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p -HYPERBOLIC ZOLOTAREV FUNCTIONS IN BOUNDARY VALUE PROBLEMS FOR A p TH ORDER DIFFERENTIAL OPERATOR

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ABSTRACT. For the self-adjoint operator of the p th derivative, a system of fundamental solutions is constructed. This system is analogous to the classical system of sines and cosines. The properties of such functions are studied. Classes of self-adjoint boundary conditions are described. For the operator of the third derivative, the resolvent is calculated and an orthonormal basis of eigenfunctions is given.

1. Introduction. The classical harmonic Fourier analysis is largely formed and developed by the spectral theory of the second-order differential operator ([4, 6, 9, 10]). Mutually enriching connections of these two directions contributed to the development of elaborate mathematical apparatus. Fourier analysis serves as an important research tool in these (and not only these) areas of analysis. For differential operators of higher orders, this technique has not been developed in its entirety, only for some particular cases [8].

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Standard trigonometric functions are eigenfunctions of the simplest second-order differential operator. As far as we know, analogues of trigonometric and hyperbolic functions for operators of higher orders have not been considered before.

In this article, for the p th order operator, a system of p independent functions is proposed, which coincides with the system of sine and cosine for the case $p = 2$. In addition, classes of self-adjoint boundary conditions are described in terms of such functions. For the operator of the third derivative, the resolvent is calculated and the orthonormal basis of eigenfunctions is presented. Note that the use of these functions for $p = 3$ makes it possible to solve both the inverse spectral problem and the inverse scattering problem for a non-local potential [11, 12].

Importance of studying differential operators of order $p > 2$ is determined, for example, by the fact that the search for $L - A$ pairs for Camassa–Holm and Degasperis–Procesi non-linear equations ([1, 2, 3, 5, 7]) leads to a third-order operator L (cubic string).

In construction of p -hyperbolic (p -trigonometric) functions, the group of p th roots of unity plays an essential role. This group partitions the complex plane into p sectors. The value of any p -function at the points of any sector is completely determined by its values at the points of one of the sectors.

The article is organized as follows.

In Section 1, the classical concept of even (odd) functions is generalized to the case $p > 2$. In Proposition 2.2, the uniqueness of the decomposition of a complex-valued function into p of k_p -even components is proved.

p -hyperbolic functions are introduced in Section 3 as a fundamental system of solutions to the simplest differential equation of p th order. The main relations between p -hyperbolic functions are given in Section 4. As an example, formulas for the case $p = 3$ are given.

For the case $p = 3$, it is proved in section 5 (Proposition 5.1) that the zeros of 3-hyperbolic functions are located on the bisectors of each of the three sectors.

Section 6 gives a general solution to the inhomogeneous Cauchy problem.

Section 7 describes the classes of self-adjoint boundary conditions for the p th order differential operation. Note that in the even case ($p = 2k$) there exist separated boundary conditions, and in the odd case ($p = 2k + 1$) there are no such boundary conditions.

As an example of the use of p -functions, an orthonormal basis is constructed in Section 8 from the eigenvectors of a 3rd order self-adjoint operator.

In Section 9, its resolvent is calculated.

2. *p*th roots of unity. *k_p*-even functions. Let $p \in \mathbb{N}$ and

$$(2.1) \quad \zeta_0 = 1, \zeta_1 = e^{i\frac{2\pi}{p}}, \dots, \zeta_{p-1} = e^{i\frac{2\pi(p-1)}{p}}$$

be all the *p*th roots of unity.

The multiplication of every complex number $z \in \mathbb{C}$ by the number ζ_m is the rotation of the complex plane around the origin through the angle $2\pi m/p$. The set of such rotations forms a group denoted by G_p . The numbers (2.1) form a commutative group with multiplication as a group operation. We will identify this group with the group G_p .

G_p is a cyclic group. Each element of the group G_p is a power of the element ζ_1 . If the number p is prime, the set of powers ζ_m^k , $k = 0, 1, \dots, p-1$ of any element $\zeta_m \neq \zeta_0 = 1$ of the group G_p is a permutation of the elements of the group.

Note that *p*th roots of unity satisfy the relations

$$(2.2) \quad \zeta_0^n + \zeta_1^n + \dots + \zeta_{p-1}^n = 0, \quad \text{for every } n \in \mathbb{Z}, n \neq 0 \pmod{p},$$

$$(2.3) \quad \zeta_j \zeta_m = \zeta_{j+m} \quad \text{where} \quad j+m := (j+m) \pmod{p},$$

$$(2.4) \quad \bar{\zeta}_j = \frac{1}{\zeta_j} = \zeta_{p-j}.$$

Moreover, from (2.2), (2.3) we get

$$(2.5) \quad \zeta_m^0 + \zeta_m^1 + \zeta_m^2 + \dots + \zeta_m^{p-1} = \begin{cases} p, & \text{if } m = 0 \pmod{p}, \\ 0, & \text{if } m \neq 0 \pmod{p}. \end{cases}$$

Let X be some set and Y be a linear space. Suppose G is the group of transformations of the set X , and

$$(2.6) \quad g \mapsto T_g, \quad T(g_1 g_2) = T(g_1) T(g_2), \quad \text{for any } g_1, g_2 \in G$$

is its representation by linear operators $T_g : Y \rightarrow Y$.

The function $f : X \rightarrow Y$ is called *invariant with respect to the representation* (2.6) if

$$T_g^{-1}f(g(x)) = f(x) \quad \text{or} \quad f(g(x)) = T_g f(x) \quad \text{for every } g \in G.$$

Example 2.1. Suppose $X = Y = \mathbb{R}$, and the two-element group $G = \{g_1, g_2\} = \{1, -1\}$ is generated by the mirror reflection of the real axis $\mathbb{R}$ about the origin. Group G has two linear representations in linear space $\mathbb{R}^1$:

- (i) *trivial* $g_1 \mapsto 1, g_2 \mapsto 1$,
- (ii) the *identity* representation $g_1 \mapsto 1, g_2 \mapsto -1$.

Functions on $\mathbb{R}$ that are invariant with respect to the representation (i) are called *even*, and those invariant with respect to the representation (ii) are called *odd*. So the same group can generate several concepts of an invariant function.

Proposition 2.1. Let $p \in \mathbb{N}$ and $\zeta_m = e^{2\pi im/p}$, $m = 0, 1, \dots, p-1$, be p th roots of unity. If $k \in \mathbb{N}$, then the map $\zeta_m \mapsto T_p^k(\zeta_m) : \mathbb{C} \rightarrow \mathbb{C}$, where

$$(2.7) \quad T_p^k(\zeta_m)z = \zeta_m^k z, \quad z \in \mathbb{C},$$

is a representation of the group G_p in the linear space $\mathbb{C}^1$. For $k = 0$, representation (2.7) is trivial: $T_p^0(\zeta_m)z \equiv z$.

Proof. It's clear that $T_p^k(\zeta_m \zeta_n) = T_p^k(\zeta_m)T_p^k(\zeta_n)$. $\square$

A set $\Omega \subseteq \mathbb{C}$ is called G_p -invariant if $z \in \Omega$ implies $\zeta_m z \in \Omega$, for every $\zeta_m \in G_p$ and for every $z \in \Omega$. In particular, the complex plane $\mathbb{C}$ is a G_p -invariant set.

Definition 2.1. A complex-valued function $w = f(z)$ defined on a G_p -invariant set Ω is called k_p -even, $k = 0, 1, \dots, (p-1)$ if it is invariant with respect to the representation T_p^k (2.7) of the group G_p , i. e.,

$$(2.8) \quad f(\zeta_m z) = \zeta_m^k f(z), \quad \text{for } m = 0, 1, \dots, (p-1).$$

Remark 2.1. For $p = 2$, the introduced concept of a k_p -even function coincides with the standard one (see Example 2.1).

We need a lemma.

Lemma 2.1. Let ζ_m ($m = 0, 1, \dots, p-1$) be the p th roots of unity, where

$p \in \mathbb{N}$. Then

$$(2.9) \quad U = \frac{1}{\sqrt{p}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta_1 & \zeta_1^2 & \cdots & \zeta_1^{p-1} \\ 1 & \zeta_2 & \zeta_2^2 & \cdots & \zeta_2^{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_{p-1} & \zeta_{p-1}^2 & \cdots & \zeta_{p-1}^{p-1} \end{pmatrix}$$

is a unitary matrix, i. e., $UU^* = I$.

Proof. We have

$$1 + \zeta_j \bar{\zeta}_j + \cdots + \zeta_j^{p-1} \bar{\zeta}_j^{p-1} = p, \quad \text{for } j = 0, 1, \dots, p-1,$$

Since $\bar{\zeta}_j = \zeta_{p-j}$, where $p-j := (p-j) \pmod{p}$, then from (2.3), (2.4) and (2.5) we obtain

$$\begin{aligned} 1 + \zeta_k \bar{\zeta}_j + \cdots + \zeta_k^{p-1} \bar{\zeta}_j^{p-1} &= 1 + \zeta_k \zeta_{p-j} + \cdots + \zeta_k^{p-1} \zeta_{p-j}^{p-1} = \\ &= 1 + \zeta_{k+p-j} + \cdots + \zeta_{k+p-j}^{p-1} = 0 \quad \text{for } k \neq j. \end{aligned}$$

This implies $UU^* = I$. $\square$

Proposition 2.2. Let $f(z)$ be a complex-valued function defined on a G_p -invariant set $\Omega \subseteq \mathbb{C}$. Then $f(z)$ is uniquely representable as

$$(2.10) \quad f(z) = f_0(z) + f_1(z) + \cdots + f_{p-1}(z),$$

where

$$(2.11) \quad f_k(z) = \frac{1}{p} \sum_{m=0}^{p-1} \frac{1}{\zeta_m^k} f(\zeta_m z)$$

is a k_p -even function, $k = 0, 1, \dots, p-1$.

