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WINDOWS AND MODULI OF BUNDLES ON A CURVE*

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Abstract. We survey some results regarding semi-orthogonal decompositions on moduli spaces of vector bundles on a curve, with an emphasis on those that use the machinery of windows. We briefly outline some of the main ideas that make windows a crucial tool for these results.

1. Introduction. The present note follows a talk given by the author during the conference "Generalized and Symplectic Geometry" that took place during August 2023 in Sunny Beach, Bulgaria.

Let C be a smooth projective curve over $\mathbb{C}$, of genus $g \geq 2$. A vector bundle V on C is said to be slope-semistable if for every nontrivial vector subbundle $W \subset V$ one has

$$\frac{\deg W}{\operatorname{rk} W} \leq \frac{\deg V}{\operatorname{rk} V}.$$

We say that V is slope-stable if the same is true with a strict inequality. For a given rank r and a fixed line bundle Λ of degree d, there is a moduli space $M_C(r, \Lambda)$

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that parametrizes isomorphism classes of slope-semistable vector bundles of rank r and determinant Λ [20]. Moreover, if $(d, r) = 1$, then $M_C(r, \Lambda)$ is a smooth projective Fano variety of index 2 with Picard group isomorphic to $\mathbb{Z}$, and there is a universal family $\mathcal{E}$ over $C \times M_C(r, \Lambda)$ [20, 19].

Throughout this note, we will only consider the case when d and r are coprime.

Definition 1.1. *We will denote by θ the ample generator of* $\operatorname{Pic} M_C(r, \Lambda)$. *Also, we normalize the universal bundle $\mathcal{E}$ so that $c_1(\mathcal{E}|_{\{x\}\times M_C(r,\Lambda)}) = \theta$.*

Definition 1.2. *We denote by $\Phi_{\mathcal{E}} : D^b(C) \to M_C(r, \Lambda)$ the Fourier–Mukai functor defined by the universal bundle $\mathcal{E}$, that is*

$$\Phi_{\mathcal{E}} = Rp_*(Lq^*(\cdot) \otimes \mathcal{E}),$$

where q and p are the canonical projections from $C \times M_C(r, \Lambda)$ to each factor.

Since $M_C(r, \Lambda)$ is Fano of index two, the vector bundles $\theta^\vee, \mathcal{O}_{M_C(r,\Lambda)}$ start a semiorthogonal decomposition on its derived category, with each of these two blocks being equivalent to $D^b(\mathrm{pt})$. What we want to do is study the remaining part of this decomposition. Observe that, for example, if $\Phi_{\mathcal{E}}$ is fully faithful, then we get a copy of $D^b(C)$ inside $D^b(M_C(r, \Lambda))$. In fact, if arranged correctly, these three blocks will constitute the first pieces of a semi-orthogonal decomposition. In what follows we present some known results that answer some questions about the decomposition of $D^b(M_C(r, \Lambda))$. Notably, for the case $r = 2$, a full semi-orthogonal decomposition is known.

2. The case $r = 2$.

When $g = 2$ and $r = 2$, with d odd, the moduli space $M_C(2, \Lambda)$ is the intersection of two quadrics in $\mathbb{P}^5$. A classical theorem by Bondal and Orlov gives a description of the derived category of $M_C(2, \Lambda)$.

Theorem 2.1 ([7, Theorem 2.9])**.** *If $g = r = 2$, and d is odd, $D^b(M_C(2, \Lambda))$ has a semi-orthogonal decomposition of the form*

$$D^b(M_C(2, \Lambda)) = \langle \theta^\vee, \mathcal{O}_{M_C(2,\Lambda)}, D^b(C)\rangle.$$

The embedding from $D^b(C)$ to $D^b(M_C(2, \Lambda))$ is obtained by the Fourier Mukai transform $\Phi_{\mathcal{E}}$. Later work by Fonarev and Kuznetsov, and then Narasimhan, showed full faithfulness of $\Phi_{\mathcal{E}}$ holds for higher genus too.

Theorem 2.2 ([10])**.** *Let $r = 2$, $g \geq 2$, d odd. For a generic curve C of genus g the functor*

$$\Phi_{\mathcal{E}} : D^b(C) \to D^b(M_C(2, \Lambda))$$

is fully faithful.

Theorem 2.3 ([18])**.** *Let $r = 2$, $g \geq 2$, d odd. For any smooth projective curve C of genus g the functor*

$$\Phi_{\mathcal{E}} : D^b(C) \to D^b(M_C(2, \Lambda))$$

is fully faithful. Moreover,

$$\theta^\vee,\ \mathcal{O}_{M_C(2,\Lambda)},\ D^b(C)$$

is the start of a semi-orthogonal decomposition on $D^b(M_C(2, \Lambda))$.

Here, by abuse of notation, we are denoting by $D^b(C)$ the essential image of $\Phi_{\mathcal{E}} : D^b(C) \hookrightarrow D^b(M_C(2, \Lambda))$. Moreover, Belmans and Mukhopadhyay showed that, by twisting this subcategory by $\theta^\vee$, one can arrange to have four semi-orthogonal blocks, now with two copies of $D^b(C)$.

Theorem 2.4 ([6])**.** *Let $r = 2$, $g \geq 4$, d odd. Then*

$$\theta^\vee,\ \theta^\vee \otimes D^b(C),\ \mathcal{O}_{M_C(2,\Lambda)},\ D^b(C)$$

is the start of a semiorthogonal decomposition on $D^b(M_C(2, \Lambda))$.

