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LENS SPACES AS DUAL COMPLEXES OF LOG CALABI-YAU PAIRS

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Abstract. We demonstrate the construction of singular log Calabi-Yau 4-folds such that the dual complex of the boundary is homeomorphic to a Lens space from a log Calabi-Yau surface with action of a finite cyclic group. We explicitly obtain the Lens spaces $L(3,1)$, $L(5,1)$, and $L(5,2)$ in this way.

1. Introduction. Given a normal crossing pair (X, Δ), we may assign the dual intersection complex $\mathcal{D}(\Delta)$, whose points correspond to components of Δ and whose higher dimensional cells correspond to connected components of the intersections of divisors of Δ. These complexes furnish a combinatorial invariant which encodes information relevant to many constructions in algebraic geometry giving connections to Hodge theory, birational geometry, non-archimedean geometry, and the mirror symmetry program.

The question of exactly what information about (X, Δ) can be reconstructed from $\mathcal{D}(\Delta)$ is an area of active research. In the context of birational

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geometry, it is known that for log crepant maps, the PL homeomorphism type of the dual complex is invariant [3]. In Hodge theory, the $\mathbb{Q}$-cohomology groups of $\mathcal{D}(\Delta)$ can be recovered from the weight filtration on the cohomology of X induced by (X, Δ) [9].

We will say that a pair (X, Δ) is log Calabi-Yau if X is proper, $K_X + \Delta$ is numerically trival and (X, Δ) is log canonical. We associate to (X, Δ) log canonical the PL homeomorphism type of the dual complex of coefficient 1 divisors of any dlt minimal model. See section 2 for details.

Under mild conditions, it is expected that if (X, Δ) is log Calabi-Yau of dimension d, then the dual complex is homeomorphic to a quotient of a sphere S^{d-1}. This is known to be true for dimension $d \leq 4$ [7]. The goal of this note is to furnish constructions of examples of log Calabi-Yau varieties of dimension 4 such that the dual complex is a Lens space $L(3, 1)$, $L(5, 1)$, or $L(5, 2)$.

More generally, suppose we have a log Calabi-Yau surface (X, Δ) with dual complex S^1, along with an action of $\mathbb{Z}/n\mathbb{Z}$ which is transitive on the dual complex generated by ψ. Choose k relatively prime to n. We construct (Y, Γ) as the quotient of $(X \times X, \pi_1^*\Delta + \pi_2^*\Delta)$ under (ψ, ψ^k). The pair (Y, Γ) is a singular log Calabi-Yau variety. We show in Theorem 3.2 that the dual complex of any dlt minimal model is homeomorphic to the Lens space $L(n, k)$. It is straightforward to find examples of log Calabi-Yau surfaces with appropriate action of $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$.

However there is no representation of the group $\mathbb{Z}/p\mathbb{Z}$ as a subgroup of the automorphisms of a rational surface for a prime $p \geq 7$ [10]. This limits the ability of the construction to produce more complicated Lens spaces, but it is conceivable there is some other construction of log Calabi-Yau 4-folds. I pose in Conjecture 4.2 that this is not possible for even $p = 7$.

2. Preliminaries. We give a short background on dual complexes in the context of birational geometry following [3, 5]. A regular cell complex of dimension m is defined inductively as follows. A dimension 0 regular cell complex is a disjoint union of vertices. Given a dimension $j-1$ cell complex C_{j-1}, we may obtain a dimension j one by attaching a collection of simplices of dimension j, such that the boundary of each simplex embeds into C_{j-1}.

For a dlt pair (X, Δ) we write $\Delta^{=1}$ to be the coefficient 1 divisors of Δ. As (X, Δ) is dlt, each component of $\Delta^{=1}$ is normal, and for any j components of the $\Delta^{=1}$, the intersection is either empty or is a disjoint union of irreducible varieties of codimension j in X. Call such a component of such an intersection a stratum v of $\Delta^{=1}$. Then the dual complex $\mathcal{D}(\Delta^{=1})$ is a regular cell complex whose vertices correspond to the components of the $\Delta^{=1}$, and whose $j-1$ cells

correspond to the codimension j strata.

