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ON THE COMPLEX POLYNOMIAL INEQUALITIES CONCERNING SOME OPERATORS

Tawheeda Akhter, Shabir Ahmad Malik

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Abstract. In this paper, we establish some new polynomial inequalities involving various operators and their relationship with the existing results. All these inequalities have been proved under the assumption that the underlying polynomial is being constrained with respect to its zeros.

1. Introduction.

Let $P(z) = \sum_{v=1}^{n} b_v z^v$ be a complex polynomial of degree at most n and $P'(z)$, the derivative of $P(z)$. Unless otherwise stated, let $\mathbb{T}_1 = \{z : |z| = 1\}$, $\mathbb{E}_1^- = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{E}_1^+ = \{z \in \mathbb{C} : |z| > 1\}$ respectively. For any holomorphic function f defined on $\mathbb{T}_1$, we write $\|f\| = \sup_{z\in\mathbb{T}_1} |f(z)|$, the supremum norm of f on $\mathbb{T}_1$. The Bernstein's classical inequality states that

$$\max_{z\in\mathbb{T}_1} |P'(z)| \le n \max_{z\in\mathbb{T}_1} |P(z)| \tag{1.1}$$

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holds for every polynomial $P(z)$. This result is best possible and the extremal polynomial for (1.1) is $P(z) = \alpha z^n$, $\alpha \neq 0$. Later, in 1930 (see [3]) Bernstein proved a classical majorization result, which states that, for any two polynomials $P(z)$ and $Q(z)$ with $\deg P \leq \deg Q$ and if $Q(z)$ has all its zeros in $\mathbb{T}_1 \cup \mathbb{E}_1^-$, then the majorization $|P(z)| \leq |Q(z)|$ on the unit circle $\mathbb{T}_1$ implies the majorization of their derivatives

$$|P'(z)| \leq |Q'(z)| \tag{1.2}$$

on $\mathbb{T}_1$. Under the same hypothesis as of (1.2) the inequality

$$\left|\frac{zP'(z)}{n} + \beta\frac{P(z)}{2}\right| \leq \left|\frac{zQ'(z)}{n} + \beta\frac{Q(z)}{2}\right| \tag{1.3}$$

holds for every β such that $|\beta| \leq 1$, n being the degree of the polynomial $Q(z)$. Inequality (1.3) is due to Malik and Vong [9]. It is easy to see that (1.3) improves upon the inequality (1.2).

Since the equality in (1.1) holds for polynomials having all their zeros at the origin, it is possible to improve upon the bound in (1.1), if we restrict to the class of polynomials having no zeros in $\mathbb{E}_1^-$. In this direction, we have the following result conjectured by Erdös [5] and later proved by Lax [7].
If $P(z)$ is a polynomial of degree n having no zeros in $\mathbb{E}_1^-$, then

$$\max_{z\in\mathbb{T}_1} |P'(z)| \leq \frac{n}{2} \max_{z\in\mathbb{T}_1} |P(z)|. \tag{1.4}$$

The result is best possible and equality in (1.4) holds for any polynomial which has all its zeros on $\mathbb{T}_1$. Under the same hypothesis Aziz and Dawood [1] proved the following refinement of inequality (1.4).

$$\|P'\| \leq \frac{n}{2}\{\|P\| - \min_{z\in\mathbb{T}_1} |P(z)|\} \tag{1.5}$$

If $P(z)$ is a polynomial of degree n, and α is any complex number, then

$$\begin{aligned} D_\alpha P(z) &= -\left[\frac{P(z)}{(z-\alpha)^n}\right]'(z-\alpha)^{n+1} \\ &= nP(z) + (\alpha - z)P'(z), \end{aligned} \tag{1.6}$$

is called the *polar derivative* of $P(z)$. Note that $D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and it generalizes the concept of "ordinary derivative" is evident and convincing from the fact that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha}=P'(z) \tag{1.7}$$

uniformly with respect to z for $\mathbb{T}_R\cup\mathbb{E}_R^-$, $R>0$.

