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MATRIX TRANSFORMS OF THE SET OF α-ABSOLUTELY CONVERGENT SEQUENCES WITH SPEED

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Abstract. We define the notion of α-absolute convergence with speed, where the speed is defined by a monotonically increasing positive sequence λ and $0 < \alpha \leq 1$. Also we present the notion of α-absolute λ-conservativity of a matrix, and the notion of improvement of α-absolute λ-convergence by a matrix. Let l_α^λ be the set of all α-absolutely λ-convergent sequences and Y a sequence space defined by another speed μ. In this paper, we give necessary and sufficient conditions for a matrix A (with real or complex entries) to map l_α^λ into Y. We also present some examples of matrices being α-absolutely λ-conservative or improving the α-absolute λ-convergence, and consider these problems in the special cases if A is the Riesz matrix (R, p_n) or the Zweier matrix $Z_{1/2}$.

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1. Introduction. Let X, Y be two sequence spaces and $A = (a_{nk})$ be an arbitrary matrix with real or complex entries. Throughout this paper we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. If for each $x = (x_k) \in X$ the series

$$A_n x := \sum_k a_{nk} x_k$$

converge and the sequence $Ax = (A_n x)$ belongs to Y, we say that A transforms X into Y. By (X, Y) we denote the set of all matrices, which transform X into Y. Let ω be the set of all real or complex valued sequences. Further we need the following well-known subspaces of ω: c – the space of all convergent sequences, c_0 – the space of all sequences converging to zero, l_∞ – the space of all bounded sequences, and

$$l_\alpha := \{x = (x_n) : \sum_n |x_n|^\alpha < \infty\}, \ \alpha > 0.$$

Let $\lambda := (\lambda_k)$ be a positive (i.e.; $\lambda_k > 0$ for every k) monotonically increasing sequence. Following Kangro [12, 13], a convergent sequence $x = (x_k)$ with

$$\lim_k x_k := s \text{ and } v_k = \lambda_k (x_k - s) \tag{1.1}$$

is called bounded with the speed λ (shortly, λ-bounded) if $v_k = O(1)$ (or $(v_k) \in l_\infty$), and convergent with the speed λ (shortly, λ-convergent) if the finite limit

$$\lim_k v_k := b$$

exists (or $(v_k) \in c$). In the following we define the notion of α-absolute convergence with speed λ.

Definition 1.1. *We say that a convergent sequence $x = (x_k)$ with the finite limit s is α-absolutely convergent with the speed λ shortly, α-absolutely λ-convergent, if $(v_k) \in l_\alpha$.*

For $\alpha = 1$ Definition 1.1 was given in [1]. In that case we say that a sequence x in Definition 1.1 is absolutely convergent with the speed λ (shortly, absolutely λ-convergent).

We denote the set of all λ-bounded sequences by l_∞^λ, the set of all λ-convergent sequences by c^λ, and the set of all α-absolutely λ-convergent sequences by l_α^λ. Moreover, let

$$c_0^\lambda := \{x = (x_k) : x \in c^\lambda \text{ and } \lim_k \lambda_k (x_k - s) = 0\}$$

and

$$l^{\lambda}_{\infty,0} = \{x = (x_k) \,:\, x \in l^{\lambda}_{\infty} \cap c_0\}.$$

It is not difficult to see that

$$l^{\lambda}_{\alpha} \subset c^{\lambda}_{0} \subset c^{\lambda} \subset l^{\lambda}_{\infty} \subset c,\ l^{\lambda}_{\infty,0} \subset l^{\lambda}_{\infty} \subset c.$$

In addition to it, for unbounded sequence λ these inclusions are strict. For $\lambda_k = O\,(1)$, we get $c^{\lambda} = l^{\lambda}_{\infty} = c$.

