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# SYNCHRONIZED ENTROPY OF TOTALLY SYNCHRONIZING GENERATED SYSTEMS

Manouchehr Shahamat

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Abstract. We introduce the notion of a minimal generator $G$ for a coded system $X$; that is a generator $G$ for $X$ such that whenever $u \in G$, then $u \notin W\left(\overline{< G \setminus \{u\} >}\right)$. An $X$ possessing such a $G$ is called a minimally generated system. We introduce a class of minimally generated totally coded shift spaces generated by certain synchronizing blocks. For such shift spaces $X$ we are able to show that if $x \in X$, then there is unique $\{\ldots, v_{-1}, v_0, v_1, v_2, \ldots\} \subset G$ such that $x = v_{-1}v_0v_1v_2$. A carefully constructed example also shows that the converse of this statement is not necessarily true.

The derived shift space $(\partial(X))$ of $X$ plays an important role in the dynamics of the system. We characterize the derived shift space and use it to give a new shorter proof for computing synchronized entropy $h_{syn}(X)$.

**1. Introduction.** Shift spaces of finite type (SFT) are among the most studied dynamical systems. An SFT $X$ is a system whose set of forbidden blocks is nite.

A block $m$ is *synchronizing* if whenever $v_1 m$ and $m v_2$ are both blocks of $X$, then $v_1 m v_2$ is a block of $X$ as well. If an irreducible system has at least one

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synchronizing block, then it is called a synchronized system and examples are *sophics*. Synchronized systems, has attracted much attention and extension of them has been of interest since that notion was introduced [2]. In Section(3), we will introduce the notion of a totally synchronizing generated system. These systems are precisely the tool that will enable us to create a bridge between dynamical systems and other mathematical branches. By creating such a connection, we will be able to introduce the basic concepts linear independence and dependence from the branch of linear algebra to dynamic systems and enter the topic of applied mathematics. See [9].

Thomsen in [11] considers a synchronized component of a general shift space and investigates the approximation of entropy from inside of this synchronized component by some certain SFT's. In fact, many results in [11] are based on this result. Minimal synchronizing blocks are main tools for our aim.

In section (4) we give another right resolving and follower separated cover for $X$, denoted by $H_G$ which is not necessarily a Fischer cover of $X$. We also give a sufficient condition on a minimal generator $G$ that guarantees that the cover $H_G$ is Fischer cover.

Thomsen proves that $\lim_{k\to\infty} h(X_k) = h_{syn}(X)$ where

$$h_{syn}(X) := \limsup_n \frac{1}{n} \log\left(\text{cardinal}\,\{a \in W_n(X) : mam \in W(X)\}\right). \tag{1}$$

where $m$ is an arbitrary synchronizing block in $W(X)$ and $X_k$'s are $SFT$ approaching $X$ from inside. For the totally synchronizing generated systems, we give another way to calculate synchronizing entropy. We will cover this in Section 5.

Now let $\text{Per}(X)$ be the set of periodic points of $X$ and set $R(X) = \overline{\text{Per}(X)}$. Also let $S(X)$ denote the set of synchronizing blocks for $R(X)$. Set

$$\partial X := \{x \in R(X) : w \subseteq x \Rightarrow w \notin S(X)\} \tag{2}$$

called the *derived shift space* of $X$. Then, $\partial(X)$ plays an important role in the dynamics of the system. As an example, a result in [10] states that in synchronized systems

$$h(X) = \max\{h_{\text{syn}}(X), h(\partial(X))\}.$$

We will characterize this situation in Section 6.

**2. Background and definitions.** This section is devoted to basic definitions for our later work. The notations has been taken from [1], [3], [5], [2] and [7] for the relevant concepts.

