Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **50** (2024), 103–124

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

A SURVEY ON PLURICLOSED AND CYT METRICS

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Communicated by J. Davidov

Abstract. A Hermitian metric on a complex manifold is called pluriclosed if the torsion of the associated Bismut connection is closed, and it is said to be CYT if the Bismut Ricci form vanishes. In this paper, we survey recent results on pluriclosed and CYT metrics and review some constructions of compact non-Kähler manifolds.

1. Introduction. In the last decade many of the important questions in Kähler geometry were answered. Together with the influence from the theoretical physics and the advances in the analysis of the nonlinear partial differential equations, this led to a growing interest in the non-Kähler complex geometry. When a compact complex manifold does not admit a Kähler metric, there isn't a canonical choice of a Hermitian structure. There are several generalizations of the Kähler condition, and most of the research in the area is focused on studying properties and relations between different type of non-Kähler metrics.

https://doi.org/10.55630/serdica.2024.50.103-124

2020 *Mathematics Subject Classification:* 53C55, 53C05, 22E25, 53C30, 53C44.
Key words: Bismut connection, Hermitian metric, pluriclosed metric, CYT metric.
Anna Fino is partially supported by Project PRIN 2022 "Geometry and Holomorphic Dynamics", GNSAGA (Indam) and by a grant from the Simons Foundation (#944448). Gueo Grantcharov is partially supported by a grant from the Simons Foundation (#853269).

This survey is based on results by the authors on certain special Hermitian metrics. We focus on pluriclosed (called also SKT), Calabi-Yau with torsion (CYT), Bismut Kähler like, and Bismut Hermitian Einstein (BHE). We also mention relations with the pluriclosed flow and recent results on Hermitian metrics with parallel Bismut torsion. The paper is complementary to the survey [22] where balanced metrics and their relations with the Hull-Strominger system and pluriclosed metrics are considered. Here we present the results and refer to the original papers for the proofs. Large part of the survey concerns particular examples, and we separate in Section 2 the main constructions which are used in the remaining sections.

In Section 3, we review the main results on pluriclosed metrics and their relation with the pluriclosed flow defined by Streets and Tian [69]. In Section 4, we survey briefly the Bismut Kähler like conditions and mention their relations to the pluriclosed flow properties as illustrated in a couple of examples - the Kodaira-Thurston surface and invariant solutions on 2-step nilpotent Lie groups. In Sections 5 and 6, we consider some curvature properties of the Bismut (also called Strominger) connection. More precisely, we review the known results about CYT and BHE metrics, and mention the result about the local structure of such spaces under the additional condition of parallel Bismut torsion (see the recent preprint [14]).

At the end, we note that for the results about special Hermitian metrics which are not included in the present survey, apart from [22, 34], we refer the reader to [62] for the locally-conformally Kähler case, and to [80] for the current program about the metrics with parallel Bismut torsion. We also realize that we do not exhaust all results in the area of this survey, and apologize for any omissions.

2. Constructions of complex non-Kähler manifolds and foliations. Apart from the standard geometric constructions leading to new spaces, which are know from Algebraic Geometry, like modifications and deformations for example, there are few others and we mention here some of the main ones for which the study of specific metrics is feasible. They are all related to additional geometric properties of the underlying smooth manifolds. In this Section. we briefly discuss three constructions for which the existence of special metrics will be considered in the rest of the survey.

Definition 2.1. *A* ***complex homogeneous space*** *is a complex manifold admitting a transitive action of a complex Lie group of biholomorphic transforma-*

tions. A complex locally homogeneous space is a complex manifold which has a cover that is homogeneous.

The most studied examples of compact complex homogeneous spaces are the generalized flag manifolds or rational homogeneous spaces which are represented as $M = G/K$, where G is a compact Lie group and K is a subgroup of maximal rank which is a centralizer of a torus in G. The basic examples of locally homogeneous spaces are given by compact quotients of nilpotent and solvable Lie groups, which are called nilmanifolds and solvmanifolds respectively. The compact complex homogeneous spaces admit a fibration, called a Tits fibration - it admits a projection onto a generalized flag manifold with fibers a complex solvmanifolds. Under an additional assumption that the space admits also an invariant volume form, the structure of the Tits fibration could be clarified:

Theorem 2.1 ([48]). *Every compact complex homogeneous space M with an invariant volume form is a principal homogeneous complex torus bundle*

$$\pi : M \to G/K \times D$$

over the product of a projective rational homogeneous space and a complex parallelizable manifold.

Examples which do not admit an invariant volume are used in [68] to show that the automorphism group of a compact complex manifold can grow exponentially with the dimension.