Proof. We have

$$f_k(\zeta_j z) = \frac{1}{p} \sum_{m=0}^{p-1} \frac{1}{\zeta_m^k} f(\zeta_m \zeta_j z) = \zeta_j^k \frac{1}{p} \sum_{m=0}^{p-1} \frac{1}{\zeta_{m+j}^k} f(\zeta_{m+j} z) = \zeta_j^k f_k(z).$$

Moreover,

$$\sum_{k=0}^{p-1} f_k(z) = \frac{1}{p} \sum_{k=0}^{p-1} \sum_{m=0}^{p-1} \frac{1}{\zeta_m^k} f(\zeta_m z) = \frac{1}{p} \sum_{m=0}^{p-1} \left(\sum_{k=0}^{p-1} \frac{1}{\zeta_m^k} \right) f(\zeta_m z).$$

From (2.4), (2.5) we get

$$\sum_{k=0}^{p-1} \frac{1}{\zeta_m^k} = \sum_{k=0}^{p-1} \zeta_m^k = \begin{cases} p, & \text{if } m = 0, \\ 0, & \text{if } m \neq 0. \end{cases}$$

Therefore,

$$\sum_{k=0}^{p-1} f_k(z) = f(z).$$

Next, prove the uniqueness. Suppose

$$\sum_{k=0}^{p-1} f_k(z) = \sum_{k=0}^{p-1} g_k(z),$$

where $f_k(z)$ and $g_k(z)$ are k_p -even functions. Then k_p -even functions $h_k(z) = f_k(z) - g_k(z)$ ($h_k(\zeta_j z) = \zeta_j^k h_k(z)$) satisfy the system of equations

$$\begin{cases} h_0(z) + h_1(z) + h_2(z) + \cdots + h_{p-1}(z) = 0 \\ h_0(z) + \zeta_1 h_1(z) + \zeta_1^2 h_2(z) + \cdots + \zeta_1^{p-1} h_{p-1}(z) = 0 \\ \dots \quad \dots \quad \dots \\ h_0(z) + \zeta_{p-1} h_1(z) + \zeta_{p-1}^2 h_2(z) + \cdots + \zeta_{p-1}^{p-1} h_{p-1}(z) = 0 \end{cases}$$

By Lemma 2.1, its matrix of coefficients is nonsingular. Then $h_k(z) \equiv 0$, $k = 0, 1, \dots, p-1$. $\square$

Corollary 2.1. *Let $p \in \mathbb{N}$. If a k_p -even function $f_k(z)$ is analytic in a neighborhood of zero, then its Taylor series has the form*

$$(2.12) \quad f_k(z) = \sum_{n=0}^{\infty} a_n z^{k+np}.$$

Proof. Due to the uniqueness of the decomposition (2.10), it is sufficient to apply the formula (2.11) to the Taylor series of the function $f_k(z)$. $\square$

Proposition 2.3. *Let $f(z)$ be a differentiable k_p -even function. Then its derivative*

$$\frac{df}{dz}(z)$$

is a $(k-1)_p$ -even function.

Proof. It is given that $f(z) = (1/\zeta_m^k)f(\zeta_m z)$. Then

$$\frac{df}{dz}(z) = \frac{1}{\zeta_m^k} \frac{df}{dz}(\zeta_m z) \zeta_m = \frac{1}{\zeta_m^{k-1}} \frac{df}{dz}(\zeta_m z). \quad \square$$

3. Fundamental systems of solutions. We are interested in solutions of the differential equation

$$(3.1) \quad \left(\frac{1}{i}D\right)^p y(x) = \lambda^p y(x) \quad (x \in \mathbb{R}, D = d/dx, \lambda \in \mathbb{C}),$$

where p is a natural number.

Proposition 3.1. *Any solution of the equation (3.1) has the form*

$$y(x) = s(i\lambda x),$$

where $s(z)$ is the solution of the equation

$$(3.2) \quad D^p s(z) = s(z), \quad (z \in \mathbb{C}, D = d/dz).$$

Proof. Introducing a new function $s(z) = y(x)$, where $z = i\lambda x$, we obtain

$$\begin{aligned} \lambda^p s(z) &= \lambda^p y(x) = (-i)^p \frac{d^p}{dx^p} y(x) = (-i)^p \frac{d^p}{dx^p} s(z) \\ &= (-i)^p \frac{d^p}{dz^p} s(z) \left(\frac{dz}{dx}\right)^p = (-i)^p i^p \lambda^p \frac{d^p}{dz^p} s(z) = \lambda^p \frac{d^p}{dz^p} s(z). \end{aligned}$$

This implies (3.2). It is obvious that (3.2) implies (3.1). $\square$

The function $s(z) = e^z$ is the only solution (up to a multiplicative constant) of equations (3.2) for all $p \in \mathbb{N}$. For each fixed $p \in \mathbb{N}$, the solution $s(z) = e^z$ is called a *principal solution* of the equation (3.2).

Definition 3.1. Let p be a natural number, and $\{\zeta_k\}_{k=0}^{p-1}$ be p th roots of unity. k_p -even components

$$(3.3) \quad s_k(z) = \frac{1}{p} \sum_{m=0}^{p-1} \frac{1}{\zeta_m^k} e^{z\zeta_m}, \quad k = 0, 1, \dots, p-1$$

of the function e^z are called *p-hyperbolic functions*.

Remark 3.1. The name “*p-hyperbolic functions*” is due to the fact that, for $p = 2$ ($\zeta_0 = 1$, $\zeta_1 = -1$), we have

$$s_0(z) = \frac{1}{2}(e^z + e^{-z}) = \cosh z, \quad s_1(z) = \frac{1}{2}(e^z - e^{-z}) = \sinh z.$$

By analogy to this case, we call

$$(3.4) \quad c_k(z) = \frac{1}{i^k} s_k(iz), \quad k = 0, 1, \dots, p-1$$

a p-trigonometric functions. By virtue of (3.4), it suffices to study only the properties of *p-hyperbolic functions*.

Note that according to Proposition 2.2,

$$s_0(z) + s_1(z) + \dots + s_{p-1}(z) = e^z.$$

This identity is a generalization of Euler’s formula for the case $p \neq 2$.

Proposition 3.2. *p-hyperbolic functions $s_k(z)$, $k = 0, 1, \dots, p-1$ are entire functions of exponential type. These functions*

- (i) *form a fundamental system of solutions of the equation (3.2),*
- (ii) *satisfy the initial data*

$$(3.5) \quad D^n s_k(z)|_{z=0} = \begin{cases} 1, & \text{if } n = k, \\ 0, & \text{if } n \neq k, \end{cases} \quad k, n = 0, 1, \dots, p-1,$$

- (iii) *have derivatives*

$$(3.6) \quad Ds_0(z) = s_{p-1}(z), \quad Ds_k(z) = s_{k-1}(z), \quad k = 1, \dots, p-1,$$

- (iv) *are represented by the Taylor series*

$$(3.7) \quad s_k(z) = \sum_{n=0}^{\infty} \frac{z^{k+pn}}{(k+pn)!}, \quad k = 0, 1, \dots, p-1,$$

(v) are real for real z , i. e.,

$$(3.8) \quad \overline{s_k(z)} = s_k(\bar{z}), \quad k = 0, 1, \dots, p-1,$$

(vi)

$$(3.9) \quad \det W[s_k(z)] = \begin{vmatrix} s_0(z) & s_1(z) & \cdots & s_{p-1}(z) \\ s'_0(z) & s'_1(z) & \cdots & s'_{p-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ s_0^{(p-1)}(z) & s_1^{(p-1)}(z) & \cdots & s_{p-1}^{(p-1)}(z) \end{vmatrix} \equiv 1.$$

Proof. (ii), (iii) and (iv) follow from Definition 3.1 and relations (2.2)–(2.5). (v) follows from (iv).

(vi). Given (iii), we get

$$(3.10) \quad W[s_k(z)] = \begin{pmatrix} s_0(z) & s_1(z) & \cdots & s_{p-1}(z) \\ s_{p-1}(z) & s_0(z) & \cdots & s_{p-2}(z) \\ \vdots & \vdots & \ddots & \vdots \\ s_1(z) & s_2(z) & \cdots & s_0(z) \end{pmatrix} = U\Lambda(z)U^*,$$

where the unitary matrix U is defined by the relation (2.9) and

$$\Lambda(z) = \text{diag}\{e^{z\zeta_0}, e^{z\zeta_1}, \dots, e^{z\zeta_{p-1}}\}.$$

Since

$$\det \Lambda(z) = e^{z(\zeta_0 + \zeta_1 + \cdots + \zeta_{p-1})} = e^0 = 1,$$

then $\det W[s_k(z)] \equiv 1$.

(i). From (iii) yields $D^p s_k(z) = s_k(z)$, $k = 0, 1, \dots, p-1$, i. e., $s_k(z)$ are solutions of equation (3.2). (vi) implies linear independence of functions. Then $\{s_k(z)\}_{k=0}^{p-1}$ is the fundamental system of solutions of the equation (3.2). $\square$

Corollary 3.1. *The function*

$$(3.11) \quad y(z) = y_0 s_0(z) + y_1 s_1(z) + \cdots + y_{p-1} s_{p-1}(z)$$

is solution of Cauchy problem

$$(3.12) \quad \begin{aligned} D^p y(z) &= y(z), \quad (D = d/dz) \\ y(0) &= y_0, \quad y'(0) = y_1, \quad \dots, \quad y^{(p-1)}(0) = y_{p-1}. \end{aligned}$$

Corollary 3.2. *For the solution (3.11) of the Cauchy problem (3.12) we have*

$$\det W[y, y', \dots, y^{(p-1)}] = \begin{vmatrix} y(z) & y'(z) & \cdots & y^{(p-1)}(z) \\ y^{(p-1)}(z) & y(z) & \cdots & y^{(p-2)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ y'(z) & y''(z) & \cdots & y(z) \end{vmatrix} = C,$$

where the constant C depends only on the initial data and does not depend on z .

Proof. Since $W[y, y', \dots, y^{(p-1)}] = Y_p^T \cdot W[s_k(z)]$, where

$$Y_p = \begin{pmatrix} y_0 & y_1 & \cdots & y_{p-1} \\ y_{p-1} & y_0 & \cdots & y_{(p-2)} \\ \vdots & \vdots & \ddots & \vdots \\ y_1 & y_2 & \cdots & y_0 \end{pmatrix},$$

and $\det W[s_k(z)] \equiv 1$, then $\det W(y, y', \dots, y^{(p-1)}) = \det Y_p$. $\square$

4. Basic identities for p -hyperbolic functions. The derivation of relations between p -hyperbolic functions is based on the properties of p roots of unity and the following simple statement.

Proposition 4.1. *Let ζ_m ($m = 0, 1, \dots, p-1$) be p roots of unity. The maps*

$$(4.1) \quad z \mapsto W_p(z) = \begin{pmatrix} s_0(z) & s_1(z) & \cdots & s_{p-1}(z) \\ s_{p-1}(z) & s_0(z) & \cdots & s_{p-2}(z) \\ \vdots & \vdots & \ddots & \vdots \\ s_1(z) & s_2(z) & \cdots & s_0(z) \end{pmatrix}$$

is a representation of the additive group of complex numbers in the linear space $\mathbb{C}^p$. Moreover, by Proposition 3.2 (vi), the main identity

$$(4.2) \quad \det W_p(z) \equiv 1$$

holds.