Given a polynomial $P(z)$, we can construct a sequence of polar derivatives or so-called higher order derivatives with respect to finitely many poles as given below

$$D_{\alpha_1}P(z)=nP(z)+(\alpha_1-z)P'(z)$$
$$D_{\alpha_2}D_{\alpha_1}P(z)=(n-1)D_{\alpha_1}P(z)+(\alpha_2-z)(D_{\alpha_1}P(z))'$$
$$\cdots\cdots\cdots$$
$$D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}P(z)=(n-t+1)D_{\alpha_{t-1}}\cdots D_{\alpha_1}P(z)+(\alpha_t-z)(D_{\alpha_{t-1}}\cdots D_{\alpha_1}P(z))',$$

for $2\le t\le n$.

Like the t^{th} ordinary derivative $P^{(t)}(z)$ of $P(z)$, the t^{th} polar derivative $D_{\alpha_t}\ldots D_{\alpha_2}D_{\alpha_1}P(z)$ of $P(z)$ is a polynomial of degree at most $n-t$. In this connection, the following results are due to Liman et al. [8] holds good.

Theorem A. *Let $Q(z)$ be a polynomial of degree n having all its zeros in $\mathbb{T}_1\cup\mathbb{E}_1^-$ and $P(z)$ is a polynomial of degree at most n. If $|P(z)|\le|Q(z)|$ on $\mathbb{T}_1$, then for all $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\ge1,|\beta|\le1$*

$$\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)\right|\le\left|zD_\alpha Q(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)Q(z)\right| \text{ for } |z|\ge1.$$

Theorem B. *Let $P(z)$ and $Q(z)$ be polynomials with degree of $P(z)$ not exceeding that of $Q(z)$. Let $Q(z)$ be a polynomial of degree n having all its zeros in $\mathbb{T}_1\cup\mathbb{E}_1^-$ and $P(z)$ is a polynomial of degree at most n. If $|P(z)|\le|Q(z)|$ on $\mathbb{T}_1$, then for any $\alpha\in\mathbb{C}$ with $|\alpha|\ge1$*

$$\left|\frac{zD_\alpha P(z)}{n}\right|+(|\alpha|-1)\left|\frac{Q(z)}{2}\right|\le\left|\frac{zD_\alpha Q(z)}{n}\right|+(|\alpha|-1)\left|\frac{P(z)}{2}\right| \text{ for } |z|\ge1.$$

Theorem C. *If $P(z)$ is a polynomial of degree n which does not vanish in $\mathbb{E}_1^-$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1, |\beta| \leq 1$ and $|z| = 1$*

$$\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)\right|$$
$$\leq \frac{n}{2}\left[\left\{\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right| + \left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\}\max_{z\in\mathbb{T}_1}|P(z)|\right.$$
$$\left. - \left\{\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right| - \left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\}\min_{z\in\mathbb{T}_1}|P(z)|\right].$$

We refer to [10] *and* [11] *for more results involving maximum modulus of sequence of polar derivatives. The following result which is due to Aziz and Shah* [2] *gives us the bound for the minimum modulus of $D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}P(z)$.*

Theorem D. *If $P(z)$ is a polynomial of degree n having all its zeros in $\mathbb{T}_1 \cup \mathbb{E}_1^-$, then for $z \in \mathbb{T}_1 \cup \mathbb{E}_1^+$*

$$\min_{z\in\mathbb{T}_1}|D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}P(z)|$$
$$\geq n(n-1)\cdots(n-t+1)|\alpha_1\alpha_2\cdots\alpha_t|\;|z|^{n-t}\min_{z\in\mathbb{T}_1}|P(z)|,$$

where α_i are real or complex numbers with $|\alpha_i| \geq 1,\ i = 1, 2, \ldots, t;\ t \leq n-1$. The result is best possible and equality holds for the polynomial $P(z) = me^{i\beta}z^n$, $m > 0$.

Let $P(z)$ be a polynomial of degree n, then the operator which is denoted as B maps a polynomial $P(z)$ into $B[P(z)]$ and is defined as

$$B[P(z)] = \lambda_0 P(z) + \lambda_1\left(\frac{nz}{2}\right)\frac{P'(z)}{1!} + \lambda_2\left(\frac{nz}{2}\right)^2\frac{P''(z)}{2!},$$

where λ_0, λ_1 and λ_2 are such that $\lambda_0 + \binom{n}{1}\lambda_1 z + \binom{n}{2}\lambda_2 z^2 \neq 0$ in $|z| > |z - \frac{n}{2}|$. This operator B is well known as B_n operator and was introduced by Q. I. Rahman [13](see also Rahman and Schmeisser [14, Lemma 14.5.7]).