First, we present some elementary concepts from [5]. Let $\mathcal{A}$ be a set of non-empty finite symbols called *alphabet*. The full $\mathcal{A}$-shift denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols in $\mathcal{A}$. Equip $\mathcal{A}$ with discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with product topology. A *block* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. It is convenient to include the sequence of no symbols, called the *empty block* which is denoted by $\varepsilon$. If $x$ is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we will denote a block of length $j-i+1$ by $x_{[i,j]} = x_i x_{i+1} \dots x_j$. If $n \geq 1$, then $u^n$ denotes the concatenation of $n$ copies of $u$, and put $u^0 = \varepsilon$. The *shift map* $\sigma$ on the full shift maps a point $x$ to the point $y = \sigma(x)$ whose $i$-th coordinate is $y_i = x_{i+1}$. By our topology, $\sigma$ is a homeomorphism. Let $\mathcal{F}$ be a collection of blocks over A. The blocks in $\mathcal{F}$ will be called the forbidden blocks. The complement of $\mathcal{F}$ is the set of *admissible blocks* or just blocks in $X$. For a full shift $\mathcal{A}^{\mathbb{Z}}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any block from $\mathcal{F}$. A *shift space* is a subset of $X$ which is shift invariant and closed (in the product topology on the full shift).

Let $W_n(X)$ denote the set of all admissible $n$-blocks. The *language* of $X$ is the collection $W(X) = \cup_n W_n(X)$. A shift space $X$ is *irreducible* if for every ordered pair of blocks $u, v \in W(X)$ there is a block $w \in W(X)$ so that $uwv \in W(X)$. It is *mixing* if for every ordered pair $u, v \in W(X)$, there is an $N$ such that for each $n \geq N$ there is a block $w \in W_n(X)$ such that $uwv \in W(X)$. A shift space $X$ is called a *shift of finite type* (SFT) if there is a finite set $\mathcal{F}$ of forbidden blocks such that $X = X_{\mathcal{F}}$. A shift is *sophic*, if it is the image of an SFT by a factor code (an onto sliding block code). Every SFT is sophic [5, Theorem 3.1.5], but the converse is not true. For instance, if $\mathcal{F} = \{10^{2n+1}1 : n \in \mathbb{N} \cup \{0\}\}$, then $X_{\mathcal{F}}$ is called the even shift which is not SFT but it is sophic [5, Page 67].

Let $G$ be a graph with edge set $\mathcal{E} = \mathcal{E}(G)$ and the set of vertices $\mathcal{V} = \mathcal{V}(G)$. The *edge shift* $X_G$ is the shift space over the alphabet $\mathcal{A} = \mathcal{E} := \mathcal{E}(G)$ defined by

$$X_G = \{\xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1})\}.$$

Each edge $e$ initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$. Let $G$ be a graph and $\mathcal{L} : \mathcal{E}(G) \to \mathcal{A}$ is a labeling of the edges of $G$ by letters in $\mathcal{A}$. For a path $\pi = \pi_0 \dots \pi_k$, $\mathcal{L}(\pi) = \mathcal{L}(\pi_0) \dots \mathcal{L}(\pi_k)$ is the label of $\pi$.

Let $\mathcal{L}_\infty(\xi)$ be the sequence of bi-infinite labels of a bi-infinite path $\xi$ in $G$ and set

$$X_{\mathcal{G}} := \{\mathcal{L}_\infty(\xi) : \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$

We say $\mathcal{G}$ is a *presentation* or *cover* for $X = \overline{X_{\mathcal{G}}}$. Where the bar is the closure in the product topology on $\mathcal{A}^{\mathbb{Z}}$. In particular, $X$ is sophic if and only if $X = X_{\mathcal{G}}$ for a finite graph $G$ [5, Proposition 3.2.10]. A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex $I$ of $G$ the edges starting at $I$ carry different labels.

Let $X$ be a shift space and $x = (x_i)_{i\in\mathbb{Z}} \in X$. Then, $x_+ = (x_i)_{i\in Z^+}$ is called a right infinite $X$-ray. Similarly $x_- = (x_i)_{i\leq 0}$ is called a left infinite $X$-ray. Let $X^+ = \{x^+ : x \in X\}$ and $X_- = \{x_- : x \in X\}$ be the sets of right infinite and left infinite $X$-rays respectively.