Proof. Recall (see (3.10)) that

$$(4.3) \quad W_p(z) = U\Lambda(z)U^*, \quad \Lambda(z) = \text{diag}\{e^{z\zeta_0}, e^{z\zeta_1}, \dots, e^{z\zeta_{p-1}}\},$$

where U is a unitary matrix from Lemma 2.1. Since

$$\Lambda(z_1 + z_2) = \Lambda(z_1) \cdot \Lambda(z_2), \quad \Lambda(0) = I,$$

then $W_p(z_1 + z_2) = W_p(z_1) \cdot W_p(z_2)$, $W_p(0) = I$. $\square$

Remark 4.1. Since the elements of the matrix $W_p(z_1 + z_2)$ are functions $s_k(z_1 + z_2)$, then from the identity $W_p(z_1 + z_2) = W_p(z_1) \cdot W_p(z_2)$ we obtain formulas for functions of the sum of the arguments. The identity $W_p(z)W_p(-z) = I$ binds the values of functions on opposite values of the arguments.

Remark 4.2. From (4.3) follows $\Lambda(z) = U^*W_p(z)U$. Comparing the diagonal elements, we obtain the *Euler identities* for p -hyperbolic functions

$$(4.4) \quad e^{z\zeta_k} = \zeta_0^k s_0(z) + \zeta_1^k s_1(z) + \dots + \zeta_{p-1}^k s_{p-1}(z), \quad k = 0, 1, \dots, p-1.$$

To conclude this section, we present the corresponding formulas for the case $p = 3$. We obtain

(i) *the main identity* ($\det W_3(z) = 1$)

$$(4.5) \quad s_0(z)^3 + s_1(z)^3 + s_2(z)^3 - 3s_0(z)s_1(z)s_2(z) = 1,$$

(ii) *addition formulas* ($W_3(z + w) = W_3(z) \cdot W_3(w)$)

$$(4.6) \quad \begin{aligned} s_0(z + w) &= s_0(z)s_0(w) + s_1(z)s_2(w) + s_2(z)s_1(w), \\ s_1(z + w) &= s_0(z)s_1(w) + s_1(z)s_0(w) + s_2(z)s_2(w), \\ s_2(z + w) &= s_0(z)s_2(w) + s_1(z)s_1(w) + s_2(z)s_0(w), \end{aligned}$$

(iii) *double argument*

$$(4.7) \quad \begin{aligned} s_0(2z) &= s_0(z)^2 + 2s_1(z)s_2(z), \\ s_1(2z) &= 2s_0(z)s_1(z) + s_2(z)^2, \\ s_2(2z) &= 2s_0(z)s_2(z) + s_1(z)^2, \end{aligned}$$

(iv) *triple argument*

$$\begin{aligned}
(4.8) \quad s_0(3z) &= 1 + 9s_0(z)s_1(z)s_2(z), \\
s_1(3z) &= 3s_0(z)^2s_1(z) + 3s_0(z)^2s_2(z) + 3s_1(z)^2s_2(z), \\
s_2(3z) &= 3s_0(z)s_1(z)^2 + 3s_0(z)s_2(z)^2 + 3s_1(z)s_2(z)^2,
\end{aligned}$$

(v) *convert product to sum*

$$\begin{aligned}
(4.9) \quad 3s_0(z)s_0(w) &= s_0(z+w) + s_0(z+\zeta_1w) + s_0(z+\zeta_2w), \\
3s_1(z)s_2(w) &= s_0(z+w) + \zeta_1s_0(z+\zeta_1w) + \zeta_2s_0(z+\zeta_2w), \\
3s_1(z)s_0(w) &= s_1(z+w) + s_1(z+\zeta_1w) + s_1(z+\zeta_2w), \\
3s_2(z)s_2(w) &= s_1(z+w) + \zeta_1s_1(z+\zeta_1w) + \zeta_2s_1(z+\zeta_2w), \\
3s_2(z)s_0(w) &= s_2(z+w) + s_2(z+\zeta_1w) + s_2(z+\zeta_2w), \\
3s_1(z)s_1(w) &= s_2(z+w) + \zeta_2s_2(z+\zeta_1w) + \zeta_1s_1(z+\zeta_2w).
\end{aligned}$$

(vi) *connection of direct and opposite arguments* ($W_3(z)W_3(-z) = I$)

$$\begin{aligned}
(4.10) \quad s_0(z)s_0(-z) + s_1(z)s_2(-z) + s_2(z)s_1(-z) &= 1, \\
s_0(z)s_1(-z) + s_1(z)s_0(-z) + s_2(z)s_2(-z) &= 0, \\
s_0(z)s_2(-z) + s_1(z)s_1(-z) + s_2(z)s_0(-z) &= 0,
\end{aligned}$$

(vii) *opposite argument*

$$\begin{aligned}
(4.11) \quad s_0(-z) &= s_0(z)^2 - s_1(z)s_2(z) \\
s_1(-z) &= s_2(z)^2 - s_0(z)s_1(z) \\
s_2(-z) &= s_1(z)^2 - s_0(z)s_2(z).
\end{aligned}$$

From here it is not difficult to obtain formulas for p -trigonometric functions $c_k(z) = s_k(iz)/i^k$.

5. Zeros of functions $s_k(z)$ for $p = 3$. Let ζ_ν , $\nu = 0, 1, 2$ be 3-roots of unity. Introduce the notation

$$l_\nu^+ = \{z \in \mathbb{C} \mid z = t \cdot \zeta_\nu, t > 0\}, \quad l_\nu^- = \{z \in \mathbb{C} \mid z = t \cdot \zeta_\nu, t \leq 0\}.$$

Rays l_ν^+ , $\nu = 0, 1, 2$ divide the complex plane $\mathbb{C}$ into three sectors

$$S_\nu = \{z \in \mathbb{C} \mid \arg \zeta_\nu < \arg z < \arg \zeta_{(\nu+1) \pmod{3}}\}, \quad \nu = 0, 1, 2.$$

The ray l_ν^- is a bisectrix of sector $S_{(\nu+1) \pmod{3}}$.

Proposition 5.1. *3-hyperbolic functions $s_k(z)$ ($k = 0, 1, 2$) do not vanish on the set*

$$(5.1) \quad \Omega = \mathbb{C} \setminus \bigcup_{\nu=0}^2 l_\nu^-.$$

In other words, the zeros of the functions $s_k(z)$ lie only on the bisectrices of each sector.

We need two lemmas

Lemma 5.1. *Let $z = x + iy$ ($x, y \in \mathbb{R}$) and $s_k(z)$, $k = 0, 1, 2$ be 3-hyperbolic functions. The zeros of the function $s_k(x + iy)$ are the roots of the system of equations*

$$(5.2) \quad \begin{cases} \cosh \frac{y\sqrt{3}}{2} \cos \left(\frac{x\sqrt{3}}{2} - \frac{2\pi k}{3} \right) = -\frac{1}{2} e^{\frac{3x}{2}} \cos \frac{3y}{2} \\ \sinh \frac{y\sqrt{3}}{2} \sin \left(\frac{x\sqrt{3}}{2} - \frac{2\pi k}{3} \right) = \frac{1}{2} e^{\frac{3x}{2}} \sin \frac{3y}{2} \end{cases}$$

Proof. We have

$$\zeta_k = \zeta_1^k = e^{i2\pi k/3} \Rightarrow \zeta_0 = 1, \quad \zeta_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \zeta_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Then from (3.3) we obtain

$$\begin{aligned} 3s_k(z) &= e^z + \frac{1}{\zeta_1^k} e^{z\zeta_1} + \frac{1}{\zeta_2^k} e^{z\zeta_2} = e^z + \bar{\zeta}_1^k e^{z\zeta_1} + \zeta_1^k e^{z\zeta_2} \\ &= e^{-\frac{z}{2}} \left(e^{\frac{3z}{2}} + e^{-i\frac{2\pi k}{3}} e^{i\frac{z\sqrt{3}}{2}} + e^{i\frac{2\pi k}{3}} e^{-i\frac{z\sqrt{3}}{2}} \right) \\ &= e^{-\frac{z}{2}} \left(e^{\frac{3z}{2}} + e^{i\left(\frac{z\sqrt{3}}{2} - \frac{2\pi k}{3}\right)} + e^{-i\left(\frac{z\sqrt{3}}{2} - \frac{2\pi k}{3}\right)} \right) \\ &= e^{-\frac{z}{2}} \left(e^{\frac{3z}{2}} + 2 \cos \left(\frac{z\sqrt{3}}{2} - \frac{2\pi k}{3} \right) \right). \end{aligned}$$

Therefore, the zeros of the function $s_k(z)$ are the roots of the equation

$$(5.3) \quad \cos \left(\frac{z\sqrt{3}}{2} - \frac{2\pi k}{3} \right) + \frac{1}{2} e^{\frac{3z}{2}} = 0.$$

It is enough to apply the identity

$$\cos(\alpha + i\beta) = \cosh \beta \cos \alpha - i \sinh \beta \sin \alpha$$

to the left side (5.3) and separate the real and imaginary parts. $\square$

Lemma 5.2. *For $y \geq 0$ the inequality*

$$(5.4) \quad \sqrt{3} \cdot \tanh \left(\frac{y\sqrt{3}}{2} \right) \geq \sin \left(\frac{3y}{2} \right)$$

is true. Equality in (5.4) holds only for $y = 0$.

Proof. The hyperbolic tangent is represented by an alternating series

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots \quad \text{for } |x| < \frac{\pi}{2}.$$

Therefore,

$$\tanh x = x - \frac{1}{3}x^3 + \Delta, \quad \text{where } \Delta > 0.$$

From this we obtain

$$(5.5) \quad \sqrt{3} \cdot \tanh \left(\frac{y\sqrt{3}}{2} \right) \geq \frac{3y}{2} - \frac{3y^3}{8} \quad \text{for } 0 \leq y \leq \frac{\pi}{\sqrt{3}}.$$

Similarly,

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \Delta_1 \quad \text{where } \Delta_1 \geq 0 \quad \text{for } x \geq 0.$$

Then

$$(5.6) \quad \frac{3y}{2} - \frac{9y^3}{16} + \frac{81y^5}{1280} \geq \sin \left(\frac{3y}{2} \right) \quad \text{for } y \geq 0.$$

Given (5.5) and (5.6), the inequality (5.4) is true if

$$(5.7) \quad \frac{3y}{2} - \frac{3y^3}{8} > \frac{3y}{2} - \frac{9y^3}{16} + \frac{81y^5}{1280}$$

is true. Find a solution to the inequality (5.7). We have

$$y^3(80 - 27y^2) > 0 \quad \Rightarrow \quad 0 < y < y_0 = \frac{4\sqrt{5}}{3\sqrt{3}} \approx 1,72 < \frac{\pi}{\sqrt{3}} \approx 1,81.$$

Therefore, the strict inequality (5.4) is true for $0 < y < y_0$. Since

$$\sqrt{3} \tanh \left(\frac{y\sqrt{3}}{2} \right) \geq \sqrt{3} \tanh \left(\frac{y_0\sqrt{3}}{2} \right) = \sqrt{3} \tanh \left(\frac{2\sqrt{5}}{3} \right) \approx 1,56 > 1$$

for $y \geq y_0$, the strict inequality (5.4) is true for all $y > 0$. $\square$

Proof of Proposition 5.1. Since $s_k(\zeta_\nu z) = \zeta_\nu^k s_k(z)$, it suffices to prove that in some closed sector $\overline{S}_\nu$ (minus the bisectrix) the function $s_k(z)$ does not vanish.