Let $e = (1, 1, \dots)$, $e^k = (0, \dots, 0, 1, 0, \dots)$, where 1 is in the k-th position. We note that

$$e, e^k \in c^{\lambda};\quad e, e^k \in l^{\lambda}_{\alpha}.$$

A matrix A is said to be conservative if $A \in (c, c)$, and regular if $A \in (c, c)$ with $\lim_n A_n x = \lim_n x_n$ for every sequence $x = (x_n) \in c$.

Definition 1.2. *We say that a matrix A is regular on l^{λ}_{α}, if $\lim_n A_n x = \lim_n x_n$ for every sequence $x = (x_n) \in l^{\lambda}_{\alpha}$.*

Let $\mu := (\mu_n)$ be another speed of convergence; i.e. a monotonically increasing positive sequence. Following Kangro ([12]), a matrix A is said to be λ-conservative if $A \in (c^{\lambda}, c^{\lambda})$, and improves the λ-convergence if $A \in (c^{\lambda}, c^{\mu})$ with $\mu_n/\lambda_n \neq O(1)$. We define the notions of the α-absolute λ-conservativity and the improvement of α-absolute λ-convergence.

Definition 1.3. *We say that a matrix A is α-absolutely λ-conservative if $A \in (l^{\lambda}_{\alpha}, l^{\lambda}_{\alpha})$.*

Definition 1.4. *We say that a matrix A improves the α-absolute λ-convergence if $A \in (l^{\lambda}_{\alpha}, l^{\mu}_{\alpha})$ with $\mu_n/\lambda_n \neq O(1)$.*

We note that for $\alpha = 1$ from Definitions 1.3 and 1.4 we obtain correspondingly the notions of the absolute λ-conservativity and the improvement of absolute λ-convergence, introduced in [1] by the authors of the present paper.

The sets $(l^{\lambda}_{\infty}, l^{\mu}_{\infty})$, (c^{λ}, c^{μ}) and $(c^{\lambda}, l^{\mu}_{\infty})$ have been described in [3] and in [11]–[14]. The sets $\left(l^{\lambda}_{\infty}, c^{\mu}\right)$, $\left(l^{\lambda}_{\infty}, l^{\mu}_{\infty,0}\right)$, $\left(l^{\lambda}_{\infty}, c^{\mu}_{0}\right)$, $\left(c^{\lambda}, l^{\mu}_{\infty,0}\right)$, $\left(c^{\lambda}, c^{\mu}_{0}\right)$, $\left(l^{\lambda}_{\infty,0}, l^{\mu}_{\infty}\right)$, $\left(l^{\lambda}_{\infty,0}, l^{\mu}_{\infty,0}\right)$, $\left(l^{\lambda}_{\infty,0}, c^{\mu}\right)$, $\left(l^{\lambda}_{\infty,0}, c^{\mu}_{0}\right)$, $\left(c^{\lambda}_{0}, l^{\mu}_{\infty}\right)$, $\left(c^{\lambda}_{0}, l^{\mu}_{\infty,0}\right)$, $\left(c^{\lambda}_{0}, c^{\mu}\right)$ and $\left(c^{\lambda}_{0}, c^{\mu}_{0}\right)$ have been characterized in [2]. The characterization of the sets $(l^{\lambda}_{1}, l^{\mu}_{1})$, $(l^{\lambda}_{1}, c^{\mu}_{0})$, $(l^{\lambda}_{1}, c^{\mu})$, $(l^{\lambda}_{1}, l^{\mu}_{\infty})$, $(l^{\lambda}_{\infty}, l^{\mu}_{1})$, $(c^{\lambda}, l^{\mu}_{1})$ and $(c^{\lambda}_{0}, l^{\mu}_{1})$ have been presented in [1].

Necessary and sufficient conditions for the λ-conservativity and the improvement of λ-convergence have been found in [12], and for the absolute λ-conservativity and the improvement of absolute λ-convergence have been found in [1]. A short overview on the convergence with speed has been presented in [4] and [14].