Definition 1. *If $P(z)$ is a polynomial of degree n and $D_\alpha P(z)$ be its polar derivative which is a polynomial of degree at most $n-1$, then*

$$B[D_\alpha P(z)] = \lambda_0 D_\alpha P(z) + \lambda_1\left(\frac{mz}{2}\right)\frac{D_\alpha P'(z)}{1!} + \lambda_2\left(\frac{mz}{2}\right)^2\frac{D_\alpha P''(z)}{2!},$$

where $0 \le m \le n-1$, λ_0, λ_1 *and* λ_2 *are such that* $\lambda_0 + \binom{m}{1}\lambda_1 z + \binom{m}{2}\lambda_2 z^2 \neq 0$ *in* $|z| > \left|z - \frac{m}{2}\right|$. *Note that, by Definition,* $B[D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} P(z)]$ *is also defined similarly.*

In this paper, we combine different ideas and techniques in order to prove some operator preserving inequalities involving the operators B, D_α and $D_{\alpha_t} \dots D_{\alpha_2} D_{\alpha_1}$ of $P(z)$ jointly. We will see that the established results provide the generalizations of Theorems A, B, C, D, and many other results in this direction as well. This paper consists of four sections. The first section which is being devoted to introduction and preliminary results. In Section 2 we will formulate the main results. Section 3 contains some auxiliary results and the proofs of the main results are presented in Section 4.

For the sake of simplicity, we use the following notations:

$$A_{\alpha_t} = (|\alpha_1| - 1)(|\alpha_2| - 1) \cdots (|\alpha_t| - 1),$$

$$N_t = n(n-1)(n-2) \cdots (n-t+1),$$

$$\Lambda_t = \alpha_t \alpha_{t-1} \cdots \alpha_1.$$

2. Main Results.

We first present the following result:

Theorem 1. *If* $P(z)$ *is a polynomial of degree* n*, having all its zeros in* $\mathbb{T}_1 \cup \mathbb{E}_1^-$*, then for every real or complex numbers* $\beta, \alpha_1, \alpha_2, \dots, \alpha_t$, $1 \le t < n$ *with* $|\alpha_i| \ge 1$, $i = 1, 2, 3, \dots, t$, $|\beta| \le 1$

$$\left| B[D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} P(z)] + \frac{N_t \beta}{2^t} A_{\alpha_t} B[P(z)] \right| \ge C N_t \left| \Lambda_t B[z^{n-t}] + \frac{\beta}{2^t} A_{\alpha_t} B[z^n] \right|,$$

where $C = \min_{z \in \mathbb{T}_1} |P(z)|$. *This result is best possible and we see that equality holds for the polynomial* $P(z) = a\, e^{i\gamma} z^n$, $a > 0$.

Remark 1. If in Theorem 1, $t \to 1$ and $\beta \to 0$, we under the same hypothesis obtain the following inequality

$$|B[D_\alpha P(z)]| \ge n \left| \alpha\ B[z^{n-t}] \right| \min_{z \in \mathbb{T}_1} |P(z)|,$$

and this result is due to Bidkham and Mezerji [4].

By the definition of B operator, it follows from Theorem 1, that

$$
\begin{aligned}
&\Bigg|\lambda_0 D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}P(z)+\lambda_1\left(\frac{mz}{2}\right)\frac{D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}P'(z)}{1!}\\
&+\lambda_2\left(\frac{mz}{2}\right)^2\frac{D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}P''(z)}{2!}\\
&+\frac{N_t\beta}{2^t}A_{\alpha_t}\left[\lambda_0 P(z)+\lambda_1\left(\frac{nz}{2}\right)\frac{P'(z)}{1!}+\lambda_2\left(\frac{nz}{2}\right)^2\frac{P''(z)}{2!}\right]\Bigg|\\
&\geq C\ N_t\Bigg|\Lambda_t\Bigg[\lambda_0 z^{n-t}+\lambda_1\left(\frac{(n-t)z}{2}\right)\frac{(n-t)z^{n-t-1}}{1!}\\
&+\lambda_2\left(\frac{(n-t)z}{2}\right)^2\frac{(n-t)(n-t-1)z^{n-t-2}}{2!}\Bigg]\\
&+\frac{\beta}{2^t}A_{\alpha_t}\left[\lambda_0 z^n+\lambda_1\left(\frac{nz}{2}\right)\frac{nz^{n-1}}{1!}+\lambda_2\left(\frac{nz}{2}\right)^2\frac{(n)(n-1)z^{n-2}}{2!}\right]\Bigg|.
\end{aligned}
\tag{2.1}
$$