If the point $z = x + iy$ belongs to $\overline{S}_1 \setminus l_0^-$, then $x \leq 0$ and $y \neq 0$, $y \in \mathbb{R}$. Taking Lemma 5.1 into account, we prove that such points do not satisfy the system of equations (5.2). From the first equation of the system, we obtain

$$\cos \left(\frac{x\sqrt{3}}{2} - \frac{2\pi k}{3} \right) = -\frac{e^{3x/2}}{2} \frac{\cos(3y/2)}{\cosh(y\sqrt{3}/2)}.$$

Thus

$$(5.8) \quad \sin^2 \left(\frac{x\sqrt{3}}{2} - \frac{2\pi k}{3} \right) = \frac{4 \cosh^2(y\sqrt{3}/2) - e^{3x} \cos^2(3y/2)}{4 \cosh^2(y\sqrt{3}/2)}.$$

Show that (5.8) does not satisfy the second equation from (5.2) for $x \leq 0$ and $y \neq 0$. Squaring both sides of the second equation, we get

$$\sinh^2 \frac{y\sqrt{3}}{2} \sin^2 \left(\frac{x\sqrt{3}}{2} - \frac{2\pi k}{3} \right) = \frac{1}{4} e^{3x} \sin^2 \frac{3y}{2}.$$

Substituting (5.8) into the last equation, we obtain

$$(5.9) \quad \tanh^2 \frac{y\sqrt{3}}{2} \left(4 \cosh^2 \frac{y\sqrt{3}}{2} - e^{3x} \cos^2 \frac{3y}{2} \right) = e^{3x} \sin^2 \frac{3y}{2}.$$

By Lemma 5.2 we have the strict inequality

$$3 \cdot \tanh^2 \frac{y\sqrt{3}}{2} > \sin^2 \frac{3y}{2} \quad \text{for } y \neq 0.$$

Then from (5.9) we obtain the strict inequality (for $x \leq 0$)

$$\begin{aligned} \tanh^2 \frac{y\sqrt{3}}{2} \left(4 \cosh^2 \frac{y\sqrt{3}}{2} - e^{3x} \cos^2 \frac{3y}{2} \right) \\ \geq 3 \cdot \tanh^2 \frac{y\sqrt{3}}{2} > \sin^2 \frac{3y}{2} \geq e^{3x} \sin^2 \frac{3y}{2}. \end{aligned}$$

Therefore, $s_k(z) \neq 0$ for $z \in \bar{S}_1 \setminus l_0^-$. $\square$

By Proposition 5.1, the zeros of the $s_0(z)$, $s_1(z)$ and $s_2(z)$ are located on l_ν^- , $\nu = 0, 1, 2$. Since $s_k(\zeta_m z) = \zeta_m^k s_k(z)$ are satisfied, it suffices to find the zeros lying only on $l_0^- = \{x \in \mathbb{R} \mid x \leq 0\}$.

Taking into account (5.3), we obtain the following assertion.

Proposition 5.2. *An infinite sequence of zeros $x_j^{(k)}$ ($j = 1, 2, \dots$) of the function $s_k(x)$ are the roots of the equation*

$$(5.10) \quad \cos \left(\frac{x\sqrt{3}}{2} - \frac{2\pi k}{3} \right) = -\frac{1}{2} e^{\frac{3x}{2}}, \quad k = 0, 1, 2.$$

The sequence $x_j^{(2)}$ interlaces with the sequence $x_j^{(0)}$ which, in its turn, interlaces with the sequence $x_j^{(1)}$. Note that $x_1^{(2)} = x_1^{(1)} = 0$ and $x_1^{(0)} < 0$.

Corollary 5.1. *If $x \rightarrow -\infty$, then asymptotically the zeros of the function $s_k(x)$ tend to the roots of the equation*

$$(5.11) \quad \cos \left(\frac{x\sqrt{3}}{2} - \frac{2\pi k}{3} \right) = 0.$$

6. General solution of the inhomogeneous equation. Consider the Cauchy problem

$$(6.1) \quad \left(\frac{1}{i} D \right)^p y(\lambda, x) = \lambda^p y(\lambda, x) + f(x), \quad x \in (0, l)$$

with initial data at zero

$$(6.2) \quad y(\lambda, 0) = y_0, \quad y'(\lambda, 0) = y_1, \dots, \quad y^{(p-1)}(\lambda, 0) = y_{p-1},$$

where $f(x)$ is the function of $x \in \mathbb{R}$, $y_s \in \mathbb{C}$ ($0 \leq s \leq p-1$) and $0 < l < \infty$.

It is easy to see that the solution $y_0(\lambda, x)$ of the problem (6.1), (6.2) with $f(x) \equiv 0$ has the form

$$(6.3) \quad y_0(\lambda, x) = \sum_{k=0}^{p-1} y_k \frac{1}{(i\lambda)^k} s_k(i\lambda x) = \sum_{k=0}^{p-1} C_k e^{i\lambda \zeta_k x}.$$

For $f(x) \neq 0$ we find the solution $y_1(\lambda, x)$ of equation (6.1) with zero initial data $y_0 = y_1 = \dots = y_{p-1} = 0$. For the method of variation of arbitrary constants, we set

$$(6.4) \quad y_1(\lambda, x) = \sum_{k=0}^{p-1} C_k(x) e^{i\lambda \zeta_k x},$$

where $C_k(x)$ are smooth functions. Easy to see that

$$\left(\frac{1}{i}D\right) y_1(\lambda, x) = \lambda \sum_{k=0}^{p-1} C_k(x) \zeta_k e^{i\lambda \zeta_k x},$$

under the additional condition

$$(6.5) \quad \sum_{k=0}^{p-1} C'_k(x) e^{i\lambda \zeta_k x} \equiv 0.$$

After the second differentiation, under the additional condition

$$(6.6) \quad \sum_{k=0}^{p-1} C'_k(x) \zeta_k e^{i\lambda \zeta_k x} \equiv 0,$$

we get

$$\left(\frac{1}{i}D\right)^2 y_1(\lambda, x) = \lambda^2 \sum_{k=0}^{p-1} C_k(x) \zeta_k^2 e^{i\lambda \zeta_k x}.$$

Repeating this trick, at the p th step we get

$$\left(\frac{1}{i}D\right)^p y_1(\lambda, x) = \lambda^p y_1(\lambda, x) + \frac{1}{i} \lambda^{p-1} \sum_{k=0}^{p-1} C'_k(x) \zeta_k^{p-1} e^{i\lambda \zeta_k x}.$$

The function $y_1(\lambda, x)$ (6.4) will be a solution to equation (6.1) if $C_k(x)$ satisfy additional conditions (of the form (6.5), (6.6) accepted at the first $p-1$ steps) and the condition

$$(6.7) \quad \sum_{k=0}^{p-1} C'_k(x) \zeta_k^{p-1} e^{i\lambda \zeta_k x} = if(x)/\lambda^{p-1}.$$

Taking into account (6.5), (6.6), and (6.7), we obtain a system of equations for the $\{C'_k(x)\}_{k=0}^{p-1}$:

$$(6.8) \quad W(\lambda, x) \begin{pmatrix} C'_0(x) \\ C'_1(x) \\ \vdots \\ C'_{p-2}(x) \\ C'_{p-1}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ if(x)/\lambda^{p-1} \end{pmatrix},$$

where

$$W(\lambda, x) = \begin{pmatrix} e^{i\lambda \zeta_0 x} & e^{i\lambda \zeta_1 x} & \dots & e^{i\lambda \zeta_{p-1} x} \\ \zeta_0 e^{i\lambda \zeta_0 x} & \zeta_1 e^{i\lambda \zeta_1 x} & \dots & \zeta_{p-1} e^{i\lambda \zeta_{p-1} x} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_0^{p-1} e^{i\lambda \zeta_0 x} & \zeta_1^{p-1} e^{i\lambda \zeta_1 x} & \dots & \zeta_{p-1}^{p-1} e^{i\lambda \zeta_{p-1} x} \end{pmatrix}.$$

Since

$$W(\lambda, x)^{-1} = \frac{1}{p} \begin{pmatrix} e^{-i\lambda \zeta_0 x} & e^{-i\lambda \zeta_0 x}/\zeta_0 & \dots & e^{-i\lambda \zeta_0 x}/\zeta_0^{p-1} \\ e^{-i\lambda \zeta_1 x} & e^{-i\lambda \zeta_1 x}/\zeta_1 & \dots & e^{-i\lambda \zeta_1 x}/\zeta_1^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-i\lambda \zeta_{p-1} x} & e^{-i\lambda \zeta_{p-1} x}/\zeta_{p-1} & \dots & e^{-i\lambda \zeta_{p-1} x}/\zeta_{p-1}^{p-1} \end{pmatrix},$$

then

$$\begin{pmatrix} C'_0(x) \\ C'_1(x) \\ \vdots \\ C'_{p-2}(x) \\ C'_{p-1}(x) \end{pmatrix} = W(\lambda, x)^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ if(x)/\lambda^{p-1} \end{pmatrix} = i \frac{f(x)}{p\lambda^{p-1}} \begin{pmatrix} e^{-i\lambda \zeta_0 x}/\zeta_0^{p-1} \\ e^{-i\lambda \zeta_1 x}/\zeta_1^{p-1} \\ \vdots \\ e^{-i\lambda \zeta_{p-2} x}/\zeta_{p-2}^{p-1} \\ e^{-i\lambda \zeta_{p-1} x}/\zeta_{p-1}^{p-1} \end{pmatrix}.$$

Therefore, for $k = 0, 1, \dots, p-1$ we have

$$C_k(x) = \frac{i}{\lambda^{p-1}} \int_0^x \frac{e^{-i\lambda\zeta_k t}}{p\zeta_k^{p-1}} f(t) dt + D_k, \quad (D_k = \text{const}).$$

Then

$$y_1(\lambda, x) = \frac{i}{\lambda^{p-1}} \int_0^x \frac{1}{p} \sum_{k=0}^{p-1} \frac{1}{\zeta_k^{p-1}} e^{i\lambda\zeta_k(x-t)} f(t) dt + \sum_{k=0}^{p-1} D_k e^{i\lambda\zeta_k x}.$$

From conditions $y_1(\lambda, 0) = y_1'(\lambda, 0) = \dots = y_1^{(p-1)}(\lambda, 0) = 0$ we obtain $D_k = 0$ for all k . Since

$$\frac{1}{p} \sum_{k=0}^{p-1} \frac{1}{\zeta_k^{p-1}} e^{i\lambda\zeta_k(x-t)} = s_{p-1}(i\lambda(x-t)),$$

then

$$(6.9) \quad y_1(\lambda, x) = \frac{i}{\lambda^{p-1}} \int_0^x s_{p-1}(i\lambda(x-t)) f(t) dt.$$

As a result we have

Lemma 6.1. *The solution of the Cauchy problem (6.1), (6.2) is the function*

$$(6.10) \quad y(\lambda, x) = \sum_{k=0}^{p-1} y_k \frac{1}{(i\lambda)^k} s_k(i\lambda x) + \frac{i}{\lambda^{p-1}} \int_0^x s_{p-1}(i\lambda(x-t)) f(t) dt.$$

7. Self-adjointness. Let M be the set of p times differentiable functions from $L^2(0, l)$ ($0 < l < \infty$). Use the Lagrange formula for differential operation $(-iD)^p$ acting on M .

$$(7.1) \quad \int_0^l (-iD)^p y(x) \overline{z(x)} dx = \frac{1}{i^p} [Q_l(y, z) - Q_0(y, z)] \\ + \int_0^l y(x) \overline{(-iD)^p z(x)} dx, \quad (y, z \in M),$$

where

$$(7.2) \quad Q_x(y, z) = y^{(p-1)}(x) \overline{z(x)} - y^{(p-2)}(x) \overline{z(x)'} + \dots + (-1)^{p-1} y(x) \overline{z(x)^{(p-1)}},$$

here $Q_l(y, z) = Q_x(y, z)|_{x=l}$, $Q_0(y, z) = Q_x(y, z)|_{x=0}$.