Again abusing the notation, by $\theta^\vee \otimes D^b(C)$ we mean objects of the form $\theta^\vee \otimes V$, where V is in the essential image of $\Phi_{\mathcal{E}}$.

In [16], Lee and Narasimhan found that, under certain hypotheses, one can also find a copy of $D^b(\mathrm{Sym}^2 C)$ inside $D^b(M_C(2, \Lambda))$.

Theorem 2.5 ([16])**.** *Let $r = 2$, d odd and suppose C is non-hyperelliptic of genus $g \geq 16$. Then there is a fully faithful functor*

$$D^b(\mathrm{Sym}^2 C) \hookrightarrow D^b(M_C(2, \Lambda)).$$

Moreover,

$$\mathcal{O}_{M_C(2,\Lambda)},\ D^b(C),\ D^b(\mathrm{Sym}^2 C)$$

is the start of a semi-orthogonal decomposition on $D^b(M_C(2, \Lambda))$.

We will discuss in §4 how to obtain the functor

$$D^b(\mathrm{Sym}^2 C) \hookrightarrow D^b(M_C(2,\Lambda))$$

appearing in this theorem.

So far we have described results where we see semi-orthogonal pieces inside $D^b(M_C(2,\Lambda))$ that are equivalent to $D^b(\mathrm{pt})$, $D^b(C)$ or $D^b(\mathrm{Sym}^2 C)$. Motivic decompositions and other cohomological computations [8, 13, 5] suggested that higher symmetric products of C should be found as well. In fact, all these are pieces of evidence towards the following conjecture, stated independently by Narasimhan and by Belmans, Galkin and Mukhopadhyay [13, 2].

Conjecture 2.6 (BGMN)**.** *$D^b(M_C(2,\Lambda))$ has a semi-orthogonal decomposition of the form*

$$\langle D^b(\mathrm{pt}),\ D^b(\mathrm{pt}),\ D^b(C),\ D^b(C), \dots$$
$$\dots,\ D^b(\mathrm{Sym}^{g-2} C),\ D^b(\mathrm{Sym}^{g-2} C),\ D^b(\mathrm{Sym}^{g-1} C)\rangle.$$

That is, there are two copies of $D^b(\mathrm{Sym}^\alpha C)$ for $\alpha = 0, \dots, g-2$ and one of $D^b(\mathrm{Sym}^{g-1} C)$.

From [17], we know that such semi-orthogonal decomposition cannot be further decomposed.

In a series of works done during recent years, the BGMN conjecture was finally proved.

Theorem 2.7 ([24, 27, 23])**.** *Conjecture* 2.6 *holds.*

Remark 2.8. In [21], Sink and Tevelev prove an analog of Theorem 2.7 for the case of even degree, where they work with a noncommutative resolution of the moduli space. In the even genus case, this shows a conjecture by Belmans [3].

In §6, we briefly discuss the part of Theorem 2.7 that has to do with building the blocks and proving semi-orthogonality between them, by using the method of windows. This proof can be found in [24]. In [27], the same result is obtained by different methods, which are also explained in [26]. For the fullness part, that is, showing that the blocks constructed exhaust $D^b(M_C(2,\Lambda))$, the reader is referred to [23].

3. The case $r > 2$. For arbitrary rank r, a semi-orthogonal decomposition of the derived category $D^b(M_C(2,\Lambda))$ is not known. However, just as in the case $r = 2$, we do know that at least we can start with $D^b(\mathrm{pt})$ and $D^b(C)$. This

was proved by Belmans and Mukhopadhyay under certain numerical conditions, and later enhanced by Lee and Moon, removing these conditions. More precisely, the results are as follows.

Theorem 3.1 ([6])**.** *For $d = 1$ and $g \geq r + 3$, the functor*

$$\Phi_{\mathcal{E}} : D^b(C) \to D^b(M_C(r, \Lambda))$$

is fully faithful. Moreover, if $g \gg 0$ then

$$\theta^\vee,\ \theta^\vee \otimes D^b(C),\ \mathcal{O}_{M_C(r,\Lambda)},\ D^b(C)$$

is the start of a semiorthogonal decomposition on $D^b(M_C(r, \Lambda))$.

Theorem 3.2 ([14])**.** *For any d with $(d, r) = 1$ and any $g \geq 2$, the functor*

$$\Phi_{\mathcal{E}} : D^b(C) \to D^b(M_C(r, \Lambda))$$

is fully faithful. Moreover, if $g \geq 6$ then

$$\theta^\vee,\ \theta^\vee \otimes D^b(C),\ \mathcal{O}_{M_C(r,\Lambda)},\ D^b(C)$$

is the start of a semiorthogonal decomposition on $D^b(M_C(r, \Lambda))$.

More recently, Lee and Moon proved the following.

Theorem 3.3 ([15])**.** *Let $g \geq 2$, $(d, r) = 1$ and suppose $r > 2\alpha$. Then there is a fully faithful functor*

$$D^b(\mathrm{Sym}^\alpha C) \hookrightarrow D^b(M_C(r, \Lambda)).$$

Remark 3.4. In particular, all symmetric products of C are Fano visitors.

In §7, we briefly mention the main ideas for the proof of Theorem 3.2, where windows are used to show full faithfulness and semi-orthogonality. Theorem 3.3 uses similar ideas, again involving windows.