If we set $\lambda_1=\lambda_2=0$ in (2.1), we obtain the following generalization of Theorem D.

Corollary 1. *If $P(z)$ is a polynomial of degree n, having all its zeros in $\mathbb{T}_1\cup\mathbb{E}_1^-$, then for every real or complex numbers $\beta,\alpha_1,\alpha_2,\ldots,\alpha_t,\ 1\leq t<n$ with $|\alpha_i|\geq 1,\ i=1,2,3,\ldots,t,\ |\beta|\leq 1$*

$$
\left|D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}P(z)+\frac{N_t\beta}{2^t}A_{\alpha_t}P(z)\right|\geq CN_t\left|\Lambda_t z^{n-t}+\frac{\beta}{2^t}A_{\alpha_t}z^n\right|,
\tag{2.2}
$$

where $C=\min_{z\in\mathbb{T}_1}|P(z)|$.

Taking $\beta=0$ in Corollary 1 we immediately get Theorem D.

If we choose $\alpha_1=\alpha_2=\cdots=\alpha_t=\alpha$, then by dividing both sides of inequality (2.2) by $|\alpha|^t$ and letting $|\alpha|\to\infty$, then according to Definition 1, $m=n-t$, therefore taking (1.7) into consideration, we obtain the following result.

Corollary 2. *If $P(z)$ is a polynomial of degree n, having all its zeros in $\mathbb{T}_1\cup\mathbb{E}_1^-$, then for every real or complex number β with $|\beta|\leq 1$*

$$
\left|P^{(t)}(z)+\frac{N_t\beta}{2^t}P(z)\right|\geq CN_t\left|z^{n-t}+\frac{\beta}{2^t}z^n\right|,
$$

where $C = \min_{z \in \mathbb{T}_1} |P(z)|$.

It is important to note that when $t \to 1$ and $\beta \to 0$ in Corollary 2, we get the following inequality which is ascribed to Aziz and Dawood [1].

$$\min_{z \in \mathbb{T}_1} |P'(z)| \geq n \min_{z \in \mathbb{T}_1} |P(z)|.$$

Again choosing $\lambda_0 = \lambda_2 = 0$ in (2.1) and since

$$D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} P(z) = (n-t+1) D_{\alpha_{t-1}} \cdots D_{\alpha_1} P(z) + (\alpha_t - z)(D_{\alpha_{t-1}} \cdots D_{\alpha_1} P(z))',$$

then it yields from Theorem 1, that:

Corollary 3. *If $P(z)$ is a polynomial of degree n which vanishes in $\mathbb{T}_1 \cup \mathbb{E}_1^-$, then for every real or complex numbers $\beta, \alpha_1, \alpha_2, \ldots, \alpha_t$, $1 \leq t < n$ with $|\alpha_i| \geq 1$, $i = 1, 2, 3, \ldots, t$, $|\beta| \leq 1$*

$$\Big| m(n-t) D_{\alpha_{t-1}} \cdots D_{\alpha_2} D_{\alpha_1} P'(z) +$$
$$(\alpha_t - z)(D_{\alpha_{t-1}} \cdots D_{\alpha_2} D_{\alpha_1} P'(z))' + \frac{N_t \beta}{2^t} A_{\alpha_t} n P'(z) \Big|$$
$$\geq C N_t \left| \Lambda_t (n-t)^2 z^{n-t-1} + \frac{\beta}{2^t} A_{\alpha_t} n^2 z^{n-1} \right|,$$

where $C = \min_{z \in \mathbb{T}_1} |P(z)|$. *This result is best possible and we see that equality holds for* $P(z) = a\, e^{i\gamma} z^n$, $a > 0$.

