)}$ and $(\triangle f)^{(i)}$ share (α_1, α_2), 0 IM and $E(0, f^{(k)}) \subseteq E(0, (\triangle f)^{(i)})$, then $\alpha_2 f^{(k)} \equiv \alpha_1 (\triangle f)^{(i)}$ for $i \geq k$.*

The folowing Example justifies the Theorem 1.1.

Example 1.1. Let $f = \sin z$, $c = \pi$, $\alpha_1 = -2$, $\alpha_2 = -4$, $i = 4$, and $k = 2$. Then it is clear that $f^{(k)}$ and $(\triangle f)^{(i)}$ share (α_1, α_2) and 0 IM. Then one can easily verify that $\alpha_1 (\triangle f)^{(i)} - \alpha_2 f^{(k)} \equiv 0$.

The following natural and intrinsic consequences are provided by Theorem 1.1:

For $k = 0$ and $i = 0$, Theorem 1.1 simplifies to

Corollary 1.1. *With c being a non-zero finite complex value and α_1, α_2 being two non-zero finite complex values, let f be a transcendental entire function with $\rho_2(f) < 1$. If f and $(\triangle f)$ share (α_1, α_2), 0 IM and $E(0, f) \subseteq E(0, (\triangle f))$, then $\alpha_2 f \equiv \alpha_1 \triangle f$.*

In the case of $\alpha_1 = \alpha_2 = \alpha$, Theorem 1.1 becomes

Corollary 1.2. *Given a transcendental entire function f with $\rho_2(f) < 1$, two non-zero finite complex numbers c and α. If $f^{(k)}$ and $(\triangle f)^{(i)}$ share α, 0 IM and $E(0, f^{(k)}) \subseteq E(0, (\triangle f)^{(i)})$, then $f^{(k)} \equiv (\triangle f)^{(i)}$.*

When $k=0$, $i=0$ and $\alpha_1=\alpha_2=\alpha$, Theorem 1.1 simplifies to

Corollary 1.3. *Let c and α be two non-zero finite complex values, and f be a transcendental entire function with $\rho_2(f)<1$. If f and $\triangle f$ share α, 0 IM and $E(0,f)\subseteq E(0,\triangle f)$ then $f\equiv\triangle f$.*

2. Lemmas.

In order to prove our findings, we require the following lemmas.

Lemma 2.1 ([7]). *Let f be a non-constant meromorphic function with $\rho_2(f)<1$, and let c be a non-zero complex number. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=S(r,f),$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Lemma 2.2 ([17]). *Let f be a non-constant meromorphic function of finite order. Then*

$$m\left(r,\frac{f'}{f}\right)=S(r,f).$$

If the order of f is infinite, then

$$m\left(r,\frac{f'}{f}\right)=O(\log(rT(r,f))),(r\to\infty,r\notin E_0),$$

where E_0 is a set whose linear measure is not greater than 2.

Remark 2.1. If k and j are positive integers with $k>j$, then

$$\frac{f^{(k)}}{f^{(j)}}=\frac{f^{(k)}}{f^{(k-1)}}\cdot\frac{f^{(k-1)}}{f^{(k-2)}}\cdot\frac{f^{(k-2)}}{f^{(k-3)}}\cdots\frac{f^{(j+1)}}{f^{(j)}},$$

which gives,

$$m\left(r,\frac{f^{(k)}}{f^{(j)}}\right)\leq m\left(r,\frac{f^{(k)}}{f^{(k-1)}}\right)+m\left(r,\frac{f^{(k-1)}}{f^{(k-2)}}\right)+m\left(r,\frac{f^{(k-2)}}{f^{(k-3)}}\right)+\cdots+m\left(r,\frac{f^{(j+1)}}{f^{(j)}}\right),$$

using this with Lemma 2.2, we get,

$$m\left(r, \frac{f^{(k)}}{f^{(j)}}\right) = S(r, f).$$

Lemma 2.3 ([18]). *Let f_1 and f_2 be two non-constant meromorphic functions in $|z| < \infty$ then,*

$$N(r, f_1 f_2) - N\left(r, \frac{1}{f_1 f_2}\right) = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right).$$

By referring the Lemma 3.4 in [10], we prove the following Lemma.