The matrix J_p of the bilinear form (7.2) has the size $p \times p$

$$(7.3) \quad J_p = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^{p-2} & \cdots & 0 & 0 \\ (-1)^{p-1} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and satisfies the conditions

$$(7.4) \quad J_p^* = (-1)^{p-1} J_p, \quad J_p^2 = I \quad (p = 2k + 1), \quad J_p^2 = -I \quad (p = 2k).$$

Case $p = 2k$. Since $(iJ_p)^* = iJ_p$ and $(iJ_p)^2 = I$, the eigenvalues of the matrix iJ_p are ± 1 . Therefore, the eigenvalues of J_p are $\pm i$.

Orthonormal eigenvectors

$$(7.5) \quad \begin{aligned} e_1^+ &= \frac{1}{\sqrt{2}} \text{col}(1, 0, \dots, 0, 0, \dots, 0, i) \\ e_2^+ &= \frac{1}{\sqrt{2}} \text{col}(0, 1, \dots, 0, 0, \dots, -i, 0) \\ &\dots \\ e_k^+ &= \frac{1}{\sqrt{2}} \text{col}(0, 0, \dots, 1, (-1)^{k-1}i, \dots, 0, 0) \end{aligned}$$

form eigen subspace $E_+^{(k)}$ ($\dim E_+^{(k)} = k$) of the operator $J_p = J_{2k}$ corresponding to the eigenvalue $\lambda = +i$, $J_{2k}e_s^+ = ie_s^+$ ($1 \leq s \leq k$).

Likewise, the orthonormal eigenvectors

$$(7.6) \quad \begin{aligned} e_1^- &= \frac{1}{\sqrt{2}} \text{col}(-1, 0, \dots, 0, 0, \dots, 0, i) \\ e_2^- &= \frac{1}{\sqrt{2}} \text{col}(0, -1, \dots, 0, 0, \dots, -i, 0) \\ &\dots \\ e_k^- &= \frac{1}{\sqrt{2}} \text{col}(0, 0, \dots, -1, (-1)^{k-1}i, \dots, 0, 0) \end{aligned}$$

form eigen subspace $E_-^{(k)}$ ($\dim E_-^{(k)} = k$) corresponding to the eigenvalue $\lambda = -i$.

Therefore,

$$(7.7) \quad E^p = E_+^p \oplus E_-^p.$$

Any vector $h \in E^p$ has a unique decomposition

$$(7.8) \quad h = h_+ + h_-, \quad h_{\pm} = P_{\pm}h,$$

where $P_{\pm}$ are orthoprojectors on $E_{\pm}$. Therefore,

$$P_+h = \sum_{s=1}^k h_s^+ e_s^+, \quad P_-h = \sum_{s=1}^k h_s^- e_s^-,$$

where $e_s^{\pm}$ are the vectors (7.5), (7.6), and

$$(7.9) \quad h_s^{\pm} = \langle h, e_s^{\pm} \rangle = \frac{\pm h_s + (-1)^{p-s} i h_{p-s+1}}{\sqrt{2}}, \quad l = \text{col}(h_1, \dots, h_p).$$

From (7.8) it follows that the quadratic form (7.8) has the form

$$(7.10) \quad \langle J_p h, h \rangle = i \{ \langle h_+, h_+ \rangle - \langle h_-, h_- \rangle \}.$$

Equality (7.10) allows us to distinguish two types of boundary conditions which ensure the self-adjointness of the operator $(-iD)^p$ for an even $p = 2k$.

The first type of boundary conditions is determined by those h that annul the quadratic form $\langle J_p h, h \rangle$. Let

$$(7.11) \quad h_V = h_+ + V h_+, \quad (h_- = V h_+),$$

where $V : E_+ \rightarrow E_-$ is an arbitrary unitary operator. Given (7.10), we get $\langle J_p h_V, h_V \rangle = 0$. The operator V has the form $V = V_+^* V_0 V_-$, where $V_{\pm} : E_{\pm}^{(k)} \rightarrow E^{(k)}$ are unitary mappings transforming $\{e_s^{\pm}\}_{s=1}^k$ into the standard basis $\{e_s\}_{s=1}^k$ of the space $E^{(k)}$ ($\dim E^{(k)} = k$) and V_0 is a unitary operator in $E^{(k)}$. Since $h_- = V h_+$, then from (7.9) it follows

$$(7.12) \quad \tilde{h}_1 + i J_k \hat{h}_1 = V_0 (-\tilde{h}_1 + i J_k \hat{h}_1),$$

where J_k is (7.3) for $p = k$ and $\tilde{h}_1, \hat{h}_1 \in E^{(k)}$ are vectors

$$(7.13) \quad \tilde{h}_1 = \text{col}(h_1, \dots, h_k), \quad \hat{h}_1 = \text{col}(h_{k+1}, \dots, h_{2k}), \quad (h = \tilde{h}_1 \oplus \hat{h}_1).$$

Rewrite (7.12) as

$$i(I - V_0)J_k \hat{h}_1 = -(I + V_0)\tilde{h}_1,$$

or

$$(7.14) \quad J_k \widehat{h}_1 = B_0 \widetilde{h}_1,$$

where

$$(7.15) \quad B_0 = i(I - V_0)^{-1}(I + V_0)$$

Since $V_0 V_0^* = I$, then $B_0^* = B_0$.

Introduce the notations

$$(7.16) \quad \begin{aligned} Y(x) &= \text{col}(y(x), \dots, y^{(k-1)}(x), y^{(k)}(x), \dots, y^{(p-1)}(x)), \\ Y_0(x) &= \text{col}(y(x), y'(x), \dots, y^{(k-1)}(x)) \in E^{(k)}, \\ Y_1(x) &= \text{col}(y^{(k)}(x), y^{(k+1)}(x), \dots, y^{(p-1)}(x)) \in E^{(k)}, \\ Y(x) &= Y_0(x) \oplus Y_1(x) \in E^{(2k)}. \end{aligned}$$

Proposition 7.1. *Let $p = 2k$, $Y_0(x)$, $Y_1(x)$ and J_k be determined by the relations (7.16) and (7.3). If B_0 , B_l are arbitrary self-adjoint $k \times k$ -matrices, then on the set of functions from $L^2(0, l)$, $0 < l < \infty$ that satisfy the boundary conditions*

$$(7.17) \quad J_k Y_1(0) = B_0 Y_0(0), \quad J_k Y_1(l) = B_l Y_0(l),$$

the operator $(-iD)^p$ is self-adjoint.

Boundary conditions (7.17) refer to *separated boundary conditions*.

The second type of self-adjoint boundary conditions is obtained as follows. The non-integral terms of the Lagrange formula (7.11) have the form

$$\begin{aligned} Q_l(y, z) - Q_0(y, z) &= \langle J_p Y(l), Z(l) \rangle - \langle J_p Y(0), Z(0) \rangle \\ &= \langle Y_+(l), Z_+(l) \rangle - \langle Y_-(l), Z_-(l) \rangle - \langle Y_+(0), Z_+(0) \rangle + \langle Y_-(0), Z_-(0) \rangle. \end{aligned}$$

This expression is zero if

$$(7.18) \quad \begin{aligned} (1) \quad &\langle Y_+(l), Z_+(l) \rangle = \langle Y_+(0), Z_+(0) \rangle, \\ (2) \quad &\langle Y_-(l), Z_-(l) \rangle = \langle Y_-(0), Z_-(0) \rangle. \end{aligned}$$

Equalities (7.18) are satisfied when

$$\begin{aligned} Y_+(l) &= V Y_+(0) \quad (Z_+(l) = V Z_+(0)), \\ Y_-(l) &= \widetilde{V} Y_-(0) \quad (Z_-(l) = \widetilde{V} Z_-(0)), \end{aligned}$$

where V and $\tilde{V}$ are unitary operators. Using (7.12) and (7.9), we rewrite these equalities in the form

$$\begin{aligned} Y_0(l) + iJ_k Y_1(j) &= V(Y_0(0) + iJ_k Y_1(0)), \\ -Y_0(l) + iJ_k Y_1(j) &= \tilde{V}(-Y_0(0) + iJ_k Y_1(0)). \end{aligned}$$

Proposition 7.2. *Let $p = 2k$, $Y_0(x)$, $Y_1(x)$ and J_k be determined by the relations (7.16) and (7.3). If V , $\tilde{V}$ are arbitrary unitary operators in $E^{(k)}$, then on the set of functions from $L^2(0, l)$, $0 < l < \infty$ that satisfy the boundary conditions*

$$(7.19) \quad \begin{aligned} 2Y_0(l) &= (V + \tilde{V})Y_0(0) + i(V - \tilde{V})J_k Y_1(0), \\ 2iJ_k Y_1(l) &= (V - \tilde{V})Y_0(0) + i(V + \tilde{V})J_k Y_1(0), \end{aligned}$$

the operator $(-iD)^p$ is self-adjoint.

Remark 7.1. If $V = \tilde{V}$, then equalities (7.19) have the form

$$(7.20) \quad Y_0(l) = VY_0(0), \quad J_k Y_1(l) = VJ_k Y_1(0),$$

and if $V = -\tilde{V}$, then

$$(7.21) \quad Y_0(l) = iVJ_k Y_1(0), \quad J_k Y_1(l) = -iVY_0(0).$$

Boundary conditions (7.19) (as well as (7.20) and (7.21)) refer to the so-called *unseparated self-adjoint boundary conditions*.