We note that the results connected with convergence, absolute convergence and boundedness with speed can be used in several applications, for example in the approximation theory. Besides, Aasma used such results for the estimation of the order of approximation of Fourier expansions in Banach spaces ([5]–[8]).

In this paper we continue the studies started in [1]–[3], [11], [12] and [13]. We describe the matrix transforms related to the α-absolute λ-convergence, giving the characterization for the sets $(l_\alpha^\lambda, l_\infty^\mu)$, $(l_\alpha^\lambda, c^\mu)$, $(l_\alpha^\lambda, c_0^\mu)$ if $0 < \alpha \leq 1$, and for the set $(l_\alpha^\lambda, l_\beta^\mu)$ if $0 < \alpha \leq \beta \leq 1$. Also we present some examples of α-absolute λ-conservative matrices, and matrices, which improve the α-absolute λ-convergence. We consider the α-absolute λ-conservativity and the improvement of the α-absolute λ-convergence in the special cases when A is the Riesz matrix (R, p_n) or the Zweier matrix $Z_{1/2}$.

2. Auxiliary results. In this section we present some results, which we need for the proof of main results. Comparing Theorem 5 of [15] with Propositions 6, 17, 28 of [19] and Theorem 3.1 of [9] (see also [16, Theorem 3.1]) we may immediately formulate the following Lemmas 2.1–2.3:

Lemma 2.1. *A matrix* $A = (a_{nk}) \in (l_\alpha, l_\infty)$ *for* $0 < \alpha \leq 1$ *if and only if*

$$a_{nk} = O(1). \tag{2.1}$$

Lemma 2.2. *A matrix* $A = (a_{nk}) \in (l_\alpha, c)$ *for* $0 < \alpha \leq 1$ *if and only if condition* (2.1) *is satisfied and*

$$\text{there exist the finite limits } \lim_n a_{nk} := a_k \text{ for all } k. \tag{2.2}$$

Moreover, the equation

$$\lim_n A_n x = \sum_k a_k x_k \tag{2.3}$$

holds for every $x = (x_k) \in l_\alpha$.

Lemma 2.3. *A matrix* $A = (a_{nk}) \in (l_\alpha, c_0)$ *for* $0 < \alpha \leq 1$ *if and only if condition* (2.1) *and condition* (2.2) *with* $a_k = 0$ *are satisfied.*

Also we have (see [17, Theorem 2.2 and Corollary 2.4] and [18, Theorem 2.2]) the following result.

Lemma 2.4. *A matrix $A = (a_{nk}) \in (l_\alpha, l_\beta)$ for $0 < \alpha \le \beta \le 1$ if and only if*

$$\sum_n |a_{nk}|^\beta = O\,(1)\,.$$

Moreover, if $0 < \alpha < \beta \le 1$, then $A = (a_{nk}) \in (l_\alpha, l_\beta)$ if and only if $A = (a_{nk}) \in (l_\beta, l_\beta)$.

3. Main results.

First we prove

Theorem 3.1. *A matrix $A = (a_{nk}) \in \left(l_\alpha^\lambda, l_\beta^\mu\right)$ for $0 < \alpha \le \beta \le 1$ if and only if condition (2.2) is satisfied, and*

$$Ae = (\mathfrak{A}_n) \in l_\beta^\mu,\ \mathfrak{A}_n := A_n e = \sum_k a_{nk} \tag{3.1}$$

$$\frac{a_{nk}}{\lambda_k} = O(1), \tag{3.2}$$

$$\frac{1}{\lambda_k^\beta} \sum_n \left[\mu_n \left|a_{nk} - a_k\right|\right]^\beta = O(1). \tag{3.3}$$