Lemma 2.4. *Consider a transcendental entire function f, let k_j $(j = 1, 2, \cdots n)$ be positive integers, a and b be two distinct small functions of f, with $a \neq \infty$ and $b \neq \infty$ and let $\nu_j = a - k_j(a - b)$ $(j = 1, 2, \cdots n)$. Set*

$$L((\triangle f)^{(i)}) = \begin{vmatrix} (\triangle f)^{(i)} - a & b - a \\ (\triangle f)^{(i+1)} - a' & b' - a' \end{vmatrix}.$$

Subsequently, we can possess,

(*i*) $L((\triangle f)^{(i)}) \not\equiv 0$;

(*ii*) $m\left(r, \dfrac{L((\triangle f)^{(i)})}{((\triangle f)^{(i)} - a)}\right) = S(r, f), \qquad m\left(r, \dfrac{L((\triangle f)^{(i)})}{((\triangle f)^{(i)} - b)}\right) = S(r, f)$;

(*iii*) $m\left(r, \dfrac{L((\triangle f)^{(i)})}{((\triangle f)^{(i)} - \nu_j)}\right) = S(r, f)$;

(*iv*) $m\left(r, \dfrac{L((\triangle f)^{(i)})}{((\triangle f)^{(i)}-\nu_1)((\triangle f)^{(i)}-\nu_2)((\triangle f)^{(i)}-\nu_3)\cdots((\triangle f)^{(i)}-\nu_n)}\right) = S(r, f)$;

(*v*) $m\left(r, \dfrac{L((\triangle f)^{(i)})((\triangle f)^{(i)})}{((\triangle f)^{(i)}-\nu_1)((\triangle f)^{(i)}-\nu_2)((\triangle f)^{(i)}-\nu_3)\cdots((\triangle f)^{(i)}-\nu_n)}\right) = S(r, f)$.

Proof. (*i*) Suppose that $L((\triangle f)^{(i)}) \equiv 0$. Then we get,

$$\frac{(\triangle f)^{(i+1)} - a'}{(\triangle f)^{(i)} - a} = \frac{a' - b'}{a - b}.$$

It follows that $(\triangle f)^{(i)} - a = t(a-b)$, where t is a non-zero constant. So $T(r,(\triangle f)^{(i)}) = T(r, t(a-b)+a) = S(r,f)$, a contradiction. Hence,

$$L((\triangle f)^{(i)}) \not\equiv 0.$$

(*ii*) Since $L((\triangle f)^{(i)}) = (a'-b')((\triangle f)^{(i)} - a) - (a-b)((\triangle f)^{(i+1)} - a')$, by Lemma 2.2, we have

$$m\left(r, \frac{L((\triangle f)^{(i)})}{((\triangle f)^{(i)} - a)}\right) \leq m(r, a'-b') + m\left(r, \frac{(a-b)((\triangle f)^{(i+1)} - a')}{(\triangle f)^{(i)} - a}\right) = S(r,f).$$

Similarly, we have

$$m\left(r, \frac{L((\triangle f)^{(i)})}{((\triangle f)^{(i)} - b)}\right) = S(r,f).$$