For a left infinite $X$-ray, say $x_-$, its follower set is $w_+(x_-) = \{x_+ \in X^+ : x_-x_+ \in X\}$. Consider the collection of all follower sets $w_+(x_-)$ as the set of vertices of a graph. There is an edge from $I_1$ to $I_2$ labeled $a$ if and only if there is an $X$-ray $x_-$ such that $x_-a$ is an $X$-ray and $I_1 = w_+(x_-), I_2 = w_+(x_-a)$. This labeled graph is called the *Krieger graph* for $X$. A block $m \in W(X)$ is *synchronizing* if whenever $um$ and $mv$ are in $W(X)$, we have $umv \in W(X)$. An irreducible shift space $X$ is *synchronized system* if it has a synchronizing block, or equivalently, if and only if it admit a countable generating graph $G$ such that $\mathcal{L}_\infty(X_G)$ is residual in $X$ [2, Theorem 1.1].

If $X$ is synchronized system with synchronizing $m$, the irreducible component of the Krieger graph containing the vertex $w_+(m)$ is denoted by $X_0^+$ and is called the *Fischer cover* of $X$. If for some $m \in W(X)$ there is a unique vertex $I$ such that $m \in F_-(m)$, then $m$ is called a *magic* block for the Fischer cover.

**3. Minimal generator.** A shift space that is the closure of the set of sequences obtained by freely concatenating the blocks in a list of countable blocks, called the set of generators, is a *coded system* [9]. Equivalently, a coded system is a shift space that can be presented by an irreducible countable labeled graph.

**Definition 3.1.** *Let $X$ be a coded system generated by $G$. Then, $G$ is called* minimal *(resp. weakly minimal), whenever $u \in G$, then $u \notin W(Z)$, (resp. $X \neq Z$) where $Z = \overline{< G \setminus \{u\} >}$. Such an $X$ is called* minimally *(resp. weakly minimally) generated system.*

Here, $< G >$ means the set of all concatenations of the elements of $G$. Clearly any minimally system is weakly minimally.

**Example 3.2.** (i) Let $X$ be the Dyke system. Set

$$G_1 := \{(),\ (()),\ [()],\ ((())),\ [(())],\ ([()]),\ [[()]],\ \ldots\}$$

$$G_2 := \{[],\ ([]),\ [[]],\ [[[]]],\ ([[]]),\ [([])],\ (([])),\ \ldots\}.$$

Then, $G = G_1 \cup G_2$ is a weakly minimal generator for the shift space $Z = \overline{< G >}$.

(ii) Let $\emptyset \neq S \subseteq \mathbb{N}$. Then, $G := \{10^n 1 : n \in S\}$ is a minimal generator for shift space $X := \overline{< G >}$.

**Definition 3.3.** *Let $X$ be a synchronized system. We will say that a block $m$ is strongly synchronizing for $X$ if whenever there are two finite paths $e$, $e'$ in the Fischer cover $X_0^+$ labeled $m$, then $e = e'$.*

An irreducible shift space with a strongly synchronizing block is called *strong synchronized*. Any strong synchronized system is synchronized since any strong synchronizing block is synchronizing block. Every strong synchronized system has a weakly minimal generator [8].

Let $X$ be a strongly synchronized system and let $S_t(X)$ (resp. $S(X)$) denote the set of all strongly synchronizing (resp. synchronizing) blocks for $X$.

The next example shows that there are weakly minimally systems that are not strong synchronized.