A particular case is when the complex homogeneous space admits a transitive compact Lie group action. If G s compact Lie group, the homogeneous space G/H for a closed subgroup H admits a complex structure if it is even-dimensional. When their fundamental group is finite, such spaces are called C-spaces by C. Wang [75] and their Tits fibration has torus as a fiber. An important example of C-spaces is due to Samelson [72], who constructed explicitly a complex structure on any even-dimensional compact Lie group G, which depends on the choice of a maximal Abelian subalgebra of the Lie algebra $\mathfrak{g}$ of G. The fibration structure of the compact complex manifolds can be generalized to the non-homogeneous spaces. The following is a result of [65] which generalizes [46]:

Theorem 2.2. *Suppose that $\pi : P \to M$ is a principal bundle with compact even-dimensional structure group G and compact complex base manifold M. If it has a connection with curvature of type (1,1), then P admits a complex structure with π a holomorphic map. If the connection 1-form is $\theta : \Lambda^1(P) \to Lie(G)$, the complex structure I is defined as the pull-back of the complex structure on M on $Ker(\theta)$ and the Samelson's complex structure on the $Ker(\pi)$.*

In fact, this construction can be generalized to an associated to P fiber bundle with complex fiber. An advantage of Theorem 2.2 is that it produces spaces of vanishing first Chern class when $\pi^*(c_1(M)) = 0 \in H^2(P,\mathbb{R})$. When G is a torus, the curvature condition can be relaxed to the topological requirement that the characteristic classes of the bundle are of type (1,1). Such construction has first appeared in Calabi–Eckmann paper [15]. The manifolds P are often called generalized Calabi–Eckmann.

The third type of spaces which we'll consider in this survey are related to the topological suspension construction. The study of the complex geometry of such spaces has started in [67] and the metric properties have been investigated in [26, 13, 14]. We recall here a definition from [26]:

Definition 2.2. *Let M be a complex manifold, and $f_1, \dots, f_{2k}$ a set of commuting holomorphic automorphisms of M. Let $\Lambda \subset \mathbb{C}^k$ be a lattice of rank $2k$, generated by $\xi_1, \dots, \xi_{2k}$. Define an action of $\mathbb{Z}^{2k} = \langle \phi_1, \dots, \phi_{2k} \rangle$ on $M \times \mathbb{C}^k$ via $\phi_j(m,z) = (f_j(m), z + \xi_j)$. In other words, $\mathbb{Z}^{2k}$ acts on $\mathbb{C}^k$ as a shift by the corresponding element ξ_j of Λ and on M by the automorphisms f_j. The quotient $(M \times \mathbb{C}^k)/\mathbb{Z}^{2k}$ is called* ***the toric suspension*** *of $(M, f_1, \dots, f_{2k})$.*

We note here that the toric suspension spaces are non-Kähler whenever f_i are not preserving a Kähler class [26], [60]. In the special case when $k = 1$ and M is a Kummer surface, it was shown in [67] that the corresponding toric suspensions are formal in the sense of Sullivan and have trivial canonical bundle. In [26], we showed that the toric suspensions carry balanced metrics as long as M is balanced and do not admit a $k-$pluriclosed metric.

A construction of the type $M \times \mathbb{R}/(f)$ where $f(x,a) = (\phi(x), a + \xi)$ for a diffeomorphism ϕ of M is an example of the so called **mapping torus**. In some cases for an odd-dimensional M such mapping tori carries a complex structure. Constructions of this type using mapping tori of products $M \times S^3$ were studied in [14]. We review their properties in Section 6.

3. Pluriclosed metrics.

The pluriclosed (or SKT) condition is essentially the only weakening of the Kähler condition which is linear in the fundamental form.

Definition 3.1. *A Hermitian metric g on a complex manifold (M^{2n}, J) is called* ***pluriclosed*** *(or* ***SKT****) if*

$$dd^c\omega = 0,$$

where $d^c = -J^{-1}dJ = -i(\overline{\partial} - \partial)$. Note that $dd^c = -2i\partial\overline{\partial}$.

There is a natural class of Hermitian metrics which exist on all compact complex manifolds, called the Gauduchon metrics, defined by the condition $\partial\overline{\partial}\omega^{n-1} = 0$. The existence on every compact complex manifold follows from the following

Theorem 3.1 ([44]). *Let (M^{2n}, g, J) be a compact Hermitian manifold of complex dimension n. Then there exists a unique $u \in \mathcal{C}^\infty(M^{2n})$ such that*

$$\partial\overline{\partial}(e^{2u}\omega)^{n-1} = 0, \qquad \int_{M^{2n}} u\, dV_g = 0.$$

As a consequence, every conformal Hermitian structure on a compact complex manifold (M^{2n}, J) contains a Gauduchon metric, i.e. a Hermitian metric $\tilde{\omega}$ such that $\partial\overline{\partial}\tilde{\omega}^{n-1} = 0$. In particular, every compact complex surface admits pluriclosed metrics.