(*iii*)

$$\begin{aligned}
& m\left(r, \frac{L(\triangle f)^{(i)}}{(\triangle f)^{(i)} - \nu_j}\right) \\
&= m\left(r, \frac{(a'-b')((\triangle f)^{(i)} - a) - (a-b)((\triangle f)^{(i+1)} - a')}{(\triangle f)^{(i)} - \nu_j}\right) \\
&\leq m\left(r, \frac{(a'-b')[(\triangle f)^{(i)} - (a - k_j(a-b))]}{(\triangle f)^{(i)} - \nu_j}\right) \\
&\qquad + m\left(r, \frac{(a-b)[(\triangle f)^{(i+1)} - (a' - k_j(a'-b'))]}{(\triangle f)^{(i)} - \nu_j}\right) + S(r,f) \\
&\leq m\left(r, \frac{(a'-b')[(\triangle f)^{(i)} - \nu_j]}{(\triangle f)^{(i)} - \nu_j}\right) + m\left(r, \frac{(a-b)[(\triangle f)^{(i+1)} - \nu_j']}{(\triangle f)^{(i)} - \nu_j}\right) + S(r,f) \\
&= S(r,f).
\end{aligned}$$

(*iv*)

$$\frac{L((\triangle f)^{(i)})}{((\triangle f)^{(i)} - \nu_1)((\triangle f)^{(i)} - \nu_2)((\triangle f)^{(i)} - \nu_3)\cdots((\triangle f)^{(i)} - \nu_j)} = \sum_{j=1}^{n} \frac{A_j L((\triangle f)^{(i)})}{(\triangle f)^{(i)} - \nu_j},$$

where $A_j = \dfrac{1}{\prod_{j\neq i}(\nu_j - \nu_i)}$ are small functions of f.

By Lemma 2.2, Remark 2.1 and above, we have

$$m\left(r, \frac{L((\triangle f)^{(i)})}{((\triangle f)^{(i)}-\nu_1)((\triangle f)^{(i)}-\nu_2)((\triangle f)^{(i)}-\nu_3)\cdots((\triangle f)^{(i)}-\nu_j)}\right)$$
$$= m\left(r, \sum_{j=1}^{n} \frac{A_j L((\triangle f)^{(i)})}{(\triangle f)^{(i)}-\nu_j}\right)$$
$$\leq \sum_{j=1}^{n} m\left(r, \frac{A_j L((\triangle f)^{(i)})}{(\triangle f)^{(i)}-\nu_j}\right) + S(r,f) = S(r,f).$$

(v)

$$\frac{L((\triangle f)^{(i)})((\triangle f)^{(i)})}{((\triangle f)^{(i)}-\nu_1)((\triangle f)^{(i)}-\nu_2)((\triangle f)^{(i)}-\nu_3)\cdots((\triangle f)^{(i)}-\nu_j)} = \sum_{j=1}^{n} \frac{A_j L((\triangle f)^{(i)})}{(\triangle f)^{(i)}-\nu_j},$$

where $A_j = \dfrac{\nu_j}{\prod_{j\neq i}(\nu_j - \nu_i)}$ are small functions of $f(z)$.

By Lemma 2.2, Remark 2.1 and above, we have

$$m\left(r, \frac{L((\triangle f)^{(i)})((\triangle f)^{(i)})}{((\triangle f)^{(i)}-\nu_1)((\triangle f)^{(i)}-\nu_2)((\triangle f)^{(i)}-\nu_3)\cdots((\triangle f)^{(i)}-\nu_j)}\right)$$
$$= m\left(r, \sum_{j=1}^{n} \frac{A_j L((\triangle f)^{(i)})}{(\triangle f)^{(i)}-\nu_j}\right)$$
$$\leq \sum_{j=1}^{n} m\left(r, \frac{A_j L((\triangle f)^{(i)})}{(\triangle f)^{(i)}-\nu_j}\right) + S(r,f) = S(r,f). \qquad \square$$

Similarly, as in Lemma 2.4, we prove the following Lemma.