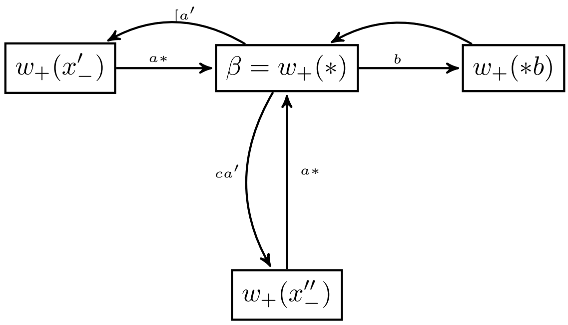


Fig. 1. A subgraph of Fischer cover $S$

**Example 3.4.** (i) Pick $S \subseteq \mathbb{N}$ such that $0 \in S$. Set $G := \{10^n : n \in S\}$ and claim that $G$ is a weakly minimal generator for $S$-gap shift $X(S)$. For all $n \in S$, $(10^n 1)^\infty \notin \overline{< G \setminus \{10^n\} >}$ and so $\overline{< G >} \neq \overline{< G \setminus \{10^n\} >}$. Also $\overline{< G >} = X(S)$ is trivial and we are done.

(ii) Let $D$ be the Dyke system. Add a new symbol $*$ (which will be a synchronizing block) to the set of four brackets and let $X$ be the shift space which consists of all bi-infinite sequences of these five symbols such that any finite subblock which does not contain a $*$ obeys the standard brackets rules [6]. We claim that $X$ is not strong synchronized system. Let $u$ be a strongly synchronizing block of $X$. So that is a synchronizing block and $* \subseteq u$. We can write $u = a * b$ where $a, b \in W(X)$. Since $*$ is a synchronizing block, there is a unique vertex $\beta$ of Fischer cover $X_0^+$ with this properties $* \in F_-(\beta)$ and $w_+(\beta) = w_+(*)$ [2, Page 146]. Since $*b \in W(X)$, so $b \in w_+(*) = w_+(\beta)$. Hence there is a finite path $\pi_b$ labeled $b$ in $X_0^+$ initiating at $w_+(*)$ and terminating at $w_+(*b)$. See Figure 1.

Pick $a' \in W(X)$ such that for each $x_- \in X^-$, $x_-a'a \in X^-$ and set $x'_- := \cdots]]][a'$, $x''_- := \cdots]]](a'$. Since $w_+(x'_-a*) = w_+(*) = w_+(x''_-a*)$ and $w_+(x'_-) \neq w_+(x''_-)$, so there are two distinct finite paths $\pi'_{a*}$ and $\pi_{a*}$ in the Krieger graph of $X$ labeled $a*$ such that $i(\pi_{a*}) = w_+(x'_-)$, $i(\pi'_{a*}) = w_+(x''_-)$ and $t(\pi_{a*}) = t(\pi'_{a*}) = w_+(*)$. But $w_+(*[a') = w_+(x'_-)$ and $w_+(*(a') = w_+(x''_-)$. So there are two finite paths initialing at $w_+(*)$ and terminating at $w_+(x'_-a*)$ and $w_+(x''_-a*)$, respectively. Hence

$$\{w_+(x'_-a*),\, w_+(x''_-a*)\} \subseteq \mathcal{V}(X_0^+).$$

So there are two finite paths labeled $a * b = u$ in $X_0^+$. This shows that $u$ is not a strongly synchronizing block and we are done.

It is easy to see that $G_* = \{*u : * \notin u \in W(X)\}$ is a weakly minimal generator for $X$.

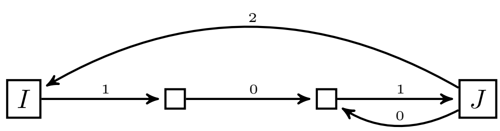


Fig. 2. The graph $H$ for the cover of a synchronized system such that $G_m$ is not a minimal generator for $X_H$

Note that if $m \in S(X)$ and $m \subseteq u$, then $u \in S(X)$. But this does not, in general $m \in S_t(X)$. This can be seen by the fact that in Figure 2, $2 \in S_t(X)$ but $012 \notin S_t(X)$.