Corollary 4. *If $P(z)$ is a polynomial of degree n having all its zeros in $\mathbb{T}_1 \cup \mathbb{E}_1^-$, then for every real or complex number β with $|\beta| \leq 1$*

$$\left| (n-t)^2 P^{(t+1)}(z) + \frac{n N_t \beta}{2^t} P'(z) \right| \geq C N_t \left| (n-t)^2 z^{n-t-1} + \frac{n^2 \beta}{2^t} z^{n-1} \right|,$$

where $C = \min_{z \in \mathbb{T}_1} |P(z)|$. *This result is best possible and equality holds for the polynomial* $P(z) = a\, e^{i\gamma} z^n$, $a > 0$.

Now we shall establish the following result which is a generalization of Theorem A.

Theorem 2. *Let $Q(z)$ be a polynomial of degree n having all its zeros in $\mathbb{T}_1 \cup \mathbb{E}_1^-$ and $P(z)$ is a polynomial of degree at most n. If $|P(z)| \le |Q(z)|$ on $\mathbb{T}_1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \ge 1,\ |\beta| \le 1$*

$$\left|B[D_\alpha P(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right|$$

$$\le \left|B[D_\alpha Q(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(z)]\right| \text{ for } |z| \ge 1.$$

If in case $\beta \to 0$, then we obtain the following result:

Corollary 5. *Let $Q(z)$ be a polynomial of degree n having all its zeros in $\mathbb{T}_1 \cup \mathbb{E}_1^-$ and $P(z)$ is a polynomial of degree at most n. If $|P(z)| \le |Q(z)|$ on $\mathbb{T}_1$, then for all $\alpha \in \mathbb{C}$ with $|\alpha| \ge 1$*

$$|B[D_\alpha P(z)]| \le |B[D_\alpha Q(z)]| \text{ for } |z| \ge 1.$$

Substituting for $B[D_\alpha P(z)]$ (see Definition 1) in Theorem 2, we obtain the following inequality

$$\begin{aligned}
&\left|\lambda_0 D_\alpha P(z) + \lambda_1\left(\frac{m_1 z}{2}\right)\frac{D_\alpha P'(z)}{1!} + \lambda_2\left(\frac{m_1 z}{2}\right)^2\frac{D_\alpha P''(z)}{2!}\right. \\
&\left.+n\beta\left(\frac{|\alpha|-1}{2}\right)\left[\lambda_0 P(z) + \lambda_1\left(\frac{dz}{2}\right)\frac{P'(z)}{1!} + \lambda_2\left(\frac{dz}{2}\right)^2\frac{P''(z)}{2!}\right]\right| \\
&\le \left|\lambda_0 D_\alpha Q(z) + \lambda_1\left(\frac{mz}{2}\right)\frac{D_\alpha Q'(z)}{1!} + \lambda_2\left(\frac{mz}{2}\right)^2\frac{D_\alpha Q''(z)}{2!}\right. \\
&\left.+n\beta\left(\frac{|\alpha|-1}{2}\right)\left[\lambda_0 Q(z) + \lambda_1\left(\frac{nz}{2}\right)\frac{Q'(z)}{1!} + \lambda_2\left(\frac{nz}{2}\right)^2\frac{Q''(z)}{2!}\right]\right|,
\end{aligned} \tag{2.3}$$

where $m_1 \le d-1,\ d \le n$ and $m \le n-1$. Inequality (2.3) includes many known polynomial inequalities as special cases, e.g., we obtain Theorem A from (2.3) if $\lambda_1 = \lambda_2 = 0$, and in view of (1.7) we obtain inequality (1.3). For $\lambda_0 = \lambda_2 = 0$, let $|\alpha| \to \infty$ and taking (1.7) into account, we get $m_1 = d-1$ and $m = n-1$. In this connection, under the same hypothesis as of Theorem 2, the following inequality is an immediate from inequality (2.3).

$$\left|(d-1)P''(z) + \frac{n\beta d}{2}P'(z)\right| \le \left|(n-1)Q''(z) + \frac{n^2\beta}{2}Q'(z)\right|.$$