Lemma 2.5 ([10]). *Consider a transcendental entire function f, let k_j $(j = 1, 2, \cdots n)$ be positive integers, where a and b are two distinct small functions of f, with $a \neq \infty$ and $b \neq \infty$ and let $\nu_j = a - k_j(a - b)$ $(j = 1, 2, \cdots n)$. Set*

$$L(f^{(k)}) = \begin{vmatrix} f^{(k)} - a & b - a \\ f^{(k+1)} - a' & b' - a' \end{vmatrix}.$$

Subsequently, we can possess,

$$\begin{array}{ll}
(i) & L(f^{(k)}) \neq 0; \\
(ii) & m\left(r, \dfrac{L(f^{(k)})}{(f^{(k)} - a)}\right) = S(r, f), m\left(r, \dfrac{L(f^{(k)})}{(f^{(k)} - b)}\right) = S(r, f); \\
(iii) & m\left(r, \dfrac{L(f^{(k)})}{(f^{(k)} - \nu_j)}\right) = S(r, f); \\
(iv) & m\left(r, \dfrac{L(f^{(k)})}{(f^{(k)} - \nu_1)(f^{(k)} - \nu_2)(f^{(k)} - \nu_3) \cdots (f^{(k)} - \nu_j)}\right) = S(r, f); \\
(v) & m\left(r, \dfrac{L(f^{(k)})(f^{(k)})}{(f^{(k)} - \nu_1)(f^{(k)} - \nu_2)(f^{(k)} - \nu_3) \cdots (f^{(k)} - \nu_j)}\right) = S(r, f).
\end{array}$$

3. Proof of Theorem 1.1. Suppose that $\alpha_2 f^{(k)} - \alpha_1(\triangle f)^{(i)} \not\equiv 0$. Considering $f^{(k)}$ and $(\triangle f)^{(i)}$ share (α_1, α_2), 0 IM and $E(0, f^{(k)}) \subseteq E(0, (\triangle f)^{(i)})$ and f is a transcendental entire function with $\rho_2(f) < 1$, using the Nevanlinna second fundamental theorem, Lemma 2.1, 2.2 and Remark 2.1 we obtain:

$$\begin{aligned}
T(r, f^{(k)}) &\leq \overline{N}\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}\left(r, f^{(k)}\right) + S(r, f) \\
&\leq N\left(r, \frac{1}{\alpha_2 f^{(k)} - \alpha_1(\triangle f)^{(i)}}\right) + S(r, f) \\
&\leq T(r, \alpha_2 f^{(k)} - \alpha_1(\triangle f)^{(i)}) + S(r, f) \\
&= m(r, \alpha_2 f^{(k)} - \alpha_1(\triangle f)^{(i)}) + S(r, f) \\
&\leq m(r, f^{(k)}) + m\left(r, \alpha_2 - \frac{\alpha_1(\triangle f)^{(i)}}{f^{(k)}}\right) + S(r, f) \\
&\leq T(r, f^{(k)}) + S(r, f).
\end{aligned}$$

Consequently,

$$T(r,f^{(k)}) = \overline{N}\left(r,\frac{1}{f^{(k)}-\alpha_1}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + S(r,f). \tag{3.1}$$

Assign

$$\psi(z) = \frac{f^{(k+1)}(\alpha_1(\triangle f)^{(i)} - \alpha_2 f^{(k)})}{(f^{(k)}-\alpha_1)f^{(k)}}. \tag{3.2}$$

$$\phi(z) = \frac{(\triangle f)^{(i+1)}(\alpha_1(\triangle f)^{(i)} - \alpha_2 f^{(k)})}{((\triangle f)^{(i)}-\alpha_2)(\triangle f)^{(i)}}. \tag{3.3}$$

Observe that the entire function f is non-constant with $\rho_2(f) < 1$ and $f^{(k)}$ and $(\triangle f)^{(i)}$ share (α_1, α_2), 0 IM and $E(0, f^{(k)}) \subseteq E(0, (\triangle f)^{(i)})$. Let z_0 be a common zero of $f^{(k)} - \alpha_1$ (resp. $f^{(k)}$) and $(\triangle f)^{(i)} - \alpha_2$ ($resp.(\triangle f)^{(i)}$) with multiplicities m and n respectively. Without loss of generality, Set $m \le n$. Then z_0 is a zero of $\alpha_2 f^{(k)} - \alpha_1(\triangle f)^{(i)}$ with multiplicity atleast m. Obviously, z_0 is the zero of $f^{(k+1)}$ with multiplicity $m-1$. Consequently, (3.2) and the sharing of (α_1, α_2), 0 IM and $E(0, f^{(k)}) \subseteq E(0, (\triangle f)^{(i)})$ by $f^{(k)}$ and $(\triangle f)^{(i)}$ support $\psi(z_0) \neq \infty$. We understand that ψ has no poles. In other words, ψ represents an entire function. Based on Lemma 2.1, 2.2, and Remark 2.1 we conclude that,