On every Hermitian manifold, there is a natural one-parameter family of Hermitian connections joining the Chern connection and the Bismut connection. More precisely,

Theorem 3.2 ([41]). *On any Hermitian manifold (M^{2n}, J, g), there exists an affine line of canonical Hermitian connections ∇^t (i.e. such that $\nabla^t J = 0$ and $\nabla^t g = 0$), completely determined by their torsion*

$$T(X, Y, Z) := g(T(X, Y), Z).$$

The family includes the Chern connection ∇^C, whose torsion T^C has trivial $(1,1)$-component and the Bismut (or Strominger) connection ∇^B, whose torsion T^B is a 3-form.

The connections ∇^B and ∇^C are related to the Levi-Civita connection ∇^{LC} by

$$g(\nabla^B_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} d^c\omega(X, Y, Z),$$

$$g(\nabla^C_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} d\omega(JX, Y, Z),$$

for every vector fields X, Y, Z on M. Therefore g is pluriclosed if and only if $dT^B = 0$.

Example 3.1 (Samelson spaces)**.** Basic examples of complex manifolds admitting a pluriclosed metric are given by $(G \times \mathbb{R}^k, J, g)$ with G a compact semisimple Lie group and J a left-invariant complex structure compatible with a bi-invariant metric g.

In this case, the torsion Bismut 3-form T^B is given by $T^B(X,Y,Z) = g([X,Y],Z)$ and the Bismut connection ∇^B is flat.

By the following result Bismut-flat Hermitian manifolds are indeed covered by Samelson spaces.

Theorem 3.3 ([76]). *The local Samelson spaces (G,J,g) are the only compact Bismut flat manifolds.*

Examples of non-Bismut flat pluriclosed manifolds are provided by compact nilmanifolds and solvmanifolds (see for instance [30, 20, 8, 59, 36, 28, 4, 63, 7, 29]). However, the existence of pluriclosed metrics on a non-Kähler C-space is quite restrictive and we have the following:

Theorem 3.4 ([27]). *Every non-Kähler C-space M admitting a pluriclosed metric is (up to a finite cover) the product of a compact Lie group and a generalized flag manifold.*

On the other side, a characterization of the existence of pluriclosed metrics on Oeljekelaus-Toma (OT) manifolds $X(K,U) := \mathbb{H}^s \times \mathbb{C}^t / U \times \mathcal{O}_K$, where $K \supseteq \mathbb{Q}$ is an algebraic number field, $\mathcal{O}_K$ is the ring of algebraic integers of K and U is an admissible subgroup of the group of totally positive units $\mathcal{O}^{*,+}$, is given in [63].

By [46], the manifold $\#_{k-1}(S^2 \times S^4)\#_k(S^3 \times S^3)$, for any positive integer $k \geq 1$, admits a pluriclosed Hermitian structure. By [45], they are all 6-dimensional simply connected compact spin manifold with torsion free cohomology and free S^1-action. The manifold $\#_{k-1}(S^2 \times S^4)\#_k(S^3 \times S^3)$ can also be described as total spaceof a T^2-bundle over the blow-up of $\mathbb{CP}^2$ at k ($k \geq 2$) points on a smooth irreducible cubic. More generally, the existence of pluriclosed metrics on the total spaces E of principal bundles over a projective manifold M with structure group an even dimensional unitary, special orthogonal or compact symplectic Lie group have been investigated in [65]. In both cases the complex structure arises from Theorem 2.2.

As in the Kähler case, the complex blow-up preserves the existence of pluriclosed metrics and can be used for resolution of orbifolds with pluriclosed metrics [33].

The existence of a pluriclosed metric on a compact complex manifold M can be characterized in terms of currents in the following way: M admits a pluriclosed metric if and only if there is no non-zero positive current of bi-dimension $(1,1)$ which is $\partial\overline{\partial}$-exact (see for instance [19]). We recall that a current of bi-dimension (p,q) on (M,J) can be locally identified with a $(n-p,n-q)$-form on M with coefficients distributions and if ω is the fundamental form of a pluriclosed metric, then ω corresponds to a real strictly positive current of bi-degree $(1,1)$, which is $\partial\overline{\partial}$-closed.