Let $G$ be a minimal generator for a shift space $X$. Let $G_{ts}$ denote the set of all $v \in G$ such that for all $u \in G$, there are not non empty blocks $a$, $b$ such that $vu = avb$ or $uv = avb$. All the elements of the set $G_{ts}$ are synchronizing [9].

Let $G$ be a minimal generator for a shift space $X$ with $G = G_{ts}$. Then, $G$ is called a *totally synchronizing generator*. Such an $X$ is called *totally synchronizing generated system*.

These systems are precisely the tool that will enable us to show that for such shift spaces $X$, each $x \in X$ can be written uniquely as $x = \ldots v_{-1}v_0v_1v_2\ldots$, where $\{\ldots, v_{-1}, v_0, v_1, v_2, \ldots\} \in G$. In fact these systems are precisely the tool that will enable us to create a bridge between dynamical systems and other mathematical branches. By creating such a connection, we will be able to introduce the basic concepts linear independence and dependence from the branch of linear algebra to dynamic systems and enter the topic of applied mathematics.

**4. Synchronized components** In this section we introduce the notion of minimal synchronizing blocks and will exploit them to identify synchronized components.

**Definition 4.1.** *A block* $m \in S(X)$ *is* minimal synchronizing block, *whenever* $w \subsetneq m$, *then* $w$ *is not synchronizing. If a shift space* $X$ *has finitely many minimal synchronizing blocks and* $S(X) \neq \emptyset$, *then we say that* $X$ *is a* FmSyn *system; otherwise, it is called an* ImSyn *system.*

**Example 4.2.** (i) The block 1 is minimal synchronizing for any $S$-gap shift $X(S)$ and no other minimal synchronizing block exists which means that this system is an FmSyn even when it is not sophic. See [8] for criteria on $S$ to have $X(S)$ non-sophic.

(ii) Let $G$ be the graph as in Figure 3 and $X = X_G = R(X)$. Then all blocks in $A = \{2, 101, 10^3 1, 10^5 1, \ldots\}$ are synchronizing blocks. However, blocks in

$$\{1, 0, 100, 1000, \ldots\} \cup \{01, 001, 0001, \ldots\}$$

are not. Therefore, no blocks in $A$ has a synchronizing subblock and so $X$ is an ImSyn.

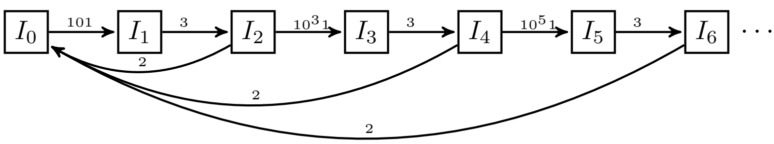


Fig. 3. Graph $G$ for the cover of an ImSyn

The next proposition tells us that a sophic space may have infinitely many minimal synchronizing block.

**Proposition 4.3.** *A sophic space may have infinitely many minimal synchronizing block.*

Proof. Define sets of words $V = \{1^{2n} : n \in \mathbb{N}\}$ and $W = \{1^{2n+1} : n \in \mathbb{N}\}$. Also consider the shift space $X$ formed as the limit of concatenations of blocks of the form $a0v_1 0v_2 0v_3 \ldots 0v_n$ where $v_i$ belong to $V$ and $n$ is any positive integer and $a0w_1 0w_2 0w_3 \ldots 0w_m$ where $w_i$ belong to $W$ and $m$ is any positive integer (in particular, notice the special symbol a at the beginning of all the blocks). The block 0 is not synchronizing, because 0110 belongs in the language of $X$ (0110 appears for example in the block $a0v0v$ where $v = 11$ belongs to $V$) and 010 belongs in the language of $X$ (010 appears for example in the block $a0w0w$ where $w = 1$ belongs to $W$), but 010110 does not belong in the language of $X$, because the separator symbol "$a$" must appear between even and odd blocks of ones. "$a$" is some symbol which is different from 0 and 1. □