When $r = 3$ and $d = 1$, there is a decomposition formula for the motive of $M_C(3, \Lambda)$ in terms of that of products of the form $\mathrm{Sym}^\alpha C \times \mathrm{Jac}\, C$, found by Gómez and Lee in [11]. However, its counterpart at the level of derived categories is an open problem. In general, a semi-orthogonal decomposition of $M_C(r, \Lambda)$ is expected to have the following shape.

Conjecture 3.5 ([11, 14])**.** *$D^b(M_C(r, \Lambda))$ has a semi-orthogonal decomposition where each block is the derived category of products of* $\mathrm{Sym}^\alpha C$ *and* $\mathrm{Jac}\, C$*.*

4. The relevant functors. We will say something about the proofs of Theorem 2.7 and Theorem 3.2. Specifically, we will address the use of windows as the main tool for showing full faithfulness of different functors and semi-orthogonality between the blocks defined by their essential images. We will not address the statement of fullness in Theorem 2.7, which uses different techniques and can be found in [23].

We have mentioned that, in the theorems above, the copies of $D^b(C)$ inside $D^b(M_C(r,\Lambda))$ are obtained using the Fourier–Mukai transform $\Phi_{\mathcal{E}} = Rp_*(Lq^*(\cdot)\otimes \mathcal{E})$, where $\mathcal{E}$ is the universal bundle (normalized as in Definition 1.1).

Now, for higher symmetric products of C, the relevant functors from $D^b(\mathrm{Sym}^\alpha C)$ to $D^b(M_C(2,\Lambda))$ are obtained doing the following construction. Let τ be the S_α-quotient map

$$\tau : C^\alpha \to \mathrm{Sym}^\alpha C$$

and consider the projections $\pi_j : C^\alpha \to C$ for $j = 1,\dots,\alpha$. Then the bundle $\bigotimes_{j=1}^{\alpha} \pi_j^*(\mathcal{E})$ has an S_α-equivariant structure. We define

$$\mathcal{E}^{\boxtimes\alpha} := \tau_*^{S_\alpha}\left(\bigotimes_{j=1}^{\alpha} \pi_j^*\mathcal{E}\right)$$

and similarly

$$\overline{\mathcal{E}}^{\boxtimes\alpha} := \tau_*^{S_\alpha}\left(\bigotimes_{j=1}^{\alpha} \pi_j^*\mathcal{E}\otimes \mathrm{sgn}\right).$$

Both $\mathcal{E}^{\boxtimes\alpha}$ and $\overline{\mathcal{E}}^{\boxtimes\alpha}$ are vector bundles on $\mathrm{Sym}^\alpha C \times M_C(r,\Lambda)$, and we can consider the Fourier–Mukai transform defined by each of them.

Definition 4.1. *We will denote by*

$$\Phi_\alpha,\ \overline{\Phi}_\alpha : D^b(\mathrm{Sym}^\alpha C) \to D^b(M_C(r,\Lambda))$$

the Fourier–Mukai transforms with kernel $\mathcal{E}^{\boxtimes\alpha}$, $\overline{\mathcal{E}}^{\boxtimes\alpha}$, respectively.

More generally, and because it will be needed later, we do an analogous definition for any vector bundle on $C \times M$, where M is any scheme over $\mathbb{C}$.

Definition 4.2. *For M a scheme over $\mathbb{C}$ and F a vector bundle on $C\times M$, we define*

$$F^{\boxtimes\alpha} := \tau_*^{S_\alpha}\left(\bigotimes_{j=1}^{\alpha}\pi_j^*F\right) \quad \text{and} \quad \overline{F}^{\boxtimes\alpha} := \tau_*^{S_\alpha}\left(\bigotimes_{j=1}^{\alpha}\pi_j^*F\otimes \operatorname{sgn}\right)$$

and denote by

$$\Phi_\alpha^F,\ \overline{\Phi}_\alpha^F : D^b(\operatorname{Sym}^\alpha C) \to D^b(M_C(r,\Lambda))$$

the corresponding Fourier–Mukai transforms thus defined.

4.1. The Bondal–Orlov criterion. To see wether Φ_α and $\overline{\Phi}_\alpha$ are fully faithful, the Bondal–Orlov criterion [7] says that it suffices to consider the skyscraper sheaves of closed points. For an effective divisor $D \in \operatorname{Sym}^\alpha C$, we denote by $(\mathcal{E}^{\boxtimes\alpha})_D$ the restriction of $\mathcal{E}^{\boxtimes\alpha}$ to $\{D\}\times M_C(r,\Lambda)$, and similarly with $(\overline{\mathcal{E}}^{\boxtimes\alpha})_D$. Note that $(\mathcal{E}^{\boxtimes\alpha})_D = \Phi_\alpha(\mathcal{O}_D)$. So the question of full faithfulness of Φ_α boils down to the computation of

$$R\operatorname{Hom}_{M_C(r,\Lambda)}\left((\mathcal{E}^{\boxtimes\alpha})_D, (\mathcal{E}^{\boxtimes\alpha})_{D'}\right) = R\Gamma_{M_C(r,\Lambda)}\left((\mathcal{E}^{\boxtimes\alpha})_D^\vee \otimes (\mathcal{E}^{\boxtimes\alpha})_{D'}\right)$$

for divisors D, D' of degree α. Namely, Φ_α will be fully faithful as long as for every $D, D' \in \operatorname{Sym}^\alpha C$ we have

$$H^p\left(M_C(r,\Lambda), (\mathcal{E}^{\boxtimes\alpha})_D^\vee \otimes (\mathcal{E}^{\boxtimes\alpha})_{D'}\right) = \begin{cases} 0 & \text{if } D \neq D' \text{ or } p<0 \text{ or } p>\alpha \\ \mathbb{C} & \text{if } p=0 \text{ and } D=D'. \end{cases} \tag{4.1}$$

For $\overline{\Phi}_\alpha$ we have the analogous condition, with $\overline{\mathcal{E}}^{\boxtimes\alpha}$ in place of $\mathcal{E}^{\boxtimes\alpha}$. This means that we need to compute cohomology groups of several bundles on $M_C(r,\Lambda)$, and in most cases we should obtain no cohomology at all.