A result of Miyaoka [32] asserts that if a (compact) complex manifold M admits a Kähler metric in the complement of a point, then M is itself Kähler. By using the extension result of [1] about positive or negative plurisubharmonic currents, it has been proved in [33] an analogous result of the Miyaokas one for pluriclosed metrics in complex dimension $n \geq 2$. More precisely,

Theorem 3.5 ([33]). *Let (M^{2n}, J) be a complex manifold of complex dimension $n \geq 2$. If $M^{2n} \setminus \{p\}$ admits a pluriclosed metric, then there exists a pluriclosed metric on the complex manifold M^{2n}.*

The pluriclosed condition is in general not stable under small deformations of the complex structure as shown by the following example.

Example 3.2 ([33]). The Iwasawa manifold $\Gamma\backslash H_3^{\mathbb{C}}$, where

$$H_3^{\mathbb{C}} = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_j \in \mathbb{C} \right\}$$

and Γ is the lattice of $H_3^{\mathbb{C}}$ defined by taking z_1, z_2, z_3 to be Gaussian integers, has a family $J_{t,s}$, $t, s \in \mathbb{R}$, $s \neq 0$, such that $J_{1,1}$ has a compatible pluriclosed metric, but for $t \neq s \neq 1$, there exist no compatible pluriclosed metrics.

Necessary conditions under which the property of being pluriclosed is stable for a smooth curve of Hermitian metrics $\{\omega_t\}$ which equals a fixed pluriclosed metric ω_0 for $t = 0$ along a differentiable family of complex manifolds $\{M_t\}$ have been recently found in [64].

Looking at the interplay of the pluriclosed condition with other types of Hermitian metrics, it is well known that a Hermitian metric which is pluriclosed and balanced is Kähler [2, 66], but the following conjecture is still open.

Conjecture 3.1. *Every compact complex manifold M admitting a balanced and a pluriclosed metric is Kähler.*

The conjecture has been until confirmed for all the known examples of compact balanced manifolds.

Remark 3.1. If M is non-compact two counterexamples for the natural generalization of this conjecture in the homogeneous invariant setting have been constructed by Freibert and Swann [36].

We recall that a Hermitian metric on a complex manifold of complex dimension n is called *astheno-Khler* if its fundamental 2-form ω satisfies the condition $\partial\overline{\partial}\omega^{n-2} = 0$. [55, 58]. In complex dimension $n = 3$, the notion of a astheno-Kähler metric coincides with that one of pluriclosed. Answering to a question posed by Székeleyhidi, Tosatti, Weinkove [71], we showed the following:

Theorem 3.6 ([27])**.** *There exists a compact simply-connected complex non-Kähler manifold admitting a balanced and an astheno-Kähler metric.*

The example is the complex homogeneous space $SU(5)/T^2$ with a complex structure studied by H. C. Wang [75]. Other non simply connected examples are given by nilmanifolds [27, 57].

3.1. The pluriclosed flow. On a compact Kähler manifold (M, J, g), as noticed already by Hamilton [52, p. 257], the Ricci flow

$$\partial_t g(t) = -Ric(g(t)), \quad g(0) = g,$$

(also called called Kähler Ricci flow) preserves the Kähler condition and reduces to a parabolic Monge-Ampére equation [16].

For a non-Kähler manifold (M, J, g), the Levi-Civita connection does not preserve the complex structure and the Ricci flow does not preserve the Hermitian condition, but one may consider other connections preserving both the complex structure and the metric (e.g. the Bismut connection).

Let $(M^{2n}, J, g_0, \omega_0)$ be a Hermitian manifold. In [69], Streets and Tian introduced the following geometric flow (also called pluriclosed flow)

$$\partial_t \omega(t) = -(\rho^B)^{1,1}(\omega(t)), \quad \omega(0) = \omega_0$$

where $\rho^B(X, Y) := \dfrac{1}{2}\sum_{i=1}^{2n} R^B(X, Y, Je_i, e_i)$.

Since the operator $\omega \mapsto -(\rho^B)^{1,1}(\omega)$ is a real quasi-linear second-order elliptic operator when restricted to pluriclosed J-Hermitian metrics, one can show the following short time existence result.

Theorem 3.7 ([69])**.** *Let (M, J) be a compact complex manifold. If ω_0 is pluriclosed, then there exists $\epsilon > 0$ such that there exists a unique solution $\omega(t)$ to the pluriclosed flow with initial condition ω_0 on the interval $[0, \varepsilon)$.*

If ω_0 is Kähler, then $\omega(t)$ is the unique solution to the Kähler Ricci flow with initial datum ω_0.