Proof. *Necessity.* Assume that $A \in \left(l_\alpha^\lambda, l_\beta^\mu\right)$. It is easy to see that $e \in l_\alpha^\lambda$ and $e^k \in l_\alpha^\lambda$. Hence conditions (2.2) and (3.1) hold. Since, from (1.1) we have

$$x_k = \frac{v_k}{\lambda_k} + s;\ \ s := \lim_k x_k,\ \ (v_k) \in l_\alpha$$

for every $x := (x_k) \in l_\alpha^\lambda$, it follows that

$$A_n x = \sum_k \frac{a_{nk}}{\lambda_k} v_k + s\mathfrak{A}_n. \tag{3.4}$$

As $(\mathfrak{A}_n) \in l_\beta^\mu$ by (3.1), then, from (3.4) we obtain that the matrix

$$A_\lambda := \left(\frac{a_{nk}}{\lambda_k}\right)$$

transforms this sequence $(v_k) \in l_\alpha$ into c. In addition, for every sequence $(v_k) \in l_\alpha$, the sequence $(v_k/\lambda_k) \in c_0$. But, for (v_k/λ_k), there exists a convergent sequence $x := (x_k)$ with $s := \lim_k x_k$, such that $v_k/\lambda_k = x_k - s$. So we have proved that, for every sequence $(v_k) \in l_\alpha$ there exists a sequence $(x_k) \in l_\alpha^\lambda$ such that $v_k = \lambda_k (x_k - s)$. Hence $A_\lambda \in (l_\alpha, c)$. This implies, by Lemma 2.2, that condition (3.2) is satisfied and the finite limit

$$\phi := \lim_n A_n x = \sum_k \frac{a_k}{\lambda_k} v_k + s \lim_n \mathfrak{A}_n$$

exists for every $x \in l_1^\lambda$. Writing

$$\mu_n(A_n x - \phi) = \mu_n \sum_k \frac{a_{nk} - a_k}{\lambda_k} v_k + s\mu_n(\mathfrak{A}_n - \lim_n \mathfrak{A}_n), \tag{3.5}$$

we obtain, by (3.1) that the matrix $A_{\lambda,\mu} \in (l_\alpha, l_\beta)$, where

$$A_{\lambda,\mu} := \left(\mu_n \frac{a_{nk} - a_k}{\lambda_k} \right).$$

Hence condition (3.3) is satisfied by Lemma 2.4.

Sufficiency. Let conditions (2.2) and (3.1)–(3.3) be fulfilled. Then relation (3.4) also holds for every $x \in l_\alpha^\lambda$ and $(\mathfrak{A}_n) \in l_\beta^\mu$ by (3.1). In addition, $A_\lambda \in (l_\alpha, c)$ and the finite limit ϕ exists for every $x \in l_\alpha^\lambda$ by Lemma 2.2, since (2.2) and (3.2) hold. Hence relation (3.5) holds for every $x \in l_\alpha^\lambda$. As (3.3) is valid, then $A_{\lambda,\mu} \in (l_\alpha, l_\beta)$ by Lemma 2.4. Therefore, due to (3.1), $A \in \left(l_\alpha^\lambda, l_\beta^\mu\right)$. □

Corollary 3.1. *A matrix $A = (a_{nk})$ is regular on l_α^λ for $0 < \alpha \le 1$ if and only if condition* (2.2) *with $a_k = 0$ and condition* (3.2) *are satisfied and*

$$\lim_n \mathfrak{A}_n = 1. \tag{3.6}$$

Proof. *Necessity.* Assume that A is regular on l_α^λ *for* $0 < \alpha \le 1$; i.e., $\lim_n A_n x = s$ for every sequence $x \in l_\alpha^\lambda$. Then condition (3.6) is satisfied, since $e \in l_\alpha^\lambda$, and relation (3.5) holds for every $x := (x_k) \in l_\alpha^\lambda$. This implies that A_λ transforms every sequence $(v_k) \in l_\alpha$ into c_0. Hence condition (2.2) with $a_k = 0$ and condition (3.2) are satisfied by Lemma 2.3.