**5. Computing synchronized entropy for a totally synchronizing generated systems.** Let $X$ be a shift space. Set $R(X) = \overline{\text{Per}X}$ and let $S(X)$ denote the set of synchronizing blocks for $R(X)$. For $s, t \in S(X)$ we write $s \sim t$ when there are blocks $u, v \in W(R(X))$ such that $sut$, $tvs \in W(R(X))$. Then, $\sim$ is an equivalence relation in $S(X)$. Note that $s \sim t$ if only if there is an $x \in R(X)$ such that $s,\ t \subseteq x$. Consider an element $\alpha \in S(X)/\sim$.

Let $x \in X$ and $p \leq s$ for integers $p$ and $s$. Set the *gap* between two blocks $x_{[p,\,q]}$ and $x_{[s,\,t]}$ to be 0 when $s \leq q$ and $s - q$ otherwise. Denote this gap by $\text{gap}(x_{[p,\,q]},\, x_{[s,\,t]})$.

**Definition 5.1** ([7]). *Let $p < s$ and $q \leq t$ and let $u = x_{[p,\,q]}, v = x_{[s,\,t]} \in \alpha \in S(X)/\sim$ be two minimal synchronizing blocks. If the only minimal synchronizing blocks in $x_{[p,\,t]}$ are $u$ and $v$, then call $u$ and $v$ the* consecutive minimal pairs *of $\alpha$ in $x$.*

Let $x \in X_{(\alpha,\,0)}$. By (3),

$$\{\text{gap}(u,\, v) :\ u,\, v \text{ are the consecutive minimal pairs of } \alpha \text{ in } x\}$$

is bounded and we will denote the maximum by $\text{maxgap}(x, \alpha)$.

Let $X_{(\alpha,\,0)}$ denote the set of elements $x \in R(X)$ as follows.

(i) For all $i \in \mathbb{Z}$ there are $u_i,\ v_i \in \alpha$ such that $u_i \subseteq x_{(-\infty,\,i]},\ v_i \subseteq x_{[i,\,+\infty)}$.

(ii) There is $M > 0$ such that if $u,\ v$ are the consecutive minimal pairs of $\alpha$ in $x$, then $\text{gap}(u,\ v) \leq M$.

(iii) $A = \{w \subset x :\ \text{w is a minimal synchronizing}\} \cap \alpha$ is finite.

In particular, if $x \in R(X)$ is periodic point with a subblock in $\alpha$, then $x \in X_{(\alpha,\,0)}$.

Thomsen in [11] proves that $X_{(\alpha,\,0)}$ is the set of elements $x \in R(X)$ for which

$$\sup_{i\in\mathbb{Z}} \left\{\inf\left\{(j-i) \geq 0 :\ \exists w \in \alpha,\ w \subseteq x_{[i,\,j]}\right\}\right\} \tag{3}$$

is finite. Here, used has the convention that $\inf \emptyset = \infty$.

Let $X$ be a shift space. The *entropy* of $X$ is defined by

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log |W_n(X)|.$$

A shift space $X$ is *almost sophic* if there are sophic shifts $X_n \subseteq X$ such that

$$\lim_{n\to\infty} h(X_n) = h(X).$$

For any synchronized system $X$, the *synchronized entropy* $h_{\text{syn}}$ of $X$ is defined by

$$h_{\text{syn}}(X) = \limsup_n \frac{1}{n} \log\left(\text{cardinal}\{a \in W_n(Y):\ mam \in W(X)\}\right), \tag{4}$$

where $m \in W(X)$ is an arbitrary synchronizing block.

