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A NOTE ON UPPER NILRADICALS AND ONE-SIDED NIL IDEALS

Santosh Kumar Pandey

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Abstract. It is not known whether or not every non-zero one-sided nil ideal of a noncommutative ring is contained in the upper nilradical of the ring. Here we give necessary and sufficient conditions under which a non-zero one-sided nil ideal of a noncommutative ring is not contained in the upper nilradical of the ring.

1. Introduction. It remains mysterious since very long to understand whether or not every non-zero one-sided nil ideal of a ring is contained in the upper nilradical of the ring. Further it may be noted that if R be a noncommutative ring and each one-sided nil ideal of R is contained in the upper nilradical $UNR(R)$ of the ring, then R satisfies the very old and most famous Koethe conjecture which is open since 1930 [1, 2, 3, 4, 5].

The Koethe conjecture has not been settled till date. As per the existing literature it can be stated in various equivalent ways including the following

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statement [1, 2, 3, 4]. Every one-sided nil ideal of a ring R is contained in a two-sided nil ideal of R.

Though it seems that it is a simple problem of ring theory however it has a very complicated history. It is now known that certain classes of rings like right Notherian rings, polynomial identity rings satisfy the Koethe conjecture but a general answer about the validity or invalidity of the conjecture is still awaited.

It may be noted that an ideal of a ring is called nil if every element of the ideal is nilpotent and an upper nil radical of a ring is defined as the sum of all nil ideals of the ring [5]. Clearly each nil ideal of a ring is contained in the upper nil radical of the ring.

In this note we discuss under what conditions a non-zero one-sided nil ideal I of R is not contained in $UNR(R)$. We obtain necessary as well as sufficient conditions under which a non-zero one-sided nil ideal I of R is not contained in $UNR(R)$.

We give our results in the next section.

2. Findings. We present our findings related to the connection between upper nilradicals and one-sided nil ideals of noncommutative rings in the form of the following important theorems.

Theorem 2.1. *Let R be a noncommutative ring with identity and I be a non-zero left nil ideal of R. If I is not contained in the upper nilradical $UNR(R)$ of R, then $u+v$ is not nilpotent for some nilpotent elements u, $v \in R$.*

Proof. If the sum of any two nilpotent elements of a ring R is nilpotent, then R satisfies the Koethe conjecture (that is each non-zero one-sided nil ideal of R is contained in the upper nilradical $UNR(R)$ of R) [3, 4]. Therefore, if I is not contained in the upper nilradical $UNR(R)$ of R, then $u+v$ is not nilpotent for some nilpotent elements $u, v \in R$. This completes the proof. □

It should be emphasized that Theorem 2.1 gives a necessary condition for a noncommutative ring in which a non-zero one-sided nil ideal is not contained in the upper nil radical of the ring.

Theorem 2.1 leads to the following important Theorem which gives necessary as well as sufficient conditions for a noncommutative ring R in which a non-zero one-sided nil ideal I is not contained in the upper nil radical $UNR(R)$ of the ring.

Theorem 2.2. *Let R be a noncommutative ring with identity and I be a non-zero left nil ideal of R. Then I is not contained in the upper nilradical $UNR(R)$ of R iff at least one of the following holds.*

(i) *$x + ab$ is not nilpotent in R for some $x, a \in I$ and $b \in R$.*

(ii) *$ab + cd$ is not nilpotent in R for some $a, c \in I$ and $b, d \in R$.*

Proof. Let R be a noncommutative ring with identity and I be a nonzero left (one-sided) nil ideal of R. Let $UNR(R)$ be the set of the upper nilradial of R. Firstly, we shall prove that if I is not contained in the upper nilradical $UNR(R)$ of R then at least one of the following is not nilpotent. (i) $x + ab$ is not nilpotent in R for some $x, a \in I$ and $b \in R$ (ii) $ab + cd$ is not nilpotent in R for some $a, c \in I$ and $b, d \in R$.

In order to prove this we proceed as follows. Evidently for each $a \in I$ and each $b \in R$ we have $ba \in I$. But for some $a \in I$ and $b \in R$, $ab \notin I$ (since I is one-sided ideal). Now it may be noted that since $ba \in I$ for each $a \in I$ and each $b \in R$ and each element of I is nilpotent, therefore ab is nilpotent. Also there will always exist some such ab because I is a left nil ideal (one-sided nil ideal).

It may be emphasized that if I is contained in $UNR(R)$, then $x + y$ is nilpotent for each $x \in I$ and each $y \in UNR(R)$. Similarly, $y + ab$ is nilpotent for each $y \in UNR(R)$ and each $a \in I$ and each $b \in R$, $x + ab$ is nilpotent for each x, $a \in I$ and $b \in R$ and $ab + cd$ is nilpotent in R for each a, $c \in I$ and b, $d \in R$.

However, if I is not contained in $UNR(R)$ even then $x + y$ is nilpotent for each $x \in I$ and each $y \in UNR(R)$ and $y + ab$ is nilpotent for each $y \in UNR(R)$ and each $a \in I$ and each $b \in R$ because the sum of an one-sided nil ideal and a two-sided nil ideal is always a nil one-sided ideal (we refer [5, page 49, Exercise 2.3 (c)]). These findings together imply that if I is not contained in $UNR(R)$ then at least one of the following holds. (i) $x + ab$ is not nilpotent in R for some x, $a \in I$ and $b \in R$ (ii) $ab + cd$ is not nilpotent in R for some a, $c \in I$ and b, $d \in R$.

Conversely, let at least one of the following holds. (i) $x+ab$ is not nilpotent in R for some x, $a \in I$ and $b \in R$ (ii) $ab + cd$ is not nilpotent in R for some a, $c \in I$ and b, $d \in R$. Then it easily follows that I is not contained in the upper nilradical $UNR(R)$ of R.

Hence the proof is complete. □

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REFERENCES

[1] O. D. Artemovych. Associative rings in which nilpotents form an ideal. *Studia Sci. Math. Hungar* **56**, *2* (2019), 177–184.

[2] T. Y. Lam. Exercises in Classical Ring Theory. Probl. Books in Math. New York, Springer-Verlag, 2003

[3] S. K. Pandey. A note on the Koethe conjecture. *Aligarh Bull. Math.* **41**, *2* (2022), 29–33 and arXiv.2202.07454 [math.RA], https://doi.org/10.48550/arXiv.2202.07454.

[4] J. Šter. Rings in which nilpotents form a subring. *Carpathian J. Math.* **32**, *2* (2016), 251–258.

[5] B. Matej. Introduction to noncommutative algebra. Universitext. Cham, Springer, 2014

Faculty of Science, Technology and Forensic
Sardar Patel University of Police, Security and Criminal Justice
Jodhpur, India
e-mail: skpandey12@gmail.com

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