Sufficiency. Let all the conditions of the present corollary are satisfied. Then relation (3.4) also holds for every $x \in l_\alpha^\lambda$. As conditions (2.2) (with $a_k = 0$) and (3.2) hold, then $A_\lambda \in (l_\alpha, c_0)$ by Lemma 2.3. Therefore $\lim_n A_n x = s$ for every $x \in l_\alpha^\lambda$ by (3.6). Thus A is regular on l_α^λ. □

Remark 3.1. If $\lambda_k = O(1)$, then condition (3.2) can be replaced by condition (2.1) in Theorem 3.1 and Corollary 3.1, and condition (3.3) by condition

$$\sum_n [\mu_n |a_{nk} - a_k|]^\beta = O(1)$$

in Theorem 3.1.

Next we present the following theorems.

Theorem 3.2. *A matrix $A = (a_{nk}) \in \left(l_\alpha^\lambda, c^\mu\right)$ for $0 < \alpha \le 1$ if and only if conditions* (2.2), (3.2) *are satisfied, $Ae \in c^\mu$, and*

$$\text{there exist the finite limits } \lim_n \mu_n(a_{nk} - a_k) := a_k^\mu \text{ for all } k, \tag{3.7}$$

$$\mu_n \frac{a_{nk} - a_k}{\lambda_k} = O(1). \tag{3.8}$$

Theorem 3.3. *A matrix $A = (a_{nk}) \in \left(l_\alpha^\lambda, c_0^\mu\right)$ for $0 < \alpha \le 1$ if and only if $Ae \in c_0^\mu$ and conditions* (2.2), (3.2), (3.7) *with $a_k^\mu = 0$, and* (3.8) *hold.*

Theorem 3.4. *A matrix $A = (a_{nk}) \in \left(l_\alpha^\lambda, l_\infty^\mu\right)$ for $0 < \alpha \le 1$ if and only if $Ae \in l_\infty^\mu$ and conditions* (2.2), (3.2), (3.8) *hold.*

As the proofs of Theorems 3.2–3.4 are similar to the proof of Theorem 3.1, we only give a short description of the proofs. As $e \in l_\alpha^\lambda$, then instead of condition (3.1), for Theorems 3.2–3.5 we correspondingly obtain $Ae \in c^\mu$, $Ae \in c_0^\mu$, and $Ae \in l_\infty^\mu$. Also, the matrix transform $A_n x$ for $x := (x_k) \in l_\alpha^\lambda$ may be presented in the form (3.4). In addition, in the proof of Theorems 3.2–3.4 (similarly to the proof of Theorem 3.1) $A_\lambda \in (l_\alpha, c)$. Hence the finite limit ϕ exists for every $x \in l_1^\lambda$ by Lemma 2.2. Hence relation (3.5) also holds for every $x \in l_\alpha^\lambda$. The role of the matrix $A_{\lambda,\mu}$ is different in the proof of each theorem: in the proof of Theorem 3.2, $A_{\lambda,\mu} \in (l_\alpha, c)$, in the proof of Theorem 3.3, $A_{\lambda,\mu} \in (l_\alpha, c_0)$, in the proof of Theorem 3.4, $A_{\lambda,\mu} \in (l_\alpha, l_\infty)$. Therefore, for completing the proof of Theorem 3.2 it is necessary to use Lemma 2.2, for completing the proof of Theorem 3.3 – Lemma 2.3, and for completing the proof of Theorem 3.4 – Lemma 2.4.

Remark 3.2. For $\alpha = \beta = 1$ Theorem 3.1, and for $\alpha = 1$ Theorems 3.2–3.4 have been proved in [1].

Remark 3.3. If $\lambda_k = O(1)$, then condition (3.2) can be replaced by condition (2.1), and condition (3.8) by condition

$$\mu_n \left| a_{nk} - a_k \right| = O(1)$$

in Theorems 3.2–3.4.