To a stratum v we associate the cell σ_v. If v is of codimension j, then there are exactly j divisors containing v, and every stratum v' properly containing v can be recovered uniquely from v and the subset of the $\Delta^{=1}$ containing v'. This furnishes the attaching map identifying the boundary of σ_v with the union of the σ'_v.

These complexes behave well with respect to crepant birational maps: Given a map $f\colon X' \to X$, if (X', Δ') and (X, Δ) are dlt pairs, and $f^*(K_X + \Delta) \sim K_{X'} + \Delta'$ then $\mathcal{D}(\Delta^{=1})$ and $\mathcal{D}(\Delta'^{=1})$ are canonically PL-homeomorphic [3, Proposition 11].

If (Y, Γ) is not dlt, we can assign to $\mathcal{D}(Y, \Gamma)$ the PL-homeomorphism type of the dual complex of the log canonical centers of any dlt minimal model. This is especially of interest when (Y, Γ) is a log canonical pair. See [2] for discussion of the problem of direct computation of the dual complex of a 3-fold log canonical pairs.

Let (X, Δ) be dlt with $\Delta = \Delta^{=1}$, and suppose G is a finite group with action on X preserving the divisor Δ. We have an induced action of G on the dual complex $\mathcal{D}(\Delta)$, and we assume that this action is free, that is, every nontrivial element of G acts with no fixed points on $\mathcal{D}(\Delta)$. To understand the quotient it is convenient to replace $\mathcal{D}(\Delta)$ with its barycentric subdivision. Such an operation corresponds to blowing up the strata of Δ in sequence by dimension, see [3, Remark 10]. Call the resulting pair (X', Δ'). The action of G extends to an action on (X', Δ'), and therefore an action on the complex $\mathcal{D}(\Delta')$, which is the barycentric subdivision of $\mathcal{D}(\Delta)$. This complex $\mathcal{D}(\Delta')$ is a simplicial complex, enjoying the property that the intersection of any two cells is a face of both.

Let (Y', Γ') be the quotient of (X', Δ') under the action of G.

Proposition 2.1. *The pair (Y', Γ') is dlt.*

P r o o f. Any lc center of (Y', Γ') is the image of an lc center of (X', Δ'), which means it is a stratum, call it v. The condition that G act freely on the dual complex implies that analytically locally around the generic point of v, the image of v is smooth and is cut out by the images of the divisors in Δ' cutting out v. Each of these divisors has an image in Δ of a different codimension by the barycentric subdivision construction. Hence their images are distinct in Y'. Thus the image of v is a smooth complete intersection of the distinct components of Γ' containing it. So (Y', Γ') is dlt. □

Proposition 2.2. *The complex $\mathcal{D}(\Gamma')$ is identified with the quotient $\mathcal{D}(\Delta')/G$.*

P r o o f. Let $\phi\colon X' \to Y'$ be the quotient map. We construct the map f exhibiting $\mathcal{D}(\Gamma')$ as $\mathcal{D}(\Delta')/G$. For any stratum w of Δ', f maps σ_w isomorphically to $\sigma_(\phi(w))$, by identifying each stratum containing w with its image under ϕ. This assignment is compatible with the face maps, and ever stratum of Γ' is the orbit of a stratum of Δ', so f gives a surjective map from $\mathcal{D}(\Delta') \to \mathcal{D}(\Gamma')$. Finally, note that two points of $\mathcal{D}(\Delta')$ have the same image under f if and only if they are related by an element of G, hence $\mathcal{D}(\Gamma')$ is identified with $\mathcal{D}(\Delta')/G$. □

We give a brief review of the construction of Lens spaces [1]: Consider the sphere S^3 as the locus in $\mathbb{C}^2$ cut out by $|z|^2 + |w|^2 = 1$. Consider positive integers k, n with k relatively prime to n. Then the action of $\mathbb{Z}/n\mathbb{Z}$ on S^3 induced by the generator $(z, w) \to (e^{2\pi \mathrm{i}/n} z, e^{2\pi k \mathrm{i}/n} w)$ is fixed point free. The topological quotient is an orientable 3-manifold called a Lens space, denoted $L(n, k)$. The Lens spaces $L(5, 1)$ and $L(5, 2)$ are a well known example of two manifolds of the same dimension with isomorphic fundamental and homology groups but not homotopy equivalent [1, VI, Theorem 10.15].