Case $p = 2k + 1$. From (7.3), it follows that $J_p^* = J_p$ and $J_p^2 = I$. Therefore, the eigenvalues of J_p are equal to ± 1 . Orthonormal eigenvectors

$$(7.22) \quad \begin{aligned} \tilde{e}_1^+ &= \frac{1}{\sqrt{2}} \text{col}(1, 0, \dots, 0, 0, \dots, 0, 1) \\ \tilde{e}_2^+ &= \frac{1}{\sqrt{2}} \text{col}(0, 1, \dots, 0, 0, \dots, -1, 0) \\ &\dots \\ \tilde{e}_k^+ &= \frac{1}{\sqrt{2}} \text{col}(0, 0, \dots, 1, 0, (-1)^{k-1}, \dots, 0, 0) \end{aligned}$$

form eigen subspace $\tilde{E}_+^{(k)}$ ($\dim \tilde{E}_+^{(k)} = k$) of the operator $J_p = J_{2k+1}$ corresponding to the eigenvalue $\lambda = +1$, $J_{2k+1} \tilde{e}_s^+ = 1 \cdot \tilde{e}_s^+$ ($1 \leq s \leq k$). Likewise, the

orthonormal eigenvectors

$$\begin{aligned}
 \tilde{e}_1^- &= \frac{1}{\sqrt{2}} \text{col}(-1, 0, \dots, 0, 0, \dots, 0, 1) \\
 \tilde{e}_2^- &= \frac{1}{\sqrt{2}} \text{col}(0, -1, \dots, 0, 0, \dots, -1, 0) \\
 &\dots \\
 \tilde{e}_k^- &= \frac{1}{\sqrt{2}} \text{col}(0, 0, \dots, -1, 0, (-1)^{k-1}, \dots, 0, 0)
 \end{aligned}
 \tag{7.23}$$

form eigen subspace $\tilde{E}_-^{(k)}$ ($\dim \tilde{E}_-^{(k)} = k$) of the operator $J_p = J_{2k+1}$ corresponding to the eigenvalue $\lambda = -1$, $J_{2k+1}\tilde{e}_s^- = -1 \cdot \tilde{e}_s^-$ ($1 \leq s \leq k$). In addition, the operator J_{2k+1} has an eigenvector

$$\tilde{e}_{k+1} = \frac{1}{\sqrt{2}} \text{col}(0, 0, \dots, 0, 1, 0, \dots, 0, 0)
 \tag{7.24}$$

and $J_p \tilde{e}_{k+1} = (-1)^k \tilde{e}_{k+1}$.

The space $E^{(p)} = E^{(2k+1)}$ is the sum of three orthogonal subspaces

$$E^{(p)} = \tilde{E}_+^{(k)} \oplus \tilde{E}_-^{(k)} \oplus \{\mu \tilde{e}_{k+1}\}.$$

Therefore, any vector $h \in E^{(p)}$ has unique decomposition

$$h = h_+ + h_- + \tilde{h} \quad (h_{\pm} = P_{\pm} h, \tilde{h} \in \{\mu \tilde{e}_{k+1}\}),
 \tag{7.25}$$

where $P_{\pm}$ are orthoprojectors onto subspaces $\tilde{E}_{\pm}$ and

$$h_{\pm} = \sum_{s=1}^k \tilde{h}_s^{\pm} \tilde{e}_s^{\pm}, \quad \tilde{h} = \mu \tilde{e}_{k+1},
 \tag{7.26}$$

where

$$\tilde{h}_s^{\pm} = \frac{\pm h_s + (-1)^{p-s} h_{p-s+1}}{\sqrt{2}} \quad (1 \leq s \leq k).
 \tag{7.27}$$

Taking into account (7.25), we obtain

$$\langle J_p h, h \rangle = \langle \tilde{h}_+, \tilde{h}_+ \rangle - \langle \tilde{h}_-, \tilde{h}_- \rangle + (-1)^k |h_{k+1}|^2.
 \tag{7.28}$$

Similarly to (7.16), we introduce the notation

$$(7.29) \quad \begin{aligned} Y(x) &= \text{col}(y(x), \dots, y^{(k-1)}(x), y^{(k)}(x), y^{(k+1)}(x), \dots, y^{(p-1)}(x)), \\ Y_0(x) &= \text{col}(y(x), y'(x), \dots, y^{(k-1)}(x)) \in E^{(k)}, \\ Y_1(x) &= \text{col}(y^{(k+1)}(x), y^{(k+2)}(x), \dots, y^{(p-1)}(x)) \in E^{(k)}, \end{aligned}$$

Proposition 7.3. *Let $p = 2k + 1$, $Y_0(x)$, $Y_1(x)$ and J_k be determined by the relations (7.29) and (7.3). If $\theta \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and B_0, B_l are arbitrary self-adjoint $k \times k$ -matrices, then on the set of functions from $L^2(0, l)$, $0 < l < \infty$ that satisfy the boundary conditions*

$$(7.30) \quad y^{(k)}(l) = \theta y^{(k)}(0), \quad J_k Y_1(0) = i B_0 Y_0(0), \quad J_k Y_1(l) = i B_l Y_0(l),$$

the operator $(-iD)^p$ is self-adjoint.

Proof. From (7.28) it follows that

$$(7.31) \quad \begin{aligned} Q_l(y, z) - Q_0(y, z) &= \langle J_p Y(l), Z(l) \rangle - \langle J_p Y(0), Z(0) \rangle \\ &= \langle Y_+(l), Z_+(l) \rangle - \langle Y_-(l), Z_-(l) \rangle + (-1)^k y^{(k)}(l) z^{(k)}(l) \\ &\quad - \langle Y_+(0), Z_+(0) \rangle \langle -Y_-(0), Z_-(0) \rangle + (-1)^k y^{(k)}(0) z^{(k)}(0). \end{aligned}$$

This expression vanishes if

$$\langle Y_+(l), Z_+(l) \rangle = \langle Y_-(l), Z_-(l) \rangle, \quad y^{(k)}(l) = \theta y^{(k)}(0), \quad (\theta \in \mathbb{T})$$

and $\langle Y_+(0), Z_+(0) \rangle = \langle Y_-(0), Z_-(0) \rangle$.

Transform, for example, the first relation. We have

$$(7.32) \quad \langle h_+, h_+ \rangle - \langle h_-, h_- \rangle = 0.$$

Obviously, this relation is satisfied for vectors of the form

$$(7.33) \quad h = h_+ + V h_+,$$

where $V : \widetilde{E}_+^{(k)} \rightarrow \widetilde{E}_-^{(k)}$ is an arbitrary unitary operator. The equality $h_- = V h_+$ means that (see (7.12))

$$\widetilde{h}_1 + J_k \widehat{h}_1 = V_0(-\widetilde{h}_1 + J_k \widehat{h}_1),$$

where J_k has the form (7.3) and

$$\tilde{h}_1 = \text{col}(h_1, \dots, h_k), \quad \hat{h}_1 = \text{col}(h_{k+2}, \dots, h_{2k+1}).$$

This implies

$$(I - V_0)J_k\hat{h}_1 = -(I + V_0)\tilde{h}_1$$

or

$$J_k\hat{h}_1 = iB_0\tilde{h}_1,$$

where $B_0 = i(I - V_0)^{-1}(I + V_0)$ is a self-adjoint operator. $\square$

Relations (7.30) are analogous to the separated boundary conditions for $p = 2k + 1$.

If in equality (7.31) we put

$$\langle Y_+(l), Z_+(l) \rangle = \langle Y_+(0), Z_+(0) \rangle, \quad y^{(k)}(l) = \theta y^{(k)}(0), \quad (\theta \in \mathbb{T})$$

and $\langle Y_-(l), Z_-(l) \rangle = \langle Y_-(0), Z_-(0) \rangle$, then $Q_l(y, z) - Q_0(y, z) = 0$. Then, similarly to Proposition 7.2, we obtain the following statement.

Proposition 7.4. *Let $p = 2k + 1$, $Y_0(x)$, $Y_1(x)$ and J_k be determined by the relations (7.29) and (7.3). If V and $\tilde{V}$ are arbitrary unitary operators in $E^{(k)}$, then on the set of functions from $L^2(0, l)$, $0 < l < \infty$ that satisfy the boundary conditions*

$$(7.34) \quad \begin{aligned} y^{(k)}(l) &= \theta y^{(k)}(0) \quad (\theta \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}), \\ 2Y_0(l) &= (V + \tilde{V})Y_0(0) + (V - \tilde{V})J_kY_1(0), \\ 2J_kY_1(l) &= (V - \tilde{V})Y_0(0) + (V + \tilde{V})J_kY_1(0), \end{aligned}$$

the operator $(-iD)^p$ is self-adjoint.

Remark 7.2. If $\tilde{V} = V$, then conditions (7.34) take the form

$$Y_0(l) = VY_0(0), \quad J_kY_1(l) = VJ_kY_1(0), \quad y^{(k)}(l) = \theta y^{(k)}(0) \quad (\theta \in \mathbb{T}).$$

For $\tilde{V} = -V$ we obtain

$$Y_0(l) = VJ_kY_1(0), \quad J_kY_1(l) = -VY_0(0), \quad y^{(k)}(l) = \theta y^{(k)}(0) \quad (\theta \in \mathbb{T}).$$

8. Orthonormal bases. Let $W_2^3(0, l)$ be a linear manifold of thrice differentiable functions $y(x)$ from $L^2(0, l)$ ($0 < l < \infty$) satisfying the boundary conditions

$$(8.1) \quad y(0) = 0, \quad y'(0) = \theta y'(l), \quad y(l) = 0, \quad (\theta \in \mathbb{T}).$$

By Proposition 7.3, $L_\theta = (-iD)^3$ on $W_2^3(0, l)$ is a self-adjoint operator.

Find the eigenfunctions of the operator L_θ . The solution of the equation

$$(8.2) \quad L_\theta y(\lambda, x) = \lambda^3 y(\lambda, x)$$

that satisfies the first boundary condition (8.1) ($y(0) = 0$) has the form

$$(8.3) \quad y(\lambda, x) = a_1 \frac{s_1(i\lambda x)}{i\lambda} + a_2 \frac{s_2(i\lambda x)}{(i\lambda)^2}.$$

Since

$$y'(\lambda, x) = a_1 s_0(i\lambda x) + a_2 \frac{s_1(i\lambda x)}{i\lambda},$$

it follows from the boundary condition $y'(0) = \theta y'(l)$ that

$$(8.4) \quad a_1 = a_1 \theta s_0(i\lambda l) + a_2 \theta \frac{s_1(i\lambda l)}{i\lambda}.$$

Remark 8.1. Note that $\lambda = 0$ is not an eigenvalue of L_θ for $\theta \neq -1$. Indeed, if $\lambda = 0$, then $y(0, x) = y_0 + y_1 x + y_2 x^2$ follows from (8.2). Since $y(0) = 0$, then $y_0 = 0$. Considering $y(l) = 0$, we get $l(y_1 + y_2 l) = 0$. Therefore, $y(0, x) = y_2(x^2 - xl)$. Since $y'(0) = \theta y'(l)$, then $-y_2 l = \theta y_2 l$. From here $\theta = -1$. If $\theta = -1$, then $y(\lambda, x) = C(x^2 - xl)$ ($C \neq 0$) is the eigenfunction of the operator L_{-1} , corresponding to the eigenvalue $\lambda = 0$.