More generally, if F is any bundle on $C\times M$, where M is a complex scheme, the functor Φ_α^F from Definition 4.2 is fully faithful if and only if the same condition (4.1) holds, with F in place of $\mathcal{E}$ and M in place of $M_C(r,\Lambda)$.

Now, if full faithfulness holds, then semi-orthogonality between these different admissible subcategories can be tested in a similar fashion. For instance, for the image of Φ_α^F to be left orthogonal to that of Φ_β^F, it suffices to prove that $R\Gamma_M\left((F^{\boxtimes\alpha})_D^\vee \otimes (F^{\boxtimes\beta})_{D'}\right) = 0$ whenever $D \in \operatorname{Sym}^\alpha C$ and $D' \in \operatorname{Sym}^\beta C$.

5. Windows. The theory of windows, as developed by Halpern-Leistner and by Ballard, Favero and Katzarkov, is a key ingredient in the theorems mentioned above. Roughly speaking, given two GIT quotients $X /\!\!/_{\mathcal{L}_\pm} G$ of the same space with different choices of stability conditions, windows will allow us to compare the cohomology groups of a same G-equivariant object on X once restricted to either of these two quotients. Here we spell out the main ideas. Details can be found in [12, 1].

Let X be a smooth projective-over-affine variety over $\mathbb{C}$ with an action of a reductive group G and let $\mathcal{L}$ be a G-equivariant ample line bundle. We consider the GIT quotient $X /\!\!/_{\mathcal{L}} G$. For simplicity, let us assume that G acts freely on the semistable locus $X^{\mathrm{ss}}(\mathcal{L})$.

5.1. Weights. Let F be a G-equivariant vector bundle on X. If a one-parameter subgroup $\lambda : \mathbb{G}_m \hookrightarrow G$ fixes a point $z \in X$, then $\mathbb{G}_m$ has to act on the fiber F_z. The eigenvalues of this action are called the *λ-weights* of F. The unstable locus $X \backslash X^{\mathrm{ss}}(\mathcal{L})$ can be given a *Kempf–Ness stratification* $\sqcup S_t$, where each stratum S_t is indexed by a pair consisting of a one-parameter subgroup λ and a connected component Z of the fixed locus X^λ. By the *set of weights* of F we mean the collection of all the λ-weights of F on Z, for every pair (λ, Z) appearing in the stratification. More generally, weights can be defined for any G-equivariant object $F \in D^b_G(X)$.

5.2. Quantization theorem. When the weights of a given object $F \in D^b_G(X)$ live in certain range, the cohomology in the GIT quotient $X /\!\!/_{\mathcal{L}} G$ can be computed directly on X.

Theorem 5.1 ([22, 12])**.** *Let $F \in D^b_G(X)$. Take a Kempf-Ness stratification of the unstable locus $X \backslash X^{\mathrm{ss}}(\mathcal{L}) = \sqcup S_t$ and or each t let η_t be the weight of $\det \mathcal{N}^\vee_{S_t/X}$. If all the weights of F on each S_t are $< \eta_t$, then*

$$H^p(X /\!\!/_{\mathcal{L}} G, F) = H^p(X, F)^G$$

for every p. More generally, let $F, K \in D^b_G(X)$ and fix an arbitrary integer w_t for each stratum. If, in every stratum, all the weights of F are $\geq w_t$ and all the weights of K are $< w_t + \eta_t$, then

$$\mathrm{Hom}_{D^b(X /\!\!/_{\mathcal{L}} G)}(F, K) = \mathrm{Hom}_{D^b_G(X)}(F, K).$$

5.3. The window subcategory. Moreover, for any given choice of integers $w = (w_t)$, there is a way to embed $D^b(X /\!\!/_{\mathcal{L}} G)$ as a *window subcategory* inside the derived category of the stack $[X/G]$, and to write down its orthogonal complement.

Theorem 5.2 ([12, 1])**.** *With the notation as above, for any choice of integers $w = (w_t)$, define $G_w \subset D^b_G(X)$ as the full triangulated subcategory consisting of objects having weights within $[w_t, w_t + \eta_t)$. Then the restriction from X to $X^{\mathrm{ss}}(\mathcal{L})$ defines an equivalence of categories*

$$G_w \cong D^b(X /\!\!/_{\mathcal{L}} G).$$

Moreover, G_w sits in a semi-orthogonal decomposition

$$D^b_G(X) = \langle D^b_{X^{\mathrm{us}}(\mathcal{L})}([X/G])_{<w},\ G_w,\ D^b_{X^{\mathrm{us}}(\mathcal{L})}([X/G])_{\geq w} \rangle,$$

where $D^b_{X^{\mathrm{us}}(\mathcal{L})}([X/G])_{<w}$ stands for the full triangulated subcategory of objects supported in the unstable locus $X^{\mathrm{us}}(\mathcal{L})$ having weights $< w_t$, and similarly for $D^b_{X^{\mathrm{us}}(\mathcal{L})}([X/G])_{\geq w}$.