3. Main construction. Define M_n as a simplicial complex on points $\Delta_{1,1}, \dots \Delta_{1,n}, \Delta_{2,1} \dots \Delta_{2,n}$. A simplex σ belongs to M_n if and only if for all $i, j \in \mathbb{Z}/n\mathbb{Z}$ whenever $\Delta_{1,i}$ and $\Delta_{1,j}$ are in σ or $\Delta_{2,i}$, and $\Delta_{2,j}$ are in σ, then i, j are consecutive in the group $\mathbb{Z}/n\mathbb{Z}$, that is either $i = j + 1$ or $j = i + 1$.

Let (X, Δ) be a surface pair such that the components of Δ form an n cycle. Then the product $(X \times X, \pi_1^*(\Delta) + \pi_2^*(\Delta))$ has dual complex identified with M_n.

Now we identify the topological realization $|M_n|$ with a subset L_n of Euclidean space $\mathbb{R}^{2n}$, with coordinates $Y_{1,1}, \dots Y_{1,n}, Y_{2,1}, \dots Y_{2,n}$. Let S_n be the simplex defined by $Y_{\mu,i} \geq 0$ and $\sum Y_{\mu,i} = 1$. Each point $\Delta^{\mu,i}$ of the simplicial complex has a corresponding coordinate $Y_{\mu,i}$. For a simplex σ, we identify $|\sigma|$ with the locus of points in S_n satisfying $Y_{\mu,i} = 0$ if $\Delta_{\mu,i} \notin \sigma$. This identification is compatible with the face relations so we have identified $|M_n|$ with a subset of S_n, which we call L_n.

Our next goal is to define a homeomorphism between L_n and the three sphere in $\mathbb{C}^2$ defined by $|z|^2 + |w|^2 = 1$. For x_1, x_2 satisfying $x_1 + x_2 \leq 1, x_1 \geq 0, x_2, \geq 0$, we define

$$f_n(x_1, x_2) = \sin(\pi/2(x_1 + x_2)) e^{(2\pi \mathrm{i}/n)(x_2/(x_1+x_2))},$$

if at least one of the $x_i \neq 0$, and define $f(0, 0) = 0$. We see that f_n is a continuous function with complex codomain. Now, for a point P of L_n, there exist i, j such that $Y_{1,i} + Y_{1,i+1} + Y_{2,j} + Y_{2,j+1} = 1$ and so all other coordinates must be zero. Define

$$F_n^{i,j}(P) = (e^{2\pi \mathrm{i}\cdot(i/n)} f_n(Y_{1,i}, Y_{1,i+1}), e^{2\pi \mathrm{i}\cdot(j/n)} f_n(Y_{2,j}, Y_{2,j+1}))$$

Proposition 3.1. *The $F_n^{i,j}$ extend to a homeomorphism $F_n \colon L_n \to S^3$.*

P r o o f. First, we note that for any point P of L_n, the function F_n is well defined. The only possible ambiguity in the first coordinate z of F_n is if exactly one of the $Y_{1,i}$ are nonzero. But in this case whichever option we use we obtain $z = \sin(\pi/2(Y_{1,i}))e^{2\pi \mathrm{i}\cdot(i/n)}$.

Next, we note that the output (z, w) of F_n always satisfies $|z_1|^2 + |z_2|^2 = 1$. So we may restrict the codomain of F_n to S^3.

To show that F_n is a homeomorphism, we make use of the well known fact that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism. As f_n was continuous, so is F_n, so we need only show that F_n is a bijection. We do so by describing how to construct the unique preimage of any point on the unit sphere $|z_1|^2 + |z_2|^2 = 1$.