In the future, we assume that $\theta \neq -1$.

From (8.4) we get

$$a_1 = a_2 \frac{\theta s_1(i\lambda l)}{i\lambda(1 - \theta s_0(i\lambda l))}.$$

Therefore, from (8.3) we obtain

$$(8.5) \quad y(\lambda, x) = \frac{a_2 [\theta s_1(i\lambda x) s_1(i\lambda l) + s_2(i\lambda x)(1 - \theta s_0(i\lambda l))]}{(i\lambda)^2(1 - \theta s_0(i\lambda l))}$$

The third boundary condition $y(l) = 0$ takes the form

$$(8.6) \quad \frac{\theta s_1(i\lambda l)^2 + s_2(i\lambda l)(1 - \theta s_0(i\lambda l))}{1 - \theta s_0(i\lambda l)} = 0$$

Lemma 8.1. *If $1 - \theta s_0(i\lambda l) = 0$ ($|\theta| = 1$), then the boundary condition (8.6) is not satisfied.*

Proof. If (8.6) is true for $s_0(i\lambda l) = \bar{\theta}$, then $s_1(i\lambda l) = 0$. From the main identity (4.5) we get

$$\bar{\theta}^3 + s_2(i\lambda l)^3 = 1 \quad \Rightarrow \quad s_2(i\lambda l) = (1 - \bar{\theta}^3)^{1/3}.$$

According to the Euler formula (4.4) we get

$$e^{i\lambda l} = \bar{\theta} + (1 - \bar{\theta}^3)^{1/3}.$$

Since $e^{i\lambda l}e^{-i\lambda l} = 1$, then

$$\frac{\theta}{(1 - \theta^3)^{1/3}} + \frac{\bar{\theta}}{(1 - \bar{\theta}^3)^{1/3}} = -1.$$

Therefore, there exists $b \in \mathbb{R}$ such that

$$(8.7) \quad \frac{\theta}{(1 - \theta^3)^{1/3}} = -\frac{1}{2} + i\frac{b}{2}.$$

Show that (8.7) is impossible for $|\theta| = 1$. Introducing the notation

$$\left(-\frac{1}{2} + i\frac{b}{2}\right)^3 = \frac{-1 + 3b^2}{8} + i\frac{3b - b^3}{8} = z = x + iy,$$

from (8.7) we obtain

$$\theta^3 = z(1 - \theta^3) \quad \Rightarrow \quad \theta^{-3} = \left(1 + \frac{x}{x^2 + y^2}\right) - i\frac{y}{x^2 + y^2}.$$

Condition $|\theta^{-3}| = 1$ is satisfied only when $x = -\frac{1}{2}$. Therefore, for b we obtain the equation

$$\frac{-1 + 3b^2}{8} = -\frac{1}{2} \quad \Leftrightarrow \quad b^2 + 1 = 0,$$

which has no real solutions. Contradiction. $\square$

We call the function

$$(8.8) \quad \Delta_\theta(\lambda) = \frac{1}{\lambda^2} [\theta s_1(i\lambda l)^2 + s_2(i\lambda l)(1 - \theta s_0(i\lambda l))]$$

the characteristic function of the operator L_θ .

Remark 8.2. From the second and third boundary conditions (8.1) for the function $y(\lambda, x)$ (8.3) we obtain a system of equations

$$(8.9) \quad \begin{cases} a_1(1 - \theta s_0(i\lambda l)) - a_2\theta s_1(i\lambda l)/i\lambda = 0 \\ a_1 s_1(i\lambda l)/i\lambda + a_2 s_2(i\lambda l)/(i\lambda)^2 = 0 \end{cases}$$

for a_1 and a_2 . The existence of nontrivial solutions for this system is equivalent to the vanishing of its determinant, that is, the characteristic function (8.8).

Taking into account (4.11), we rewrite the characteristic function in the form

$$(8.10) \quad \Delta_\theta(\lambda) = \frac{1}{\lambda^2} [s_2(i\lambda l) + \theta s_2(-i\lambda l)].$$

Lemma 8.2. *The characteristic function (8.10) is an entire function of exponential type that satisfies the conditions*

$$(8.11) \quad \begin{aligned} \text{(i)} \quad & \overline{\Delta_\theta(\bar{\lambda})} = \Delta_{\bar{\theta}}(-\bar{\lambda}) = \bar{\theta} \Delta_\theta(\bar{\lambda}), \\ \text{(ii)} \quad & \Delta_\theta(-\lambda) = \theta \Delta_{\bar{\theta}}(\lambda), \\ \text{(iii)} \quad & \Delta_\theta(\lambda \zeta_j) = \Delta_\theta(\lambda), \quad j = 0, 1, 2. \end{aligned}$$

If $\theta \neq -1$, then zeros of the function $\Delta_\theta(i\lambda l)$ are simple and have the form

$$(8.12) \quad \mathcal{Z}_0 = \{ \zeta_1^s \lambda_n(\varphi), \quad -\zeta_1^s \lambda_n(-\varphi), \quad s = 0, 1, 2, \quad n \in \mathbb{N} \},$$

where $\varphi = \arg \theta$ ($|\varphi| < \pi$), $\{\lambda_n(\varphi)\}_0^\infty$ ($\{-\lambda_n(-\varphi)\}_0^\infty$) are the positive (negative) zeros of $\Delta_\theta(\lambda)$, numbered in ascending (descending) order.

Zeros $\lambda_n(\varphi)$ and $-\lambda_n(-\varphi)$ belong to the interval

$$\left(\frac{\pi}{2}(2n-1) - \varphi, \frac{\pi}{2}(2n+1) - \varphi \right), \left(-\frac{\pi}{2}(2n+1) + \varphi, -\frac{\pi}{2}(2n-1) + \varphi \right).$$

The following asymptotics exist,

$$(8.13) \quad \lambda_n(\varphi) = \frac{2\pi n}{l} - \varphi - \frac{\pi}{3l} + o\left(\frac{1}{n}\right) \quad (n \rightarrow \infty)$$

$$-\lambda_n(-\varphi) = -\frac{2\pi n}{l} + \varphi + \frac{\pi}{3l} + o\left(\frac{1}{n}\right) \quad (n \rightarrow \infty).$$

Proof. Identities (8.11) follow from Definition 3.1, Proposition 3.2 (v). Using the Taylor series (3.8) it is easy to show that

$$\Delta_\theta(\lambda)|_{\lambda=0} = -l^2(1+\theta)/2 \neq 0 \quad \text{for } \theta \neq -1.$$

First, find the zeros lying on the lines $l_k = \{z \in \mathbb{C} \mid z = x\zeta_k, x \in \mathbb{R}\}$, $k = 0, 1, 2$. Later we will show that there are no other zeros.

It follows from (8.11) (ii), (iii) that it suffices to find the zeros $\lambda_n(\varphi)$ of the function $\Delta_\theta(\lambda)$ that lie on $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$.

Using $\theta = e^{i\varphi}$, we represent the characteristic function in the form

$$\Delta_\theta(\lambda) = \frac{2e^{i\varphi/2}}{3\lambda^2} \sum_{k=0}^2 \zeta_k \cos\left(\lambda\zeta_k l - \frac{\varphi}{2}\right).$$

Since $\theta \neq -1$, then the equation $\Delta_\theta(\lambda) = 0$ is equivalent to the equation

$$\sum_{k=0}^2 \zeta_k \cos\left(\lambda\zeta_k l - \frac{\varphi}{2}\right) = 0.$$

From this we get

$$\begin{aligned} \cos\left(\lambda l - \frac{\varphi}{2}\right) - \cos\left(\frac{\lambda l + \varphi}{2}\right) \cosh \frac{\lambda l \sqrt{3}}{2} \\ - \sqrt{3} \sin\left(\frac{\lambda l + \varphi}{2}\right) \sinh \frac{\lambda l \sqrt{3}}{2} = 0, \end{aligned}$$

or

$$\begin{aligned} (8.14) \quad \cos\left(\frac{\lambda l + \varphi}{2}\right) \left(\cos\left(\frac{\lambda l}{2} - \varphi\right) - \cosh \frac{\lambda l \sqrt{3}}{2} \right) \\ = \sin\left(\frac{\lambda l + \varphi}{2}\right) \left(\sin\left(\frac{\lambda l}{2} - \varphi\right) + \sqrt{3} \sinh \frac{\lambda l \sqrt{3}}{2} \right). \end{aligned}$$

Note that if $\varphi \in (0, \pi)$, then

$$\Phi(x) := \sin(x - \varphi) + \sqrt{3} \sinh x \sqrt{3} > 0 \quad \text{for } x > 0.$$

If $\varphi \in (-\pi, 0)$, then $\Phi(x)$ has a simple zero x_0 lying in $(0, \pi)$. In this case $\Phi(x) < 0$, for $x \in (0, x_0)$, and $\Phi(x) > 0$, for $x \in (x_0, \infty)$.