5.4. Windows in the wall-crossing. As a consequence, we can compare what happens when we restrict objects in $D^b_G(X)$ to different GIT chambers on either side of a GIT wall. If the weights satisfy the right numerical condition, cohomology will be the same on both restrictions. Furthermore, when one window is strictly smaller than the other, the derived category of one side of the wall is embedded in that of the other.

Theorem 5.3 ([12, 1])**.** *Let $\mathcal{L}_\pm$ be two G-equivariant line bundles on either side of a codimension-one GIT wall $\mathcal{L}_0$. Suppose the wall-crossing is balanced, with $X^{\mathrm{ss}}(\mathcal{L}_0) = \sqcup S^\pm_t \cup X^{\mathrm{ss}}(\mathcal{L}_\pm)$. Let $F, K \in D^b_G(X)$ such that, on the strictly semistable locus of the wall, all the weights satisfy*

$$-\eta^-_t < \mathrm{weight}_{\lambda_t} K - \mathrm{weight}_{\lambda_t} F < \eta^+_t.$$

Then

$$\mathrm{Hom}_{D^b(X /\!\!/_{\mathcal{L}_+} G)}(F, K) = \mathrm{Hom}_{D^b_G(X)}(F, K) = \mathrm{Hom}_{D^b(X /\!\!/_{\mathcal{L}_-} G)}(F, K).$$

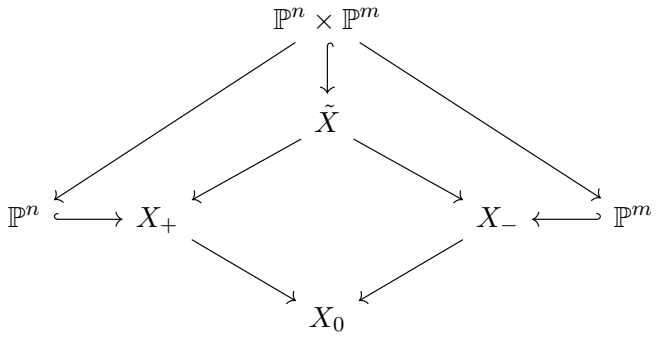
Moreover, if the weights of ω_X on the strictly semistable locus of the wall are all < 0, then $[w_t, \eta^+_t) \subset [w_t, \eta^-_t)$ and we get an embedding

$$D^b(X^{\mathrm{ss}} /\!\!/_{\mathcal{L}_+} G) \hookrightarrow D^b(X^{\mathrm{ss}} /\!\!/_{\mathcal{L}_-} G)$$

given by $G^+_w \subset G^-_w$.

Remark 5.4. Here we have assumed that G acts freely on the semistable locus of a given GIT chamber, so that in this case the GIT quotient $X /\!\!/_{\mathcal{L}} G$ is isomorphic to the corresponding quotient stack $[X^{\mathrm{ss}}(\mathcal{L})/G]$. In the general case, the results mentioned in the present section need to be stated in terms of the derived category of the stack $[X^{\mathrm{ss}}(\mathcal{L})/G]$ rather than $X /\!\!/_{\mathcal{L}} G$.

Example 5.5. Let $\mathbb{G}_m$ act on $X = \mathbb{A}^{n+1} \times \mathbb{A}^{m+1}$ by $(tx, t^{-1}y)$ and choose linearizations $\mathcal{L}_\pm = \mathcal{O}_X(\pm 1)$. We get a standard flip



A window on the left hand side lives in weights $[w, w+n+1)$, and a window on the right hand side lives in weights $[w, w+m+1)$. If $n \leq m$, we get an embedding $D^b(X_+) \subset D^b(X_-)$ from $G_w^+ \subset G_w^-$.

Remark 5.6. In the case of $M_C(r, \Lambda)$, we only have one possible stability condition, namely slope-stability. This means that windows cannot be applied directly to our case. However, we can try to rewrite the original questions in terms of a different moduli problem where we do have variation of GIT.

6. Stable pairs for $r = 2$. In the case of $r = 2$, instead of studying the space $M_C(2, \Lambda)$ directly, we will deal with moduli spaces of stable pairs, where the stability condition depends on a real parameter σ. These spaces were introduced and studied by Thaddeus [25].

Definition 6.1. *Let C be a smooth projective curve of genus $g \geq 2$. Fix a number σ. A pair (E, ϕ) of a rank-two bundle with a section $\phi \in H^0(E)$ is said to be σ-semistable if for every line subbundle $L \subset E$ one has*

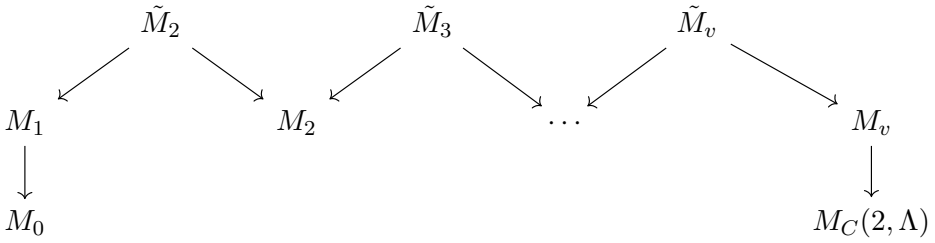
$$\deg L \leq \frac{1}{2} \deg E - \sigma \qquad \text{if } \phi \in H^0(L),$$
$$\deg L \leq \frac{1}{2} \deg E + \sigma \qquad \text{if } \phi \notin H^0(L).$$

As usual, Λ is a fixed line bundle on C of degree d.