Note that the Bismut Ricci form ρ^B is related to the Chern Ricci form ρ^C by the following relation

$$\rho^B = \rho^C + dd^*\omega, \tag{1}$$

where $\rho^C = \frac{1}{2} i\partial\overline{\partial} \log\det g(t)$, $dd^*\omega = d(J\theta)$, and θ is the Lee form.

Natural possible candidates for canonical metrics on non-Kähler manifolds are given by the static metrics introduced in [69], i.e. pluriclosed metrics such that $(\rho^B)^{1,1} = \lambda\omega$ for a real number λ.

Theorem 3.8 ([77]). *Let g be a static point of the pluriclosed flow on a compact complex manifold (M, J), i.e. a pluriclosed metric such that $(\rho^B)^{1,1} = \lambda\omega$. Then*

(*i*) *If $\lambda \neq 0$, the metric has to be Kähler Einstein;*

(*ii*) *If $\lambda = 0$, then $\rho^B = 0$ (in this case the metric is also called Calabi-Yau with torsion (shortly CYT – see Definition 5.1).*

Natural problems related to the pluriclosed flow are the following:

(a) describe the maximal smooth existence time T;

(b) study the limiting behavior at the time T;

(c) find non-trivial examples of compact complex manifolds with a pluriclosed metric such that $\rho^B = 0$, but not Bismut flat.

Related to the problem (c), in Section 6, we will review the recent construction of (not Bismut flat) pluriclosed metrics with $\rho^b = 0$ given in [14].

For the first two problems, in analogy to the Kähler case, one can consider the real $(1, 1)$ Aeppli cohomology

$$H^{1,1}_{\mathcal{A},\mathbb{R}} := \frac{\{\operatorname{Ker} i\partial\overline{\partial} : \Lambda^{1,1} \to \Lambda^{2,2}\}}{\{\partial\overline{\eta} + \overline{\partial}\eta \mid \eta \in \Lambda^{1,0}\}}$$

and define the (1, 1) Aeppli positive cone

$$\mathcal{P} := \{[\psi] \in H^{1,1}_{\mathcal{A},\mathbb{R}} \mid \exists\omega \in [\psi],\, \omega > 0\},$$

which consists precisely of the (1, 1) Aeppli classes represented by pluriclosed metrics.

For a general complex manifold (M, J), one has that the first Chern class $c_1(M^{2n})$, which belongs to the space

$$H^{1,1}_{BC,\mathbb{R}} := \frac{\{\operatorname{Ker} d : \Lambda^{1,1} \to \Lambda^{2,2}\}}{\{i\partial\overline{\partial} f \mid f \in \mathcal{C}^\infty\}},$$

can be viewed as an element of $H^{1,1}_{\mathcal{A},\mathbb{R}}$.

As in the Kähler-Ricci flow case, one can prove [69] the following equality for the real $(1,1)$ Aeppli class

$$[\omega(t)] = [\omega_0] - t\, c_1(M^{2n}).$$

As consequence, Streets and Tian [69] showed that the maximal time T of existence of smooth solution for the pluriclosed flow with initial condition g_0 satisfies the following relation:

$$T \leq \tau^*(\omega_0) := \sup\{t \geq 0 \mid [\omega_0] - t\, c_1(M^{2n}) \in \mathcal{P}\}.$$

The following conjecture is still open.

Conjecture 3.2 ([69])**.** *Let (M, J, g_0) be a compact complex manifold with pluriclosed metric. The maximal smooth solution to the pluriclosed flow with initial condition g_0 exists on $[0, \tau^*(\omega_0))$.*

In the homogeneous setting, the pluriclosed flow has been studied on nilpotent Lie groups in [21].

Theorem 3.9 ([21])**.** *The pluriclosed flow on a nilpotent simply connected Lie group (G, J) starting from a left-invariant pluriclosed metric g has a long-time solution.*

For nilpotent Lie groups, the solutions of the pluriclosed flow converge in the Gromov-Hausdorff sense, after a suitable normalization, to self-similar solutions of the flow [7]. More recently, in [38], it has been shown that any invariant solution to the pluriclosed flow on a solvmanifold exists for all positive times.