Remark 3.4. If $\mu_k = O(1)$, then conditions (3.7) and (3.8) are redundant in Theorems 3.2–3.4.

Corollary 3.2. *Condition* (3.2) *can be replaced by condition*

$$\frac{a_k}{\lambda_k} = O(1) \tag{3.9}$$

in Theorems 3.1–3.4.

Proof. It is easy to see that condition (3.9) follows from (2.2) and (3.2). From the other side, conditions (2.2), (3.3) and (3.9) or (2.2), (3.8) and (3.9) imply the validity of (3.2). Indeed, first from (3.3) or from (3.8) we obtain that

$$\frac{a_{nk} - a_k}{\lambda_k} = O(1), \tag{3.10}$$

since (μ_n) is bounded from below due to $\mu_n \geq \mu_0 > 0$ for every n. As

$$\frac{a_{nk}}{\lambda_k} = \frac{a_{nk} - a_k}{\lambda_k} + \frac{a_k}{\lambda_k},$$

then

$$\frac{|a_{nk}|}{\lambda_k} \leq \frac{|a_{nk} - a_k|}{\lambda_k} + \frac{|a_k|}{\lambda_k}.$$

Moreover, the finite limits a_k exist by (2.2). Hence condition (3.2) is satisfied by (3.9) and (3.10). □

4. $\boldsymbol{\alpha}$-absolutely $\boldsymbol{\lambda}$-conservative matrices and the improvement of $\boldsymbol{\alpha}$-absolute $\boldsymbol{\lambda}$-convergence. In this section we study the α-absolute λ-conservativity of matrices and the improvement of α-absolute λ-convergence by matrices. First we present some examples on α-absolute λ-conservative matrices, and on matrices, improving the α-absolute λ-convergence.

Example 4.1. Let λ be defined by

$$\lambda_k := (k+1)^r, \ r \geq 0. \tag{4.1}$$

Then a lower triangular matrix $A = (a_{nk})$, defined by

$$a_{nk} := \frac{k+1}{(n+1)^c}; \; c \in \mathrm{R}, \tag{4.2}$$

is α-absolutely λ-conservative $(0 < \alpha \leq 1)$ if

$$1 \leq r < c - \frac{1}{\alpha} - 2. \tag{4.3}$$

For proving it, we show that all conditions of Theorem 3.1 for $\mu_n = \lambda_n$ are satisfied. It is easy to see that conditions (2.2) and (3.2) with $a_k = 0$ hold, and

$$T_k := \frac{1}{\lambda_k^\alpha} \sum_n [\mu_n \, |a_{nk}|]^\alpha = \frac{1}{(k+1)^{\alpha(r-1)}} \sum_{n=k}^{\infty} \frac{1}{(n+1)^{\alpha(c-r)}} = O(1)$$

for $\alpha(r-1) > 0$ and $\alpha(c-r) > 0$; i.e., $T_k = O(1)$ for

$$1 \leq r < c - 1/\alpha. \tag{4.4}$$

As (4.4) follows from (4.3), then condition (3.3) is satisfied. Finally we show that condition (3.1) holds. As

$$A_n e = \sum_{k=0}^{n} \frac{k+1}{(n+1)^\alpha} = \frac{1}{2} \frac{(n+2)(n+1)}{(n+1)^c},$$

then

$$\lim_n A_n e = 0,$$

since $c > 2$ by (4.3). Hence

$$S := \sum_n [\lambda_n \, |A_n e|]^\alpha = \left(\frac{1}{2}\right)^\alpha \sum_n \left(\frac{n+2}{n+1}\right)^\alpha \frac{1}{(n+1)^{\alpha(c-r-2)}} = O(1)$$

for

$$\alpha(c - r - 2) > 1. \tag{4.5}$$

As (4.5) follows from (4.3), then condition (3.1) holds. We note that the collection $\{\alpha, c, r\}$ satisfying (4.3) is not empty. For example, if $\alpha = 3/4$ and $c = 5$, then condition (4.3) holds for$1 \leq r < 5/3$.