Theorem 3. *If the polynomial $P(z)$ has no zeros in $\mathbb{E}_1^-$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1$, $|\beta| \leq 1$ and for $|z| = 1$*

$$\left|B[D_\alpha P(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right|$$

$$\leq \frac{n}{2}\left[\left|\alpha B[z^{n-1}] + \beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right| + |\lambda_0|\left|1+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right]\max_{z\in\mathbb{T}_1}|P(z)|.$$

Remark 2. By taking $\beta = 0$ in Theorem 3, we get a result due to Bidkham and Mezerji[4, Theorem 2]. Also if we take $\lambda_1 = \lambda_2 = 0$, we obtain a result which is due to Liman et al.[8, Corollary 3]. Besides all this, Theorem 3 includes many well known polynomial inequalities as special cases by choosing λ_0, λ_1 and λ_2 appropriately, see, e.g., inequality (1.4).

Next we will present the following generalization of Theorem C.

Theorem 4. *If $P(z)$ is a polynomial of degree n having no zeros in $\mathbb{E}_1^-$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1, |\beta| \leq 1$ and $|z| = 1$*

$$\left|B[D_\alpha P(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right|$$

$$\leq \frac{n}{2}\left[\left\{\left|\alpha B[z^{n-1}] + \beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right| + |\lambda_0|\left|1+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\}\max_{z\in\mathbb{T}_1}|P(z)|\right.$$

$$\left.-\left\{\left|\alpha B[z^{n-1}] + \beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right| - |\lambda_0|\left|1+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\}\min_{z\in\mathbb{T}_1}|P(z)|\right].$$

Remark 3. A result due to Bidkham and Mezerji[4, Theorem 3] immediately follows from Theorem 4 if $\beta = 0$. In addition, if we substitute $B[D_\alpha P(z)]$(see Definition 1) in Theorem 4, and take $\lambda_1 = \lambda_2 = 0$, we get a result due to Liman et al.[8, Theorem 3]. Similarly, inequality (1.5) is also obtained from Theorem 4.

Finally we present the following generalization of Theorem B. To be more precise, we prove.

Theorem 5. *Let $Q(z)$ be a polynomial of degree n and $Q(z) = 0$ in $\mathbb{T}_1 \cup \mathbb{E}_1^-$ and $P(z)$ is a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ on $\mathbb{T}_1$, then for all $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$*

$$\left|\frac{B[D_\alpha P(z)]}{n}\right| + (|\alpha|-1)\left|\frac{B[Q(z)]}{2}\right| \leq \left|\frac{B[D_\alpha Q(z)]}{n}\right| + (|\alpha|-1)\left|\frac{B[P(z)]}{2}\right|, \quad \textit{for } |z| \geq 1.$$

Remark 4. By Definition 1, Substituting for $B[D_\alpha P(z)]$ in Theorem 5, we have

$$\begin{aligned}
&\frac{1}{n}\left|\lambda_0 D_\alpha P(z) + \lambda_1\left(\frac{m_1 z}{2}\right)\frac{D_\alpha P'(z)}{1!} + \lambda_2\left(\frac{m_1 z}{2}\right)^2\frac{D_\alpha P''(z)}{2!}\right| \\
&+ \frac{(|\alpha|-1)}{2}\left|\lambda_0 Q(z) + \lambda_1\left(\frac{nz}{2}\right)\frac{Q'(z)}{1!} + \lambda_2\left(\frac{nz}{2}\right)^2\frac{Q''(z)}{2!}\right| \\
&\leq \frac{1}{n}\left|\lambda_0 D_\alpha Q(z) + \lambda_1\left(\frac{mz}{2}\right)\frac{D_\alpha Q'(z)}{1!} + \lambda_2\left(\frac{mz}{2}\right)^2\frac{D_\alpha Q''(z)}{2!}\right| \\
&+ \frac{(|\alpha|-1)}{2}\left|\lambda_0 P(z) + \lambda_1\left(\frac{dz}{2}\right)\frac{P'(z)}{1!} + \lambda_2\left(\frac{dz}{2}\right)^2\frac{P''(z)}{2!}\right|,
\end{aligned} \tag{2.4}$$

where $m_1 \leq d-1,\ d \leq n$ and $m \leq n-1$. As earlier, if we fix $\lambda_1 = \lambda_2 = 0$ in (2.4), we get a result ascribed to Liman et al. [8, Theorem 2]. Again if we choose $\lambda_0 = \lambda_2 = 0$ in (2.4), we get

$$\frac{m_1}{n}|D_\alpha P'(z)| + \frac{n(|\alpha|-1)}{2}|Q'(z)| \leq \frac{m}{n}|D_\alpha Q'(z)| + \frac{d(|\alpha|-1)}{2}|P'(z)| \tag{2.5}$$