$$\begin{aligned} T(r,\psi(z)) = m(r,\psi(z)) &= m\left(r,\frac{f^{(k+1)}(\alpha_1(\triangle f)^{(i)} - \alpha_2 f^{(k)})}{(f^{(k)}-\alpha_1)f^{(k)}}\right) \\ &\le m\left(r,\frac{f^{(k+1)}}{(f^{(k)}-\alpha_1)}\right) + m\left(r,\frac{((\triangle f)^{(i)} - f^{(k)})}{f^{(k)}}\right) \\ &= S(r,f). \end{aligned} \tag{3.4}$$

From the first fundamental theorem, Lemma 2.1, 2.2, Remark 2.1 and by Hypothesis, we obtain

$$\begin{aligned} T(r,\phi(z)) &= m(r,\phi(z)) = m\left(r,\frac{(\triangle f)^{(i+1)}(\alpha_1(\triangle f)^{(i)} - \alpha_2 f^{(k)})}{((\triangle f)^{(i)}-\alpha_2)(\triangle f)^{(i)}}\right) \\ &\le m\left(r,\frac{(\triangle f)^{(i+1)}\alpha_1(\triangle f)^{(i)}}{((\triangle f)^{(i)}-\alpha_2)(\triangle f)^{(i)}}\right) + m\left(r,\frac{(\triangle f)^{(i+1)}(\alpha_2 f^{(k)})}{((\triangle f)^{(i)}-\alpha_2)(\triangle f)^{(i)}}\right) \end{aligned}$$

$$
\begin{aligned}
&\le m\left(r,\frac{(\triangle f)^{(i+1)}\alpha_1}{((\triangle f)^{(i)}-\alpha_2)}\right)+m\left(r,\frac{(\triangle f)^{(i+1)}\alpha_2}{((\triangle f)^{(i)}-\alpha_2)}\right)\\
&\qquad +m\left(r,\frac{f^{(k)})}{(\triangle f)^{(i)}}\right)+S(r,f)\\
&\le T\left(r,\frac{(\triangle f)^{(i)}}{f^{(k)}}\right)+S(r,f)\\
&\le m\left(r,\frac{(\triangle f)^{(i)}}{f^{(k)}}\right)+N\left(r,\frac{(\triangle f)^{(i)}}{f^{(k)}}\right)+S(r,f)\\
&= S(r,f). \qquad (3.5)
\end{aligned}
$$

Thus by the Nevanlinna first fundamental theorem, (3.4), $E(0,f^{(k)})\subseteq E(0,(\triangle f)^{(i)})$, Lemma's 2.1,2.2 and Remark 2.1, we have