Example 4.2. Let λ be defined by (4.1), and μ by

$$\mu_n := (n+1)^t, \ t > 1.$$

Then the matrix $A = (a_{nk})$, defined by (4.2), improves the α-absolute ($0 < \alpha \leq 1$) λ-convergence if

$$1 \leq r < t < c - \frac{1}{\alpha} - 2. \tag{4.6}$$

Indeed, as in Example 4.1, conditions (2.2) and (3.2) with $a_k = 0$ hold. In this case

$$T_k = \frac{1}{(k+1)^{\alpha(r-1)}} \sum_{n=k}^{\infty} \frac{1}{(n+1)^{\alpha(c-t)}} = O(1)$$

and

$$S = \left(\frac{1}{2}\right)^{\alpha} \sum_{n} \left(\frac{n+2}{n+1}\right)^{\alpha} \frac{1}{(n+1)^{\alpha(c-t-2)}} = O(1),$$

if

$$r \geq 1, \ \alpha(c-t) > 1 \ \alpha(c-t-2) > 1. \tag{4.7}$$

As (4.7) follows from (4.6), then conditions (3.1) and (3.3) are satisfied.

Hence all conditions of Theorem 3.1 hold and $\mu_n/\lambda_n \neq O(1)$. We note that the collection $\{\alpha, c, r, t\}$ satisfying (4.6) is not empty. For example, if $\alpha = 3/4$ and $c = 5$, then condition (4.3) holds for $1 \leq r < t < 5/3$.

As from the point of view of applications, regular matrices are still the most important, let us look at some examples of them. Let (p_n) be a sequence of nonzero real numbers and $P_n = p_0 + \cdots + p_n \neq 0$. Then the Riesz matrix (R, p_n), defined by a lower triangular matrix $A = (a_{nk})$, is given by equalities [4, p. 29 or p. 131]

$$a_{nk} = \frac{p_k}{P_n}, \ k \leq n.$$

Proposition 4.1. *The Riesz matrix* (R, p_n) *with* $\lim_n P_n = \infty$ *is* α*-absolutely* λ*-conservative if and only if*

$$\left[\frac{p_k}{\lambda_k}\right]^{\alpha} \sum_{n=k}^{\infty} \left[\frac{\lambda_n}{P_n}\right]^{\alpha} = O(1). \tag{4.8}$$

P r o o f. For the proof it is sufficient to show that all conditions of Theorem 3.1 are satisfied for $\mu_n = \lambda_n$ and $A = (R, p_n)$. It is easy to see that in this case $Ae = e$ and $a_k = 0$; so conditions (3.1) and (3.2) are satisfied. Therefore the validity of condition (3.3) implies condition (2.2). In addition, condition (3.3) takes now the form (4.8). □

Corollary 4.1. *The Riesz matrix* (R, p_n) *with* $\lim_n P_n = \infty$ *and* $p_n > 0$ *does not* α*-absolutely* λ*-conservative for any unbounded* λ*.*

P r o o f. As $p_n > 0$ and $\lim_n P_n = \infty$, then the series

$$\sum_n \frac{p_n}{P_n}$$

diverges by Dini's test. Therefore there exist $M > 0$ such that

$$P_n < Mp_n(n+1)ln(n+1).$$

Hence

$$L_k := \sum_{n=k}^{\infty} \left[\frac{\lambda_n}{P_n}\right]^{\alpha} > \frac{1}{M^{\alpha}} \sum_{n=k}^{\infty} \frac{\lambda_n^{\alpha}}{p_n^{\alpha}[(n+1)ln(n+1)]^{\alpha}}.$$

For the boundedness of L_k it is necessary that $[p_n/\lambda_n]^{\alpha} \neq O(1)$. But in this case condition (4.8) does not hold. Therefore, (R, p_n) does not α-absolutely λ-conservative for any unbounded λ. □