4. Bismut Kähler-like conditions. In general, the Bismut connection ∇^B does not satisfy the first Bianchi identity, since

$$\begin{aligned}\mathfrak{S}_{X,Y,Z}\, R^B(X,Y,Z,U) &= dT^B(X,Y,Z,U) + (\nabla^B_U T^B)(X,Y,Z)\\ &\quad -\mathfrak{S}_{X,Y,Z}\, g(T^B(X,Y), T^B(Z,U)),\end{aligned}$$

for every vector fields X, Y, Z, U on M. One can then introduce for the Bismut connection (or more generally, a Hermitian connection) the so called Kähler-like condition, in the sense that its curvature tensor obeys all the symmetries of the curvature of a Kähler manifold. More precisely,

Definition 4.1. *A Hermitian metric g on a complex manifold (M, J) is **Bismut Kähler-like** (shortly **BKL**) if the Bismut connection ∇^B satisfies the first Bianchi identity*

$$\mathfrak{S}_{X,Y,Z}\, R^B(X,Y,Z) = 0$$

and the type condition

$$R^B(X,Y,Z,W) = R^B(JX,JY,Z,W),$$

for every vector fields X, Y, Z, W.

In [4], it has been conjecture that if for a compact Hermitian manifold (M, J, g) the Bismut connection ∇^B is Kähler-like, then the metric g has to be pluriclosed.

The conjecture has been proved recently by Zhao and Zheng in [79].

Theorem 4.1 ([79]). *For every Hermitian manifold (M, J, g) the Bismut connection ∇^B is Kähler-like if and only if the metric g is pluriclosed and $\nabla^B T^B = 0$.*

Recently a study of the Hermitian manifolds with parallel Bismut torsion without the pluriclosed condition have been initiated in [80]. Another natural problem is to study the behaviour of the Bismut Kähler-like condition along the pluriclosed flow. The case of complex surfaces was studied in [31].

Note that for complex surfaces T^B is related to the Lee form θ by the following relation: $T^B = - * \theta$.

We recall the following

Definition 4.2. *A Hermitian metric g on a complex manifold M^{2n} is a* **Vaisman metric** *if $d\omega = \theta \wedge \omega$ for some d-closed 1-form θ with $\nabla^{LC}\theta = 0$, where ∇^{LC} is the Levi-Civita connection of g.*

As a consequence, Vaisman metrics are Gauduchon and $|\theta|$ is constant. For complex surfaces, it was shown the following characterization:

Theorem 4.2 ([31]). *Let (M^4, J) be a complex surface. Then a Hermitian metric g is Vaisman if and only if g is pluriclosed and the Bismut connection ∇^B satisfies the first Bianchi identity.*

Compact Vaisman surfaces have been classified by Belgun in [11] and they are non-Kähler properly elliptic surfaces, Kodaira surfaces, and Class 1 or elliptic Hopf surfaces.

In [31], the behaviour of the Vaisman condition on complex surfaces has been studied in relation to the pluriclosed flow, showing in particular the following:

Theorem 4.3 ([31]). *If (M^4, J) admits a Vaisman metric g_0 with constant scalar curvature, then the pluriclosed flow starting with ω_0 preserves the Vaisman condition.*

For the proof of this result, it has been used that if (M^4, J, g) is a compact Vaisman surface, then $\rho^C = h\, d(J\theta)$ for some $h \in \mathcal{C}^\infty(M^4)$ and that the scalar curvature of g is constant if and only if h is constant (note that in such a case $c_1(M^4) = 0$).

One of the Vaisman surface is given by the Kodaira-Thurston surface $KT = (\Gamma\backslash Nil^3) \times S^1$, where

$$Nil^3 = \left\{ \left[\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right] \mid x, y, z \in \mathbb{R} \right\}$$

and Γ is the lattice in Nil^3 of matrices having integers entries.

Therefore the Kodaira-surface KT is parallelizable and has a global left-invariant co-frame

$$e^1 = dy, \quad e^2 = dx, \quad e^3 = dw, \quad e^4 = dz - xdy$$

satisfying the structure equations

$$de^1 = de^2 = de^3 = 0, \quad de^4 = e^1 \wedge e^2.$$

Remark 4.1. The Kodaira-Thurston surface KT can also be viewed as the total space of a principal torus bundle over a 2-torus:

$$\pi_{xy} \colon M \to T^2_{xy}.$$

By the description of T^2-invariant Vaisman metrics, it has also been shown that on the Kodaira-Thurston surface there exist T^2-invariant Vaisman metrics which have non-constant scalar curvature [25]. In relation to the pluriclosed flow, one has the following:

Theorem 4.4 ([25])**.** *If ω_0 is the fundamental form of a T^2-invariant Vaisman metric on the Kodaira-Thurston surface, then the pluriclosed flow starting with ω_0 preserves the Vaisman condition if and only ω_0 has constant scalar curvature.*

Other results in complex dimension $n > 2$ and in the homogeneous setting have been obtained in [31] for almost abelian Lie groups and more recently in [32] for nilpotent Lie groups. In particular, it turns out that if a 6-dimensional nilpotent Lie group (G, J) admits a Bismut Kähler-like metric, then the left-invariant complex structure J has to be abelian, i.e. $[JX, JY] = [X, Y]$ for every left-invariant vector fields X and Y. In relation to the pluriclosed flow one has the following:

Theorem 4.5 ([32])**.** *Let (G, J, g_0) be a 2-step nilpotent Lie group with a left-invariant Bismut Kähler-like Hermitian structure and let $g(t)$ be the solution* *to the pluriclosed flow starting from g_0. Then $g(t)$ is Bismut Kähler-like for every t.*

5. CYT metrics. In this section, we consider Hermitian metrics satisfying weaker condition than Bismut-flat.

Definition 5.1. *Hermitian manifold (M, J, g) of complex dimension n is said to be **Calabi-Yau with Torsion** (shortly **CYT**) if $\rho^B = 0$ or equivalently if the restricted holonomy group of its Bismut connection is contained in $SU(n)$.*

Chronologically, the CYT metrics are one of the first special non-Kähler metrics considered in the literature. They are introduced by the physicists C. Hull [54] and A. Strominger [70] in relation with models in string theory. Since ρ^C represents the first Chern class $c_1(M)$, by (1), the Ricci-Bismut form ρ^B also represents $c_1(M)$. Therefore the existence of a CYT metric can only happen if $c_1(M) = 0$.

Unlike the Bismut-flat spaces, there are many examples of compact CYT manifolds. In analogy with the Calabi conjecture, they have led the authors of [50] to a conjecture that *any* compact complex manifold with vanishing firs Chern class admits a CYT metric. But, a counterexample of this conjecture was quickly found in [23] on a compact nilmanifold. It is known that the Calabi-Yau property for Kähler manifolds is stable under small deformations of the complex structure. However, in [23], it was proven the following:

Fact. Unlike the condition $c_1(M) = 0$, the CYT condition is not stable under small deformations.

Apart from $c_1(M) = 0$, another general restriction for the existence of CYT metrics is based on the Kodaira dimension. The following was proved independently for HKT metrics, which are also CYT by the second author, and communicated to the authors of [2]

Theorem 5.1 ([2]). *The plurigenera of a compact CYT manifold are* 0 *or* 1.

In both cases, the proof is based on a conformal change and Gauduchon's plurigenera vanishing theorem [43, 44].

Theorem 5.2 ([47]). *Every compact complex homogeneous space M with $c_1(M) = 0$, after an appropriate deformation of the complex structure, has a homogeneous CYT metric if it has an invariant volume form.*

A natural problem, as mentioned in Subsection 3.1, is to investigate if there exist pluriclosed metrics which are CYT, but not Bismut flat.

Remark 5.1. Many CYT homogeneous examples have non-zero Bismut curvature.

Every non-Kähler C-space M is a compact complex manifold with a transitive action by a compact Lie group of biholomorphisms and finite fundamental group. By [75], M admits a transitive action of a compact semisimple Lie group G. The complex structure on G is the Samelson one determined by a maximal torus T, and we have the Tits bundle $G \to G/T$. The construction also fits the principal toric bundle construction from Theorem 2.2

In the semisimple case, we proved the following:

Theorem 5.3 ([24]). *Let G be a compact semisimple Lie group of even dimension and J a left-invariant complex structure on G. If there exists a left-invariant pluriclosed metric g compatible with J, then J must be compatible with a bi-invariant metric.*

Moreover, every pluriclosed and CYT invariant metric which is also right-invariant by the action of the maximal torus T is a bi-invariant one (hence Bismut flat).

The second part of the previous theorem was extended recently in [9].

Theorem 5.4 ([9]). *The condition "right T-invariant" for the metric in Theorem 5.3 is not necessary for the conclusion in it.*

To prove the previous theorem, Barbaro uses the following convergence result of the pluriclosed flow on Bismut flat manifold.

Theorem 5.5 ([39]). *Let (M, J, ω_{BF}) be a compact Bismut flat manifold. Given a pluriclosed metric ω_0 such that $[\partial\omega_0] = [\partial\omega_{BF}] \in H^{2,1}_{\overline{\partial}}(M, J)$, the solution to the pluriclosed flow with initial data ω_0 exists on $[0, \infty)$ and converges to a Bismut flat metric ω_∞.*

Moreover, he calculates the $(1, 1)$-Aeppli cohomology group:

Theorem 5.6 ([9]). *Let G be a semisimple Lie group equipped with a bi-invariant metric g and a compatible Samelson complex structure J. Then $H^{1,1}_A(G, J) = \mathbb{C}^s$ where s is the number of irreducible components of (G, J) as complex manifold.*

Using the uniqueness of the solution to the pluriclosed flow and the above result, he was able to prove Theorem 5.4.