Theorem 6.2 ([25])**.** *Given $\sigma \in (0, d/2]$, there exists a moduli space $M_C(2, \Lambda; \sigma)$ parametrizing σ-semistable pairs (E, ϕ) with fixed determinant* $\det E =$

Λ. *Moreover, when* $\sigma \notin d/2+\mathbb{Z}$, *the space* $M_C(2,\Lambda;\sigma)$ *carries a universal bundle* $\mathcal{F}$ *and a universal section* $\varphi \in H^0(\mathcal{F})$.

Thaddeus gives an explicit way to construct $M_C(2,\Lambda;\sigma)$ using GIT. Indeed, for different values of σ, these spaces arise from variation of GIT, which is described by chambers M_0, $M_1,\dots,$ M_v and a diagram

(6.1)


where M_0 is a projective space, the map $M_1 \to M_0$ is a blow-up along a copy of $C \hookrightarrow M_0$, and $M_{i-1} \dashrightarrow M_i$ are small birational modifications for $1 < i \le v = \lfloor (d-1)/2 \rfloor$. The last morphism $M_v \to M_C(2,\Lambda)$ forgets the section in the pair, and it is birational if $d = 2g-1$. The GIT walls occur at $\sigma \in d/2+\mathbb{Z}$.

6.1. Koszul resolutions. Let $M_i = M_C(2,\Lambda;\sigma)$ be one of these spaces. One important feature is that, if we consider the locus of pairs (E,ϕ) where ϕ vanishes along a given effective divisor D, this again consists of a moduli space of stable pairs, now with the same parameter σ but different determinant bundle.

Theorem 6.3 ([25, §1.9])**.** *For any effective divisor* D *on* C *of degree* α, *one can embed*

$$M_{i-\alpha}(\Lambda(-2D)) \hookrightarrow M_i(\Lambda)$$

as the locus of pairs (E,ϕ) *such that* ϕ *vanishes along* D.

Let $x \in C$ and let $(\mathcal{F},\varphi)$ be the universal pair on $M_i(\Lambda)$. Write $\mathcal{F}_x$ for the restriction of $\mathcal{F}$ to $\{x\} \times M_i(\Lambda)$ and $\varphi_x \in H^0(\mathcal{F}_x)$ for the corresponding restriction of φ. Then the zero locus of φ_x is precisely $M_{i-1}(\Lambda(-2x)) \subset M_i(\Lambda)$. Furthermore, it is not difficult to show that there is an exact sequence

$$0 \to \Lambda^2\mathcal{F}_x^\vee \to \mathcal{F}_x^\vee \to \mathcal{O}_{M_i(\Lambda)} \to \mathcal{O}_{M_{i-1}(\Lambda(-2x))} \to 0, \tag{6.2}$$

where the map $\mathcal{F}_x^\vee \to \mathcal{O}_{M_i(\Lambda)}$ is given by φ_x.

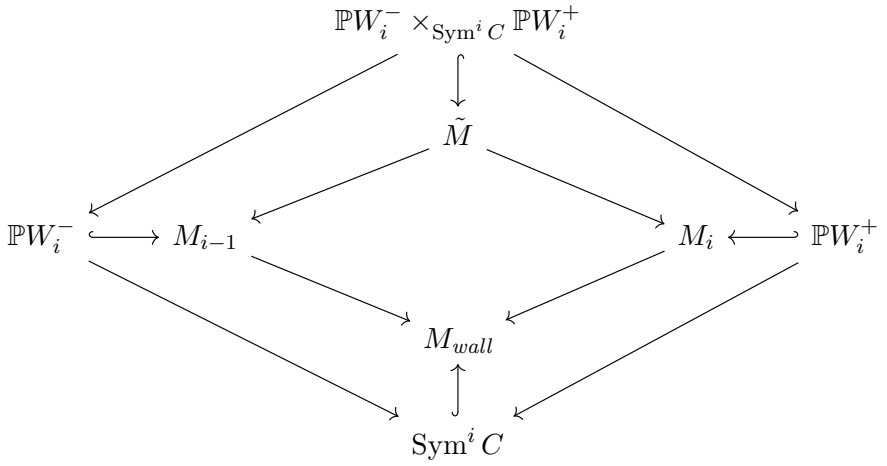
6.2. Wall-crossing. Every two consecutive chambers M_{i-1}, M_i can be written as GIT quotients $X /\!\!/_{\mathcal{L}_\pm} PGL_\chi$ of the same space by some projective linear group under different stability conditions. There is a PGL_χ-equivariant bundle $\mathcal{F}$ on X that restricts to the universal bundle on both sides of the wall.

Recall that, from Theorem 5.3, if F, K are PGL_χ-equivariant objects in X satisfying the correct weights conditions in the strictly semistable locus on the wall, we will have

$$\mathrm{Hom}_{D^b(M_{i-1})}(F, K) = \mathrm{Hom}_{D^b(M_i)}(F, K).$$

In our case, there is only one Kempf–Ness stratum, where the strictly semistable locus on the wall consists essentially of vector bundles that split as $L \oplus L'$, together with a section $\phi \in H^0(L)$. These points are fixed by a diagonal one-parameter subgroup of $G = PGL_\chi$, where $\chi = h^0(L) + h^0(L')$. One can compute the widths η of a given window, as well as the weights of the needed vector bundles, using the following description.