If $z_\mu \neq 0$, then $\dfrac{z_\mu}{|z_\mu|}$ is a point on the unit circle. If it is the root of unity $e^{(2\pi \mathrm{i})(j/n)}$, then $Y_{\mu,j}$ is nonzero but all the other $Y_{\mu,j'}$ are 0. If the phase of $\dfrac{z_\mu}{|z_\mu|}$ lies strictly between the roots of unity numbered j and $j+1$, then $Y_{\mu,j}$ and $Y_{\mu,j+1}$ are nonzero and their ratio can be uniquely recovered from the phase.

If the other coordinate $z_\lambda = 0$, then all $Y_{\lambda,j} = 0$ and we have uniquely recovered a point of L_n. Both z_1 and z_2 cannot be 0.

If z_1 and z_2 are nonzero, we have seen that we can recover uniquely from the phases the ratios of $Y_{1,i}$ to $Y_{1,i+1}$ and that of $Y_{2,j}, Y_{2,j+1}$. The quantity $Y_{1,i} + Y_{1,i+1}$ is recovered uniquely from the magnitude of z_1, and this is enough information to determine the rest. □

Now, let Let (X, Δ) be an snc log Calabi-Yau surface, such that the dual complex of Δ is an n-cycle. Let ψ be an order n automorphism of X which cyclically permutes the components of Δ. We assume also that ψ fixes no divisor of Δ. Let $X \times X$ be the product, and let Δ_1, Δ_2 be the two pullbacks of Δ. Choose $1 \leq k < n$ relatively prime to n, and let Ψ_k be the automorphism given by ψ on the first factor and ψ^k on the second.

Define (Y, Γ) to be the log canonical pair given by the quotient of $(X \times X, \Delta_1 + \Delta_2)$ by $\mathbb{Z}/n\mathbb{Z}$ generated by Ψ_k. As Ψ_k fixes no divisor of $X \times X$, $K_Y + \Gamma$ is numerically trivial.

Theorem 3.2. *The dual complex of a dlt minimal model of (Y, Γ) is homeomorphic to the Lens space $L(n, k)$.*

P r o o f. We identify the dual complex of $(X \times X, \Delta_1 + \Delta_2)$ with L_n, and the action of Ψ_k sends $Y_{1,i}$ to $Y_{1,i+1}$, and $Y_{2,j}$ to $Y_{2,j+k}$. We apply F_n to identify this action with an action on S^3. Let $(z_1, z_2) = F_n(P)$. Then

$$F_n(\Psi_k(P)) = (e^{2\pi \mathrm{i}/n} z_1, e^{2\pi k \mathrm{i}/n} z_2).$$

The quotient of S^3 under this action is the Lens space $L(n, k)$.

Let (Z, Δ') be the iterated blow up of $(X \times X, \Delta_1 + \Delta_2)$, starting with the 0-dimensional strata and at the ith stage blowing up along the strict transform of the dimension i strata of $\Delta_1 + \Delta_2$. We identify $\mathcal{D}(\Delta')$ with the barycentric subdivision of $\mathcal{D}(\Delta_1 + \Delta_2)$. Then by Proposition 2.1, the quotient (Y', Γ') of (Z, Δ') is dlt, and by Proposition 2.2, $\mathcal{D}(\Gamma')$ is identified with $L(n, k)$. The pair (Y, Γ') is a dlt minimal model of (Y, Γ). □

We can construct the Lens space $L(3, 1)$ by taking X to be $\mathbb{P}^2$, D the sum of the three coordinate lines and letting $\mathbb{Z}/3\mathbb{Z}$ act by cyclic action on the coordinates.

For $L(5, 1)$, and $L(5, 2)$, we consider $X = \mathbb{P}^2$ blown up at 4 points. Take L for the class of a pullback of a line, and Δ_i for the exceptional divisors.