**Lemma 5.2.** *Let $G$ be a minimal generator for the coded system $X$ and $v_0 \in G_{ts}$. Then,*

$$h_{syn}(X) = \limsup_n \frac{1}{n} \log |\{(v_1, \ldots, v_k) \in G^k : \Sigma_{i=1}^k |v_i| = n\}|.$$

P r o o f. Since $v_0 \in S_t(X)$([8]), so

$$h_{\text{syn}}(X) = \limsup_n \frac{1}{n} \log |\{u \in W_n(X):\ v_0uv_0 \in W(X)\}|.$$

There is exactly one path labeled $v_0$ in $\mathcal{H}_G = X_0^+$, so $v_0uv_0 \in W(X)$ if and only if $u$ is a finite concatenation of elements in $G$ and so we are done. □

**Proposition 5.3.** *Let $G$ be a minimal generator generator for the coded system $X$ and $v \in G_{ts}$. Then,*

$$h_{syn}(X) = \lim_{n\to\infty} \frac{1}{n gcd(X_{([v],0)})} \log |\{(v_1, \ldots, v_k) \in G^k : \Sigma_{i=1}^k |v_i| = n gcd(\mathcal{L}(\mathcal{H}_G))\}|.$$

P r o o f. Since $v \in S(X)$ ([8]), so $X$ is synchronized system. Thus by [11, Lemma 3.5], $\overline{X_{([v],0)}} = X$ and so by [11, Proposition 3.2],

$$h_{\text{syn}}(X) = \lim_{n\to\infty} \frac{1}{n\text{period}(X_{([v],0)})} \log |\{u \in W_{n\text{period}(X_{([v],0)})}(X):\ vuv \in W(X)\}|. \tag{5}$$

But by Lemma 5.2,

$$\{u \in W_N(X) : vuv \in W(X)\} = \{(v_1, \ldots, v_k) \in G^k : \Sigma_{i=1}^k |v_i| = N\}$$

where $N = n\text{period}(X_{([v],0)})$ and we are done. □

An implication of the above proposition is that if $X$ is mixing synchronized, then by [11, Lemma 3.6], $\gcd(X_{([v],0)}) = 1$ and so

$$h_{\text{syn}}(X) = \lim_{n\to\infty} \frac{1}{n} \log |\{(v_1, \ldots, v_k) \in G^k : \Sigma_{i=1}^k |v_i| = n\}|.$$

**6. Computing derived shift space.** Let $\partial(X)$ be as in (2) and denote the set of synchronizing blocks for $R(\partial^k(X))$ by $S(\partial^k(X))$. Define the *depth* of $X$ to be $\text{Depth}(X) = \sup\{k \in \mathbb{N} : \partial^k(X) \neq \emptyset\}$. If $\alpha \in S(\partial^k(X))/\sim$, then $(\partial^k(X))_{(\alpha,0)}$ will be denoted by $X_{(\alpha,k)}$. Recall that when $X$ is a sophic shift with non-wandering part $R(X)$, we can consider the shift space

$$\partial X = \{x \in R(X) : x \text{ consists of no blocks that are synchronizing for } R(X)\}$$

which is called the derived shift space of $X$. Since $\partial X$ is a shift space we can continue, and consider $\partial(\partial X) = \partial^2 X$, $\partial(\partial^2 X) = \partial^3 X$, etc. We define the *depth* of $X$ to be

$$\mathit{Depth}(X) = \sup\{n \in \mathbb{N} : \partial^n X \neq \emptyset\}.$$

The irreducible components at level 0 of $\partial^k X$ will be called the irreducible components of $X$ at level $k$. They will be denoted by $X_{(\alpha,k)}$ [10].

Call an *internal subblock* of $u = a_1 a_2 \ldots a_k$ to be a subblock of $a_2 \ldots a_{k-1}$ which we denote it by $u^0$.

**Proposition 6.1.** *Let $X$ be a coded system generated by $G$. Then,*

$$\partial X = \{\cup_i u_i : u_i \subseteq v_{i_1} \ldots v_{i_n} \textit{ s.t. } v_{i_j} \in G, u_i = (u_{i+1})^0 \textit{ and } u_i \notin S(X)\}.$$

Proof. Set $x := \cup_i u_i$ where $u_i \in W(X)$, $u_i = (u_{i+1})^0$ and $u_i \notin S(X)$. If there is a $m \in S(X)$ such that $m \subseteq x$, then $m \subseteq u_i$ for some $i \in \mathbb{N}$. This is a contradiction and so $x \in \partial X$.