If we divide (2.5) both sides by $|\alpha|$ and make $|\alpha| \to \infty$, then $m_1 = d-1$ and $m = n-1$, and we obtain the following more generalized version of inequality (1.2).

$$\frac{d-1}{n}|P''(z)| + \frac{n}{2}|Q'(z)| \leq \frac{n-1}{n}|Q''(z)| + \frac{d}{2}|P'(z)|.$$

3. Lemmas. For the proofs of the main results, we need the following lemmas.

Lemma 1. *If all the zeros of an nth degree polynomial $P(z)$ lie in a circular region C and if none of the points $\alpha_t, \alpha_{t-1}, \ldots, \alpha_1$ lie in the region C, then each of the polar derivatives*

$$D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} P(z), \quad t = 1, 2, \ldots, n-1$$

has all of its zeros in region C.

This lemma follows by repeated applications of Laguerre's theorem [12].

Lemma 2. *If all the zeros of a polynomial $P(z)$ of degree n lie in a circle $z \in \mathbb{T}_1 \cup \mathbb{E}_1^-$, then all the zeros of the polynomial $B[P(z)]$ also lie in the circle $z \in \mathbb{T}_1 \cup \mathbb{E}_1^-$.*

For the Lemma 2 we refer to Rahman and Schmeisser [14, Def.14.5.1 and Lemma 14.5.7].

Lemma 3. *If all the zeros of a polynomial $P(z)$ of degree n lie in $z \in \mathbb{T}_1 \cup \mathbb{E}_1^-$, then for $|\alpha| \geq 1$, the polynomial $B[D_\alpha P(z)]$ also has all its zeros in $z \in \mathbb{T}_1 \cup \mathbb{E}_1^-$.*

Lemma 3 is ascribed to Bidkham and Mezerji [4].

Lemma 4. *If all the zeros of a polynomial $P(z)$ of degree n lie in a circle $z \in \mathbb{T}_1 \cup \mathbb{E}_1^-$, then for $|\alpha_1| \geq 1,\ |\alpha_2| \geq 1, \ldots, |\alpha_t| \geq 1$ all the zeros of the polynomial $B[D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} P(z)]$ also lie in the circle $z \in \mathbb{T}_1 \cup \mathbb{E}_1^-$.*

Proof. By combining Lemma 1 and Lemma 3, the proof of Lemma 4 follows. □

Next lemma is due to Jain [6].

Lemma 5. *If $P(z)$ is a polynomial of degree n, having all its zeros in $\mathbb{T}_1 \cup \mathbb{E}_1^-$, then for $|\alpha_1| \geq 1, |\alpha_2| \geq 1, \ldots, |\alpha_t| \geq 1,\ (t < n)$*

$$|D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} P(z)| \geq \frac{N_t}{2^t} A_{\alpha_t} |P(z)|.$$

Lemma 6. *If all the zeros of an n-th degree polynomial $P(z)$ lie in $\mathbb{T}_1 \cup \mathbb{E}_1^-$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$,*

$$|D_\alpha P(z)| \geq \frac{n(|\alpha| - 1)}{2} |P(z)|,\ |z| = 1.$$

Equality holds only for the polynomials having all zeros on $|z| = 1$. This lemma is ascribed to Shah[15].

Lemma 7. *If the polynomial $P(z)$ has no zeros in $\mathbb{E}_1^-$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1,\ |\beta| \leq 1$*

$$\left| B[D_\alpha P(z)] + n\beta \left(\frac{|\alpha| - 1}{2} \right) B[P(z)] \right| \leq \left| B[D_\alpha Q(z)] + n\beta \left(\frac{|\alpha| - 1}{2} \right) B[Q(z)] \right|,$$

for $|z| \geq 1$, *where* $Q(z) = z^n\overline{P(1/\bar{z})}$.