$$
\begin{aligned}
2T(r,f^{(k)}) &= T(r,(f^{(k)})^2)\\
&= T\left(r,\frac{1}{\psi(z)}\right)+T\left(r,f^{(k+1)}(\alpha_1(\triangle f)^{(i)}-\alpha_2 f^{(k)})+\psi(z)f^{(k)}\alpha_1\right)\\
&\le m(r,f^{(k)})+m\left(r,\frac{f^{(k+1)}\alpha_1(\triangle f)^{(i)}-\alpha_2 f^{(k+1)}f^{(k)}+\psi(z)f^{(k)}\alpha_1}{f^{(k)}}\right)\\
&\qquad +S(r,f)\\
&\le m(r,f^{(k)})+m(r,(\triangle f)^{(i)})+m\left(r,\frac{f^{(k+1)}\alpha_1(\triangle f)^{(i)}-\alpha_2 f^{(k+1)}f^{(k)}}{f^{(k)}(\triangle f)^{(i)}}\right)\\
&\qquad +S(r,f)\\
&\le T(r,f^{(k)})+T(r,(\triangle f)^{(i)})+m\left(r,\frac{\alpha_1 f^{(k+1)}}{f^{(k)}}\right)+m\left(r,\frac{f^{(k+1)}}{(\triangle f)^{(i)}}\right)\\
&\qquad +S(r,f)\\
&\le T(r,f^{(k)})+T(r,(\triangle f)^{(i)})+m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\frac{f^{(k)}}{(\triangle f)^{(i)}}\right)+S(r,f)\\
&\le T(r,f^{(k)})+T(r,(\triangle f)^{(i)})+m\left(r,\frac{(\triangle f)^{(i)}}{f^{(k)}}\right)+N\left(r,\frac{(\triangle f)^{(i)}}{f^{(k)}}\right)\\
&\qquad +S(r,f)\\
&\le T(r,f^{(k)})+T(r,(\triangle f)^{(i)})+S(r,f).
\end{aligned}
$$

Thus,

$$T(r, f^{(k)}) \leq T(r, (\triangle f)^{(i)}) + S(r, f). \tag{3.6}$$

On the other hand,

$$\begin{aligned} T(r, (\triangle f)^{(i)}) &= m(r, (\triangle f)^{(i)}) + N(r, (\triangle f)^{(i)}) \\ &\leq m\left(r, \frac{(\triangle f)^{(i)}}{f^{(k)}}\right) + m(r, f^{(k)}) \\ &\leq T(r, f^{(k)}) + S(r, f) \end{aligned}$$

$$T(r, (\triangle f)^{(i)}) \leq T(r, f^{(k)}) + S(r, f). \tag{3.7}$$

By using (3.6) and (3.7) we get,

$$T(r, (\triangle f)^{(i)}) = T(r, f^{(k)}) + S(r, f). \tag{3.8}$$

Now let m and n be two positive integers and let $z_1 \in S_{(m,n)}(\alpha_1, \alpha_2)$ $(S_{(m,n)}(0,0))$. i.e, z_1 be a common zero of $f^{(k)} - \alpha_1$ $(resp. f^{(k)})$ and $(\triangle f)^{(i)} - \alpha_2$ $((\triangle f)^{(i)})$ with multiplicities m and n, respectively, (3.2) and (3.3) imply that $n\psi(z_1) - m\phi(z_1) = 0$.

If $n\psi(z) - m\phi(z) \equiv 0$ for some positive integers m and n. It follows that $n\psi(z) \equiv m\phi(z)$. Then by calculating we have,

$$n\left(\frac{f^{(k+1)}}{f^{(k)} - \alpha_1} - \frac{f^{(k+1)}}{f^{(k)}}\right) \equiv m\left(\frac{(\triangle f)^{(i+1)}}{(\triangle f)^{(i)} - \alpha_2} - \frac{(\triangle f)^{(i+1)}}{(\triangle f)^{(i)}}\right). \tag{3.9}$$

It suggests that,

$$\left(\frac{f^{(k)} - \alpha_1}{f^{(k)}}\right)^n \equiv A\left(\frac{(\triangle f)^{(i)} - \alpha_2}{(\triangle f)^{(i)}}\right)^m \tag{3.10}$$

A is a non-zero constant. As a result, m equals n; otherwise, (3.8) would be contradictory.

$$B\left(\frac{f^{(k)} - \alpha_1}{f^{(k)}}\right) \equiv \left(\frac{(\triangle f)^{(i)} - \alpha_2}{(\triangle f)^{(i)}}\right) \tag{3.11}$$

where $B \neq 1$ is a non-zero constant. Thus we have

$$\frac{\alpha_2}{(\triangle f)^{(i)}} = \frac{(B-1)f^{(k)} + (-\alpha_1 B)}{f^{(k)}}.$$