Conversely, let $x \in \partial X$. Then, for each $i \in \mathbb{N}$, $x_{[-i,i]} \in W(X)$ is a block with no synchronizing subblock and a subblock of $v_{i_1} \ldots v_{i_n}$ for some $v_{i_j} \in G$. Set $u_i := x_{[-i,i]}$ and then the conclusion is immediate. □

A *minimal synchronizing* block is a block whose proper subblocks are not synchronizing.

**Corollary 6.2.** *Assume that all elements of the generator $G$ of a coded system $X$ are minimal synchronizing block. Then,*

$$\partial X = \{\cup_n u_n : u_n \subsetneqq v \text{ for some } v \in G \text{ and } u_n = (u_{n+1})^0\}.$$

**Example 6.3.** Let $X$ be a shift space over $\mathcal{A}$ generated by $G := \{u_i := ab^ic^ia : i \in \mathbb{N}\}$, where $a \in \mathcal{A}$, $\{b, c\} \subseteq W(X)$ and $a \notin bc$. Then,

(i) $G = G_{ts}$.

(ii) $W(X) - S(X) = \{b^ic^j : i, j \geq 0\} \cup \{a\}$ and so $\partial X = \{b^\infty, c^\infty, b^\infty c^\infty\}$.

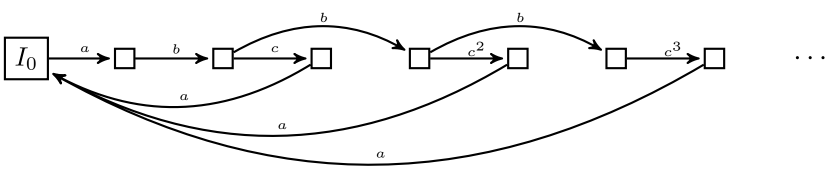


Fig. 4. The graph $\mathcal{G}_{u_i \hookrightarrow a_i}$; merged graph $\mathcal{H}_G$ from $\mathcal{G}_{u_i \hookrightarrow a_i}$ have two irreducible components $\{b^\infty\}$ and $\{c^\infty\}$ at lavel 0 for $\partial(\overline{< G >})$

Example 6.3 gives a cover of a strongly synchronizing of depth 2, that contains one irreducible component $X$ at level 0 and two irreducible components $\{b^\infty\}$ and $\{c^\infty\}$ at level 1 (Figure 4). One can easily do this for infinitely many irreducible components. In fact let $P = \cup P_i$ be the set of all prime numbers such that $|P_i| = \infty$ and $P_i \cap P_j = \emptyset$ for all $i, j \in \mathbb{N}$. Set $G_i := \{2(10^{2i-1}1)^n 2 : n \in P_i\}$, where $i \in \mathbb{N}$ and Suppose that $X$ be a shift space generated by $G := \cup_i G_i$. Then, it is not hard to see that $G = G_{ts}$ and $\partial X = \{(10^{2i-1}1)^\infty : i \geq 1\} \cup \{0^\infty\}$.

**Corollary 6.4.** *Suppose that $X$ is a shift space with a minimal generator $G$ and $v \in G$. Then, $|V_v := \{I \in \mathcal{V}(\mathcal{H}_G) : t(\pi_v) = I\}| < \infty$.*

*If $w_+(I_0) = w_+(I)$ for all $I \in V_v$, then $v \in G_{ts}$ and so is a strongly synchronizing block.*

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*Department of Mathematics*
*Islamic Azad University*
*Dezful Branch*
*Dezful, Iran*
*e-mail:* m.shahamat@iaud.ac.ir

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