Let x_0 be zero of $\Phi(x)$ ($\varphi \in (-\pi, 0)$). Then (8.14) implies $\cos(x_0 + \varphi/2) = 0$. Therefore $\lambda_0(\varphi) = (\pi - \varphi)/2$ is the zero of the characteristic function $\Delta_\theta(\lambda)$. For $\lambda > \lambda_0(\varphi)$, the inequality $\sin\left(\frac{\lambda l}{2} - \varphi\right) + \sqrt{3} \sinh \frac{\lambda l \sqrt{3}}{2} > 0$ holds, and (8.14) is equivalent to the equation

$$(8.15) \quad \tan\left(\frac{\lambda l + \varphi}{2}\right) = f(\lambda, \varphi),$$

where

$$(8.16) \quad f(\lambda, \varphi) = \frac{\cos\left(\frac{\lambda l}{2} - \varphi\right) - \cosh\left(\frac{\lambda l \sqrt{3}}{2}\right)}{\sin\left(\frac{\lambda l}{2} - \varphi\right) + \sqrt{3} \sinh\left(\frac{\lambda l \sqrt{3}}{2}\right)} \quad (\lambda > \lambda_0(\varphi)).$$

The derivative with respect to λ of this function is

$$f'(\lambda, \varphi) = l \frac{1 - \cos\left(\frac{\lambda l}{2} - \varphi\right) \cosh \frac{\lambda l \sqrt{3}}{2} - \sqrt{3} \sin\left(\frac{\lambda l}{2} - \varphi\right) \sinh \frac{\lambda l \sqrt{3}}{2}}{\left(\sin\left(\frac{\lambda l}{2} - \varphi\right) + \sqrt{3} \sinh\left(\frac{\lambda l \sqrt{3}}{2}\right)\right)^2}$$

Since the inequalities

$$\cos\left(\frac{\lambda l}{2} - \varphi\right) < \cosh \frac{\lambda l \sqrt{3}}{2}, \quad \sin\left(\frac{\lambda l}{2} - \varphi\right) < \sinh \frac{\lambda l \sqrt{3}}{2}$$

hold for $\lambda > \lambda_0(\varphi)$, then

$$f'(\lambda, \varphi) < l \frac{1 - \cos^2\left(\frac{\lambda l}{2} - \varphi\right) - \sqrt{3} \sin^2\left(\frac{\lambda l}{2} - \varphi\right)}{\left(\sin\left(\frac{\lambda l}{2} - \varphi\right) + \sqrt{3} \sinh\left(\frac{\lambda l \sqrt{3}}{2}\right)\right)^2} < 0.$$

Therefore, $f(\lambda, \varphi)$ monotonically decreases on the semi-axis $(\lambda_0(\varphi), \infty)$ and $f(\lambda, \varphi) \rightarrow -\frac{1}{\sqrt{3}}$ as $\lambda \rightarrow \infty$. Then the equation (8.15) for $\lambda > \lambda_0(\varphi)$ on each of the intervals

$$\left(\frac{\pi}{l}(2n-1) - \varphi, \frac{\pi}{l}(2n+1) - \varphi\right) \quad (n \in \mathbb{N})$$

has a single simple root

$$\lambda_n(\varphi) = \frac{2\pi n}{l} - \varphi - \varepsilon_n \quad \left(0 < \varepsilon_n < \frac{\pi}{l}\right).$$

Asymptotics (8.13) follows from (8.15) and the relation

$$f(\lambda, \varphi) \rightarrow -\frac{1}{\sqrt{3}} \quad (\lambda \rightarrow \infty).$$

Finally, note that $\Delta_\theta(\lambda)$ has no zeros outside the lines $l_s = \{z \in \mathbb{C} \mid z = x\zeta_1^s, x \in \mathbb{R}\}$ ($s = 0, 1, 2$). Indeed, if $w \notin \bigcup l_s$ is zero, then the equation (8.9) has a non-zero solution. Then there is an eigenfunction with complex eigenvalue $w^3 \notin \mathbb{R}$, which is impossible, since the operator L_θ is self-adjoint. $\square$

The eigenfunctions of the operator L_θ have the form

$$(8.17) \quad u(\mu_n, x) = \frac{\theta s_1(i\mu_n x) s_1(i\mu_n l) + s_2(i\mu_n x)(1 - \theta s_0(i\mu_n l))}{a_n(\mu_n)},$$

where $\mu_n \in \mathcal{Z}_0$ (see (8.12)) and the numbers $a_n(\mu_n)$ are chosen from the normalization condition $\|u(\mu_n, x)\|_{L^2(0,l)} = 1$.

Corollary 8.1. *On the domain $D \subset L^2(0, l)$ of the operator L_θ , the functions (8.17) form an orthonormal basis.*

Remark 8.3. The boundary conditions

$$(8.18) \quad y(0) = ih_0 y''(0), \quad y'(0) = \theta y'(l), \quad y(l) = ih_l y''(l), \quad (h_j \in \mathbb{R}, \theta \in \mathbb{T})$$

for the operation $(-iD)^3$ also generate a self-adjoint operator L_{θ, h_0, h_l} . Its eigenfunctions also form an orthonormal basis on the corresponding domain of the operator L_{θ, h_0, h_l} .

9. Operator resolvent. Calculate the resolvent $R_{L_\theta}(\lambda^3) = (L_\theta - \lambda^3 I)^{-1}$.

Let $y = R_{L_\theta}(\lambda^3)f$, i. e., $L_\theta y = \lambda^3 y + f$. By Lemma 6.1, $y(\lambda, x)$ has the form

$$(9.1) \quad y(\lambda, x) = y_1 \frac{s_1(i\lambda x)}{i\lambda} + y_2 \frac{s_2(i\lambda x)}{(i\lambda)^2} + i \int_0^x \frac{s_2(i\lambda(x-t))}{\lambda^2} f(t) dt$$

and satisfies the boundary condition $y(\lambda, 0) = 0$. Since

$$y'_x(\lambda, x) = y_1 s_0(i\lambda x) + y_2 \frac{s_1(i\lambda x)}{i\lambda} - \int_0^x \frac{s_1(i\lambda(x-t))}{\lambda} f(t) dt,$$

then from the boundary conditions $y'(0) = \theta y'(l)$, $y(l) = 0$ we obtain the system of equations

$$\begin{cases} y_1(\theta s_0(i\lambda l) - 1) + y_2 \theta \frac{s_1(i\lambda l)}{i\lambda} = \theta \int_0^l \frac{s_1(i\lambda(l-t))}{\lambda} f(t) dt \\ y_1 \frac{s_1(i\lambda l)}{i\lambda} + y_2 \frac{s_2(i\lambda l)}{(i\lambda)^2} = -i \int_0^l \frac{s_2(i\lambda(l-t))}{\lambda^2} f(t) dt. \end{cases}$$

The determinant of this system coincides with the characteristic function $\Delta_\theta(\lambda)$ (see 8.8). Then

$$\begin{aligned} y_1 &= \frac{\theta}{\lambda^3 \Delta_\theta(\lambda)} \int_0^l [s_2(i\lambda(l-t))s_1(i\lambda l) - s_1(i\lambda(l-t))s_2(i\lambda l)] f(t) dt, \\ y_2 &= \frac{i}{\lambda^2 \Delta_\theta(\lambda)} \int_0^l [s_2(i\lambda(l-t))(1 - \theta s_0(i\lambda l)) + \theta s_1(i\lambda(l-t))s_1(i\lambda l)] f(t) dt. \end{aligned}$$

Then (9.1) becomes

$$\begin{aligned} (9.2) \quad y(\lambda, x) &= \frac{i}{\lambda^4 \Delta_\theta(\lambda)} \\ &\times \left\{ \theta s_1(i\lambda x) \int_0^l [s_1(i\lambda(l-t))s_2(i\lambda l) - s_2(i\lambda(l-t))s_1(i\lambda l)] f(t) dt \right. \\ &- s_2(i\lambda l) \int_0^l [s_2(i\lambda(l-t))(1 - \theta s_0(i\lambda l)) + \theta s_1(i\lambda(l-t))s_1(i\lambda l)] f(t) dt \\ &\left. + \int_0^x s_2(i\lambda(x-t)) [s_2(i\lambda l) + \theta s_2(-i\lambda l)] f(t) dt \right\}. \end{aligned}$$

To simplify this expression, we need the lemma

Lemma 9.1. *Let $s_k(z)$ ($k = 0, 1, 2$) be 3-hyperbolic functions. Then*

$$\begin{aligned} (9.3) \quad & s_1(x)s_1(l-t)s_2(l) - s_1(x)s_2(l-t)s_1(l) \\ & + s_2(x)s_2(l-t)s_0(l) - s_2(x)s_1(l-t)s_1(l) \\ & = s_2(-t)s_2(x-l) - s_2(x-t)s_2(-l) \end{aligned}$$

for any $x, l, t \in \mathbb{C}$.

Proof. Using (4.6), we transform the left side of (9.3):

$$A = s_1(x)s_2(l) \left[s_0(l)s_1(-t) + \underline{s_1(l)s_0(-t)} + s_2(l)s_2(-t) \right]$$

$$\begin{aligned}
& -s_1(x)s_1(l) \left[\underline{s_0(l)s_2(-t)} + s_1(l)s_1(-t) + s_2(l)s_0(-t) \right] \\
& + s_2(x)s_0(l) \left[s_0(l)s_2(-t) + \underline{\underline{s_1(l)s_1(-t)}} + s_2(l)s_0(-t) \right] \\
& - s_2(x)s_1(l) \left[\underline{\underline{s_0(l)s_1(-t)}} + s_1(l)s_0(-t) + s_2(l)s_2(-t) \right] \\
& = s_2(-t) \left[s_1(x) (s_2(l)^2 - s_0(l)s_1(l)) + s_2(x) (s_0(l)^2 - s_1(l)s_2(l)) \right] \\
& + s_1(-t)s_1(x) (s_0(l)s_2(l) - s_1(l)^2) + s_0(-t)s_2(x) (s_0(l)s_2(l) - s_1(l)^2).
\end{aligned}$$

Using (4.11), we obtain

$$\begin{aligned}
A &= s_2(-t) [s_1(x)s_1(-l) + s_2(x)s_0(-l)] - \\
& s_1(-t)s_1(x)s_2(-l) - s_0(-t)s_2(x)s_2(-l) = \\
& s_2(-t) [s_1(x)s_1(-l) + s_2(x)s_0(-l)] - \\
& s_2(-l) [s_1(-t)s_1(x) + s_0(-t)s_2(x)].
\end{aligned}$$

Applying (4.6) again, we get

$$A = s_2(-t)s_2(x-l) - s_2(x-t)s_2(-l).$$

□

Applying the identity (9.3) to the integrand in (9.2), we obtain

$$\begin{aligned}
(9.4) \quad y(\lambda, x) &= \frac{i}{\lambda^4 \Delta_\theta(\lambda)} \left\{ \int_0^l [\theta s_2(i\lambda(x-l))s_2(-i\lambda t) - \theta s_2(i\lambda(x-t))s_2(-i\lambda l) \right. \\
& \quad \left. - s_2(i\lambda x)s_2(i\lambda(l-t))] f(t) dt \right. \\
& \quad \left. + \int_0^x s_2(i\lambda(x-t))[s_2(i\lambda l) + \theta s_2(-i\lambda l)] f(t) dt \right\}.
\end{aligned}$$

Theorem 9.1. *The resolvent of the operator L_θ has the form*

$$\begin{aligned}
(9.5) \quad (R_{L_\theta}(\lambda^3)f)(x) &= \frac{i}{\lambda^4 \Delta_\theta(\lambda)} \\
& \times \left\{ \int_0^x s_2(i\lambda(x-t))s_2(i\lambda l) f(t) dt - \theta \int_x^l s_2(i\lambda(x-t))s_2(-i\lambda l) f(t) dt \right\}
\end{aligned}$$

$$+ \int_0^l [\theta s_2(i\lambda(x-l))s_2(-i\lambda t) - s_2(i\lambda x)s_2(i\lambda(l-t))] f(t)dt \Big\},$$

where $\Delta_\theta(\lambda)$ is determined by the relation (8.10).

Formula (9.5) allows one to find spectral projections onto eigensubspaces of L_θ as a residue of the resolvent at the zeros of the characteristic function $\Delta_\theta(\lambda)$.

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