In particular, the cohomological condition $[\partial\omega_0] = [\partial\omega_{BF}] \in H^{2,1}_{\overline{\partial}}$ holds. Another consequence is that the Bismut flat metrics on compact manifolds with finite fundamental group are globally stable under the pluriclosed flow.

6. BHE manifolds with parallel Bismut torsion. We recall the following

Definition 6.1. *If a Hermitian metric g on a complex manifold (M, J) is pluriclosed and CYT, then g is called **Bismut Hermitian Einstein** (or shortly **BHE**).*

Note that BHE metrics are the only non-Kähler static points for the pluriclosed flow [77]. Up to now, the only known examples of BHE manifolds are the trivial ones, i.e. one of the following

(a) Kähler-Ricci flat manifolds;

(b) Bismut flat spaces (which are trivially CYT);

(c) products of (a) and (b).

Under the additional assumption that the torsion of the Bismut connection is parallel, we can see that locally these are the only ones:

Theorem 6.1 ([14]). *Let (M, I, h) be a compact BHE manifold such that $\nabla^B T^B = 0$.*

Then its Riemannian holomorphic universal cover $(\tilde{M}, \tilde{I}, \tilde{h})$ is holomorphically isometric to $(M_1, J_1, g_1) \times (M_2, J_2, g_2)$, where (M_1, J_1, g_1) is Kähler Ricci flat and (M_2, J_2, g_2) is a Bismut flat space.

Recently G. Barbaro, F. Pediconi, and N. Tardini [10] were able to extend the local classification to the manifolds having only pluriclosed metric with parallel Bismut torsion:

Theorem 6.2 ([10]). *Let (M^{2n}, J, g) be a complete, simply-connected Hermitian manifold. Then (M, J, g) is pluriclosed with parallel Bismut torsion if and only if it decomposes as a product of Hermitian irreducible factors, each of them being either Kähler or a Riemannian product $\mathbb{R}^\ell \times \Pi_{i=1}^s S_i \times K$ endowed with a standard complex structure, where $\mathbb{R}^\ell$ is the ℓ-dimensional flat Euclidean space, each S_i is a Sasaki 3-dimensional manifold, K is a compact semisimple Lie group of rank r with a bi-invariant metric and $\ell \leq s + r$.*

Starting from Kähler manifolds and using the mapping torus construction reviewed in Section 2, we have

Theorem 6.3 ([14]). *Let (K, J, g) be a compact Kähler manifold and let ψ be a Kähler isometry. Then the toric suspension (mapping torus) $M_f = (K \times S^3)_f$ with $f = (\psi, Id_{S^3})$ admits a pluriclosed structure (I, h) Moreover, if g is (non-flat) Ricci flat, then (M_f, I, h) is BHE manifold with non-flat Bismut connection.*

Remark 6.1. (i) Note that $\nabla^B T^B = 0$ since the curvature R^B of the Bismut connection satisfies the first Bianchi identity and the Hermitian structure is pluriclosed.

(ii) Since (M_f, I) is the total space of the holomorphic fibre bundle

$$(K, J) \hookrightarrow (M_f, I) \to (S^3 \times S^1, J_-),$$

(M_f, I) does not admit any balanced metric because $S^3 \times S^1$ is non-Kähler.

One can prove that the manifold M_f is diffeomorphic to a product. Indeed,

Proposition 6.1 ([14]). *M_f is diffeomorphic to $M_\psi \times S^3$, where M_ψ is the mapping torus of the Kähler manifold K with respect to the Kähler isometry ψ.*

As a consequence M_f, is formal (in the sense of Sullivan [73]). Moreover, since the first Betti number $b_1(M_f)$ is odd, the complex manifold (M_f, I) does not admit any Kähler metric.

A non-trivial BHE manifold can be constructed starting from a K3 surface.

Example 6.1 ([14]). Let (K, J, g) be a K3 surface endowed with a Ricci flat metric. and ψ a Kähler isometry of (K, J, g) different from the identity. Then the mapping torus $M_f = (K \times S^3)_f$ is a BHE manifold. Since

$$b_2(M_f) < 22 = b_2(K \times S^3 \times S^1)$$

$M_f = (K \times S^3)_f$ provides the first non-trivial example of compact BHE manifold.

Thus, it remains open the problem if there exist compact BHE manifolds whose Bismut torsion is not parallel.

Acknowledgements. The authors would like to thank J. Davidov for the careful reading of the manuscript and his useful suggestions.

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Received July 21, 2024
Accepted August 8, 2024