Theorem 6.4 ([25, §3])**.** *The wall-crossing between two consecutive chambers M_{i-1} and M_i is given by a flip*



where $\mathbb{P}W_i^\pm$ are projective bundles over $\mathrm{Sym}^i C$.

Using windows and wall-crossing, one can prove the following.

Proposition 6.5 ([24, Proposition 3.18])**.** *If $d \leq 2g-1$, for each i there is a fully faithful embedding $D^b(M_{i-1}) \hookrightarrow D^b(M_i)$ and a semi-orthogonal decomposition of the form*

$$D^b(M_i) = \langle D^b(M_{i-1}), D^b(\mathrm{Sym}^i C), \ldots, D^b(\mathrm{Sym}^i C)\rangle$$

with $d+g-3i-1$ *copies of* $D^b(\mathrm{Sym}^i C)$.

6.3. The fully faithful functors. As in Definition 4.2, consider the bundles $\mathcal{F}^{\boxtimes\alpha}$ and $\overline{\mathcal{F}}^{\boxtimes\alpha}$ on $\mathrm{Sym}^\alpha C \times M_i$ and the corresponding Fourier–Mukai transforms $\Phi_\alpha^{\mathcal{F}}$ and $\overline{\Phi}_\alpha^{\mathcal{F}}$ from $D^b(\mathrm{Sym}^\alpha C)$ into $D^b(M_i)$.

Theorem 6.6 ([24])**.** *If* $\deg\Lambda \le 2g-1$*, then the functors*

$$\Phi_\alpha^{\mathcal{F}},\ \overline{\Phi}_\alpha^{\mathcal{F}} : D^b(\mathrm{Sym}^\alpha C) \to D^b(M_i)$$

are fully faithful for $i \ge \alpha$.

As mentioned in §4.1, the proof of full faithfulness requires computations of $H^p\left(M_i, (\mathcal{F}^{\boxtimes\alpha})_D^\vee \otimes (\mathcal{F}^{\boxtimes\alpha})_{D'}\right)$ and $H^p\left(M_i, (\overline{\mathcal{F}}^{\boxtimes\alpha})_D^\vee \otimes (\overline{\mathcal{F}}^{\boxtimes\alpha})_{D'}\right)$, where we should obtain the right hand side of (4.1). To this end, we use the fact that the restrictions of $\mathcal{F}^{\boxtimes\alpha}$ and $\overline{\mathcal{F}}^{\boxtimes\alpha}$ to a divisor $D = x_1 + \ldots + x_\alpha$ are deformations of $\bigotimes_{j=1}^{\alpha} \mathcal{F}_{x_j}$, so that

$$h^p\left((\mathcal{F}^{\boxtimes\alpha})_D\right),\ h^p\left((\overline{\mathcal{F}}^{\boxtimes\alpha})_D\right) \le h^p\left(\bigotimes_{j=1}^{\alpha} \mathcal{F}_{x_j}\right)$$

by semicontinuity, and then apply $R\Gamma$ to Koszul resolutions of the form (6.2), twisted by suitable bundles. This effectively allows us to reduce computations in $M_i(\Lambda)$ to computations in $M_{i-1}(\Lambda(-2x))$. After induction arguments, computations of cohomology groups of several vector bundles on M_α, and then windows into M_i, $i \ge \alpha$, Theorem 6.6 is proved.

6.4. The semi-orthogonal blocks. Without loss of generality, we can fix $d = \deg\Lambda = 2g-1$. To this point, we have constructed a collection of admissible subcategories in $D^b(M_v)$, namely the essential images of $\Phi_\alpha^{\mathcal{F}}$ and $\overline{\Phi}_\alpha^{\mathcal{F}}$, for $\alpha \le v = g-1$. Each of these is equivalent to $D^b(\mathrm{Sym}^\alpha C)$. Furthermore, we can use the same techniques to prove that these admissible subcategories, after twisting by suitable bundles, can be arranged in a way that they are semi-orthogonal to each other.

Recall the diagram (6.1). When $d = 2g-1$ the last map $\pi : M_v \to M_C(2,\Lambda)$ is a birational morphism and π^* is fully faithful. Using the fact that $\pi^*\mathcal{E}$ is equal to $\mathcal{F}$ twisted by a line bundle, and looking at the condition (4.1), one can see that full faithfulness of Φ_α follows from that of $\Phi_\alpha^{\mathcal{F}}$, and full faithfulness of $\overline{\Phi}_\alpha$ follows from that of $\overline{\Phi}_\alpha^{\mathcal{F}}$.