Then the following 5 classes each admit a -1 curve, and together form an anticanonical cycle, such that the dual complex is the standard cycle:

(1) $\Delta_1 = L - E_2 - E_4$
(2) $\Delta_2 = L - E_1 - E_3$
(3) $\Delta_3 = E_3$
(4) $\Delta_4 = L - E_2 - E_3$
(5) $\Delta_5 = E_2$

In fact there is an action of $\mathbb{Z}/5\mathbb{Z}$ on the pair (X, Δ) which is transitive on the Δ_i. This comes from the realization of X as the moduli space $\overline{\mathcal{M}_{0,5}}$ [6]. This moduli space admits the action of $\mathbb{Z}/5\mathbb{Z}$ via the cyclic permutation of the 5 marked points on $\mathbb{P}^1$.

Each -1 curve in X corresponds to a locus $A_{i,j}$ corresponding to stable rational curves with the marked points p_i and p_j on the same component with exactly one other special point, corresponding to where that component intersects the next component of the stable curve. The induced action on the $A_{i,j}$ apply the cyclic permutations to the indices $1, \ldots 5$. To show the action is transitive on $\mathcal{D}(\Delta)$, we must identify the Δ_i with the $A_{i,j}$.

Proposition 3.3. *The locus Δ_i corresponds to the locus $A_{i,i+2}$*

P r o o f. We interpret the map to $\mathbb{P}^1$ induced by the pencil of conics through the 4 points as the total family $\overline{\mathcal{M}_{0,5}} \to \overline{\mathcal{M}_{0,4}}$ [6]. The four sections

represent the four marked points, and the point of X itself is the location of the fifth point. When this point is on one of the E_i, we must create a new component of the fiber containing both p_i and p_5. Thus $E_i = A_{i,5}$. On the other hand, for the other components each $L - E_i - E_j$ corresponds to the $A_{k,l}$ where k and l are the remaining indices from the first four in order.

We can then confirm our list of Δ_i correspond exactly the the $A_{i,i+2}$. □

As a result applying Theorem 3.2 we may produce a cyclic action on $(X \times X, \pi_1^*(\Delta) + \pi_2^*(\Delta))$ such that the resulting log canonical log Calabi-Yau pair has dual complex which is a Lens space, either $L(5,1)$ or $L(5,2)$ depending on how we construct the action.

4. Comments. Our examples of log Calabi-Yau 4-fold pairs with dual complex $L(p,q)$ require a surface log Calabi-Yau pair with action of $\mathbb{Z}/p\mathbb{Z}$. The condition of having a rational surface with an action of a finite cyclic group is quite restrictive. If we restrict to groups of prime order such actions exist only for $p \leq 5$ [10].

In higher dimensions not every rationally connected variety is rational, so we cannot even use the study of the Cremona group to rule out examples. The following result was in a previous version of this paper stated as a conjecture, but Mirko Mauri has pointed out this follows from work of Moraga.

Proposition 4.1. *For sufficiently large prime p, there is no dlt log-Calabi-Yau 4-fold pair whose dual complex is homeomorphic to $L(p,k)$ for any k.*

P r o o f. Follows from Corollary 1 of [8]. □

Instead we make the conjecture for $p = 7$:

Conjecture 4.2. *There is no dlt log-Calabi-Yau 4-fold pair whose dual complex is homeomorphic to $L(7,k)$ for any k.*

Our examples always admit a dlt minimal model (Y', Γ') where every component of Γ' has coefficient 1. As Lens spaces are orientable, by [4, Proposition 5.5] we must have $K_{Y'} + \Gamma' \sim 0$. On the other hand the quotient construction implies Y' admits a torsion divisor whose algebra of sections induces the cover X', likewise for Y.

Any example of a dlt 4-fold pair (Y', Γ') with $\mathcal{D}(\Gamma')$ homeomorphic to $L(p,q)$ must admit a quasi-étale cover $(\tilde{Y}', \tilde{\Gamma}')$, corresponding to the universal cover of $L(p,q)$ by the sphere S^3, by [7, Theorem 2.5].

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