The matrix (R, p_n) is regular for $\lim_n P_n = \infty$ and $p_n > 0$ (see [9, Corollary 17.1]). Therefore, the question may arise as to whether it exists at all a regular α-absolutely λ-conservative matrix. The answer is yes. To confirm the assertion, let's look at the Zweier matrix $Z_{1/2}$, defined by the lower triangular matrix $A = (a_{nk})$, where (see [10, p. 14] or [4, p. 3]) $a_{00} = 1/2$ and

$$a_{nk} = \begin{cases} \frac{1}{2}, & \text{if } k = n-1 \text{ and } k = n; \\ 0, & \text{if } k < n-1 \end{cases}$$

for $n \geq 1$. The method $A = Z_{1/2}$ is regular (see [4, p. 3]).

Proposition 4.2. *The Zweier matrix* $Z_{1/2}$ *is* α*-absolutely* λ*-conservative if and only if* $\lambda_{k+1}/\lambda_k = O(1)$*.*

P r o o f. For the proof it is sufficient to show that all conditions of Theorem 3.1 are satisfied for $\mu_n = \lambda_n$ and $A = Z_{1/2}$. As $A_0 e = 1/2$ and $A_n e = 1$

for $n \geq 1$, similarly to the proof of Proposition 4.1, it is possible to show that conditions (3.1) and (2.2) hold, and condition (3.2) follows from (3.3). Condition (3.3) takes now the form

$$\left(\frac{1}{2}\right)^{\alpha}\left[1+\left(\frac{\lambda_{k+1}}{\lambda_k}\right)^{\alpha}\right]=O(1),$$

which is equivalent to $\lambda_{k+1}/\lambda_k = O(1)$. This completes the proof. □

Remark 4.1. It is not difficult to see that λ, defined by (4.1), satisfies the condition of Proposition 4.2.

Next we prove the following result.

Lemma 4.1. *If a matrix $A = (a_{nk})$ improves the α-absolute λ-convergence ($0 < \alpha \leq 1$), then*

$$\lim_r \sum_{n=k}^{r} |a_{nk} - a_k|^{\alpha} = 0, \ r > k \tag{4.9}$$

for every k.

Proof. If a matrix $A = (a_{nk})$ improves the α-absolute λ-convergence, then $\mu_n/\lambda_n \neq O(1)$ by definition. Suppose (4.9) does not hold. Then, for every $\epsilon > 0$ and for every k, there exists a sequence of indexes $\left(i_r^k\right)$, such that

$$\sum_{n=k}^{i_r^k} |a_{nk} - a_k|^{\alpha} \geq \epsilon.$$

Hence

$$\frac{1}{\lambda_k^{\alpha}} \sum_{n=k}^{i_r^k} [\mu_n |a_{nk} - a_k|]^{\alpha} \geq \epsilon \left[\frac{\mu_k}{\lambda_k}\right]^{\alpha}$$

(since μ is monotonically increasing). As a matrix A improves the α-absolute λ-convergence, then

$$\frac{1}{\lambda_k^{\alpha}} \sum_{n=k}^{i_r^k} [\mu_n |a_{nk} - a_k|]^{\alpha} = O(1) \tag{4.10}$$

by Theorem 3.1. But condition (4.10) can be satisfied only in the case if $\mu_n/\lambda_n = O(1)$, which contradicts $\mu_n/\lambda_n \neq O(1)$. Thus condition (4.9) holds. □

As $a_k = 0$ for every regular matrix (see [10, pp 46–47] or [9, pp 17–19]), from Lemma 4.1 we immediately obtain the following corollary.

Corollary 4.2. *Any regular matrix cannot improve the α-absolute λ-convergence for any unbounded λ.*

Remark 4.2. Propositions 4.1, 4.2, Corollaries 4.1, 4.2 and Lemma 4.1 for $\alpha = 1$ were proved in [1].

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