Since $f(z)$ is an entire function with $\rho_2(f) < 1$, such that

$$f^{(k)} \neq \frac{-\alpha_1 B}{1-B}.$$

Clearly,

$$\frac{-\alpha_1 B}{1-B} \neq \alpha_1, 0. \tag{3.12}$$

Therefore, we have

$$\begin{aligned} 2T(r, f^{(k)}) &\leq \overline{N}\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{f^{(k)} - \frac{-\alpha_1 B}{1-B}}\right) + S(r,f) \\ &\leq \overline{N}\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r,f). \end{aligned}$$

Which contradicts (3.1).

Hence, $n\psi(z) \not\equiv m\phi(z)$ for any two positive numbers, m and n. Therefore, we have

$$\begin{aligned} \overline{N}_{(m,n)}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) &\leq \overline{N}\left(r, \frac{1}{n\psi(z) - m\phi(z)}\right) \\ &\leq T(r, n\psi(z) - m\phi(z)) + S(r,f) \\ &\leq T(r, \psi(z)) + T(r, \phi(z)) + S(r,f) \\ &= S(r,f) \end{aligned} \tag{3.13}$$

for all positive integers m and n. Thus, using (3.1), (3.8) and (3.13), we obtain

$$
\begin{aligned}
T(r, f^{(k)}) &= \overline{N}\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \\
&\leq \overline{N}_{1)}\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) + \overline{N}_2\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) + \overline{N}_3\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) \\
&\quad + \overline{N}_4\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) + \overline{N}_{(5}\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) + \overline{N}_{1)}\left(r, \frac{1}{f^{(k)}}\right) \\
&\quad + \overline{N}_2\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}_3\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}_4\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}_{(5}\left(r, \frac{1}{f^{(k)}}\right) \\
&\quad + S(r, f) \\
&\leq \sum_{n=1}^{4}\sum_{m=1}^{4}\left(\overline{N}_{(m,n)}\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{f^{(k)}}\right)\right) \\
&\quad + \overline{N}_{(5}\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) + \overline{N}_{(5}\left(r, \frac{1}{(\triangle f)^{(i)} - \alpha_2}\right) + \overline{N}_{(5}\left(r, \frac{1}{f^{(k)}}\right) \\
&\quad + \overline{N}_{(5}\left(r, \frac{1}{(\triangle f)^{(i)}}\right) + S(r, f) \\
&\leq \frac{1}{5}\left[N\left(r, \frac{1}{f^{(k)} - \alpha_1}\right) + N\left(r, \frac{1}{f^{(k)}}\right)\right] \\
&\quad + \frac{1}{5}\left[N\left(r, \frac{1}{(\triangle f)^{(i)} - \alpha_2}\right) + N\left(r, \frac{1}{(\triangle f)^{(i)}}\right)\right] + S(r, f) \\
&\leq \frac{4}{5}T(r, f^{(k)}) + S(r, f)
\end{aligned}
$$

which gives a contradiction to the hypothesis, that f is transcendental entire function.

Thus $\alpha_2 f^{(k)} \equiv \alpha_1 (\triangle f)^{(i)}$. □

4. Open problem. Here, it is natural to pose the open quenstion that is it possible to prove the following Theorem:

"Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$, let c be a non-zero finite complex value, and α_1, α_2, β_1, β_2 be non-zero finite values such that $\alpha_1 \not\equiv \beta_1$ and $\alpha_2 \not\equiv \beta_2$. If $f^{(k)}$ and $(\triangle f)^{(i)}$ share (α_1, α_2) and (β_1, β_2) IM, then $(\alpha_2 - \beta_2) f^{(k)} - (\alpha_1 - \beta_1)(\triangle f)^{(i)} \not\equiv \alpha_2\beta_1 - \alpha_1\beta_2$".

"Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$, let c be a non-zero finite complex value, and $\alpha_1, \alpha_2, \beta_1, \beta_2$ be small functions of f, such that $\alpha_1 \not\equiv \beta_1$, and $\alpha_2 \not\equiv \beta_2$. If $f^{(k)}$ and $(\triangle f)^{(i)}$ share (α_1, α_2) and (β_1, β_2) IM, then $(\alpha_2 - \beta_2) f^{(k)} - (\alpha_1 - \beta_1)(\triangle f)^{(i)} \not\equiv \alpha_2\beta_1 - \alpha_1\beta_2$, for $i \leq k$."

REFERENCES

[1] T. B. Cao. Difference analogues of the second main theorem for meromorphic functions in several complex variables. *Math Nachr.* **287**, *5–6* (2014), 530–545.

[2] S. Chen. On uniqueness of meromorphic functions and their difference operator with partially shared values. *Comput. Methods Funct. Theory* **18**, *3* (2018), 529–536.

[3] Z.-X. Chen, H.-X. Yi. On sharing values of meromorphic functions and their differences. *Results Math.* **63**, *1–2* (2013), 557–565.

[4] Y.-M. Chiang, S.-J. Feng. On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. *Ramanujan J.* **16**, *1* (2008), 105–129.

[5] R. S. Dyavanal. Uniqueness and value-sharing of differential polynomials of meromorphic functions. *J. Math. Anal. and Appl.* **372**, *1* (2010), 252–261.

[6] R. S. Dyavanal, A. M. Hattikal. Uniqueness of difference-differential polynomials of entire functions sharing one value. *Tamkang J. Math.* **47**, *2* (2016) 193–206.

[7] R. G. Halburd, R. J. Korhonen. Nevanlinna theory for the difference operator. *Ann. Acad. Sci. Fenn. Math.* **31**, *2* (2006), 463–478.

[8] W. K. Hayman. Meromorphic Functions. Oxford Math. Monogr. Oxford, Clarendon Press, 1964.

[9] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo. Uniqueness of meromorphic functions sharing values with their shifts. *Complex Var. Elliptic Equ.* **56**, *1–4* (2011), 81–92.

[10] X. Huang. Unicity on entire function concerning its differential-difference operators. *Results Math.* **76**, *3* (2021), Paper No. 147, 17 pp.

[11] X. Huang, Uniqueness of entire functions sharing two pairs of values with its difference operator. arXiv: 2109.15183, 2021, `https://arxiv.org/abs/2109.15183`

[12] X. Huang, B. Deng, M. L. Fang. Entire functions that share two pairs of small functions. *Open Math.* **19**, *1* (2021), 144–156.

[13] X. Huang, M. L. Fang. Unicity of entire functions concerning their shifts and derivatives. *Comput. Methods Funct. Theory* **21**, *3* (2021), 523–532.

[14] I. Lahiri. Value distribution of certain differential polynomials, *Int. J. Math. Math. Sci.* **28**, *2* (2001), 83–91.

[15] X. Qi, L. Yang. Uniqueness of meromorphic functions concerning their shifts and derivatives. *Comput. Methods Funct. Theory* **20**, *1* (2020) 159–178.

[16] L. Sheng, D. Mei, B. Chen. Uniqueness of entire functions sharing two values with their difference operators. *Adv. Difference Equ.* (2017), Paper No. 390, 9 pp.

[17] C.-C. Yang, H.-X. Yi. Uniqueness Theory of Meromorphic Functions. Math. Appl., vol. **557**. Dordrecht, Kluwer Academic Publishers Group, 2003.

[18] L. Yang. Value Distribution Theory. Berlin, Springer-Verlag; Beijing, Science Press Beijing, 1993.

[19] Q. C. Zhang. Uniqueness of meromorphic functions with their derivatives. *Acta Math. Sinica (Chinese Ser.)* **45**, *5* (2002), 871–876.

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