Finally, and for similar reasons, semi-orthogonality of these blocks in $D^b(M_v)$ implies semi-orthogonality of the corresponding blocks in $D^b(M_C(2,\Lambda))$, after twisting by adequate line bundles. The end result is a list of semi-orthogonal blocks of the form

$$\begin{array}{ccccc} \theta^\vee \otimes D^b(\mathrm{pt}), & (\theta^\vee)^2 \otimes D^b(\mathrm{Sym}^2 C), & (\theta^\vee)^3 \otimes D^b(\mathrm{Sym}^4 C), & \dots, \\ \dots, & (\theta^\vee)^3 \otimes D^b(\mathrm{Sym}^5 C), & (\theta^\vee)^2 \otimes D^b(\mathrm{Sym}^3 C), & \theta^\vee \otimes D^b(C), \\ D^b(\mathrm{pt}), & \theta^\vee \otimes D^b(\mathrm{Sym}^2 C), & (\theta^\vee)^2 \otimes D^b(\mathrm{Sym}^4 C), & \dots, \\ \dots & (\theta^\vee)^2 \otimes D^b(\mathrm{Sym}^5 C), & \theta^\vee \otimes D^b(\mathrm{Sym}^3 C), & D^b(C) \end{array}$$

where each $D^b(\mathrm{Sym}^\alpha C)$ is the image of Φ_α, if α is even, or the image of $\overline{\Phi}_\alpha$, if α is odd. For $\alpha \leq g-2$, the category $D^b(\mathrm{Sym}^\alpha C)$ appears twice, while $D^b(\mathrm{Sym}^{g-1} C)$ appears only once.

A priori, there could be an orthogonal complement $\mathcal{P}$ to the admissible subcategory generated by all these blocks. The fact that $\mathcal{P} = 0$ was proved in [23] using completely different techniques.

7. Parabolic bundles for $r > 2$. We will briefly sketch the key ingredients for the proof of Theorem 3.2, which concerns the case $r > 2$. For this case, we consider the following spaces.

Definition 7.1. *Fix two points $q_1, q_2 \in C$. Given two weights $a_1, a_2 \in [0,1]$, we denote by $M_{(C;q_1,q_2)}(r,\Lambda;a_1,a_2)$ the space of parabolic bundles on $(C;q_1, q_2)$ with a flag of rank $r-1$ on q_1 and a flag of rank 1 on q_2, and weights a_1, a_2, respectively.*

For a fixed pair of points (q_1,q_2), the spaces $M_{(C;q_1,q_2)}(r,\Lambda;a_1,a_2)$ are obtained as GIT quotients, related to each other by variation of GIT as the choice of the stability parameter (a_1,a_2) varies.

When the weights are small, say $a_1 = a_2 = \epsilon$, we have

$$M_{(C;q_1,q_2)}(r,\Lambda;\epsilon,\epsilon) \cong \mathbb{P}(\mathcal{E}_{q_1}^\vee) \times_{M_C(r,\Lambda)} \mathbb{P}(\mathcal{E}_{q_2}). \tag{7.1}$$

The key idea to Theorem 3.2 is then to use (7.1) to rewrite the cohomology of certain vector bundles on $M_C(r,\Lambda)$ as cohomology of vector bundles on $M_{(C;q_1,q_2)}(r,\Lambda;\epsilon,\epsilon)$. The latter cohomology groups are thus obtained by doing computations in some other space $M_{(C;q_1,q_2)}(r,\Lambda;a_1,a_2)$ and then using windows.

For example, to prove that $\Phi_{\mathcal{E}} : D^b(C) \to D^b(M_C(r,\Lambda))$ is fully faithful, one needs to find $H^p(M_C(r,\Lambda), \mathcal{E}_{q_1} \otimes \mathcal{E}_{q_2}^\vee)$. But

$$H^p(M_C(r,\Lambda), \mathcal{E}_{q_1} \otimes \mathcal{E}_{q_2}^\vee) = H^p\left(M_{(C;q_1,q_2)}(r,\Lambda;\epsilon,\epsilon), \mathcal{O}(1,1)\right) \tag{7.2}$$

using (7.1). A windows argument will tell us that the right hand side of (7.2) is equal to the cohomology of a corresponding bundle in a different GIT chamber $M_{(C;q_1,q_2)}(r,\Lambda;a_1,a_2)$, where classical vanishing techniques may be available. For details, see [14].

8. More on parabolic bundles. We finish by mentioning an open problem about the derived category of another moduli space of parabolic bundles, which can also be seen as the moduli space of bundles on a stacky curve.

Choose $p_1,\dots,p_{2g+1}\in\mathbb{P}^1$ and let $\mathcal{M}=M_{(\mathbb{P}^1;p_1,\dots,p_{2g+1})}(2,0;1/2,\dots,1/2)$ denote the moduli space of parabolic vector bundles of rank two on $(\mathbb{P}^1;p_1,\dots,p_{2g+1})$ of degree zero, with weights $1/2$ at each marked point. Let $\mathcal{C}$ denote a stacky $\mathbb{P}^1$ with $\mathbb{Z}/2\mathbb{Z}$ structure at each p_i. The space $\mathcal{M}$ is also equal to the moduli space of rank-two vector bundles on the stacky curve $\mathcal{C}$, and can be obtained as the Fano scheme of $(g-2)$-dimensional linear spaces inside the intersection of two quadrics in $\mathbb{P}^{2g}$. Fonarev has conjectured the following.

Conjecture 8.1 ([9])**.** *The derived category of $\mathcal{M}$ has a semi-orthogonal decomposition of the form*

$$\langle D^b(\mathrm{pt}),\ D^b(\mathcal{C}),\ D^b(\mathrm{Sym}^2\mathcal{C}),\dots,D^b(\mathrm{Sym}^{g-1}\mathcal{C})\rangle$$

where $\mathrm{Sym}^\alpha\mathcal{C}$ denotes the root stack obtained from $\mathbb{P}^k$ by extracting the square root from $2g+1$ hyperplanes in general position.

Fonarev has provided evidence towards this conjecture in [9], by computing the Grothendieck groups on both sides. More evidence backing this conjecture, as well as other conjectures regarding the derived categories of other Fano schemes on the intersection of two quadrics, can be found in [4].

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