Proof. Since $P(z) \neq 0$ in $\mathbb{E}_1^-$. Therefore all the zeros of $Q(z) = z^n\overline{P(1/\bar{z})}$ lie in $\mathbb{T}_1 \cup \mathbb{E}_1^-$ and also for $|z| = 1$

$$|P(z)| = |Q(z)|.$$

Applying Theorem 2 for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1$, $|\beta| \leq 1$, we have

$$\left|B[D_\alpha P(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right| \leq \left|B[D_\alpha Q(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(z)]\right|.$$

This completes the proof of Lemma 7. □

Lemma 8. *The inequality*

$$\left|B[D_\alpha P(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right| \\ \leq n\left|\alpha B[z^{n-1}] + \beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right|\max_{z\in\mathbb{T}_1}|P(z)|$$

holds for every polynomial $P(z)$ *of degree* n *and for all* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| \geq 1$, $|\beta| \leq 1$.

Proof. Let $M = \max_{z\in\mathbb{T}_1}|P(z)|$. If we take $Q(z) = Mz^n$ in Theorem 2, we have $D_\alpha\{Mz^n\} = nM\alpha z^{n-1}$, $B[nM\alpha z^{n-1}] = nM\alpha B[z^{n-1}]$ and $B[Mz^n] = MB[z^n]$. Therefore, from Theorem 2, we get

$$\left|B[D_\alpha P(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right| \\ \leq n\left|\alpha B[z^{n-1}] + \beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right|\max_{z\in\mathbb{T}_1}|P(z)|.$$

This completes the proof of Lemma 8. □

Lemma 9. *For every polynomial* $P(z)$ *of degree* n *and for all* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| \geq 1$, $|\beta| \leq 1$, *we have*

$$\left|B[D_\alpha P(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right| + \left|B[D_\alpha Q(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(z)]\right| \\ \leq n\left[|\lambda_0|\ \left|1 + \beta\left(\frac{|\alpha|-1}{2}\right)\right| + \left|\alpha B[z^{n-1}] + \beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right|\right]\max_{z\in\mathbb{T}_1}|P(z)|,$$

where $Q(z) = z^n \, \overline{P(1/\bar{z})}$.

P r o o f. Let $M = \max_{z\in\mathbb{T}_1} |P(z)|$. Then by Rouche's theorem all the zeros of $S(z) = P(z) + \gamma M z^n$ lie in $|z| < 1$, for every γ with $|\gamma| > 1$. If $T(z) = z^n \, \overline{S(1/\bar{z})}$, then for $|z| = 1$, we have

$$|T(z)| < |\delta S(z)|$$

for any $\delta \in \mathbb{C}$ with $|\delta| > 1$. Therefore, it yields by Rouche's theorem that all the zeros of $T(z) + \delta S(z)$ lie in $|z| < 1$. Thus applying Lemma 6 for every α with $|\alpha| > 1$ and $|z| = 1$, we get

$$|D_\alpha(T(z) + \delta S(z))| \geq \frac{n(|\alpha| - 1)}{2} |T(z) + \delta S(z)|.$$

Equivalently for every $\beta \in \mathbb{C}$ with $|\beta| < 1$ and for $|z| = 1$, we have

$$|n\beta(|\alpha| - 1)(T(z) + \delta S(z))| < 2|D_\alpha(T(z) + \delta S(z))|.$$

Since $|\alpha| > 1$, so by Lemma 1 for $k = 1$, all the zeros of $D_\alpha(T(z) + \delta S(z))$ lie in $|z| < 1$. Therefore, by Rouche's theorem all the zeros of $2D_\alpha(T(z) + \delta S(z)) + n\beta(|\alpha| - 1)(T(z) + \delta S(z))$ lie in $|z| < 1$. Since B is linear, we have by Lemma 3 all the zeros of

$$F(z) = 2B[D_\alpha T(z)] + n\beta(|\alpha| - 1)B[T(z)] + \delta \left\{2B[D_\alpha S(z)] + n\beta(|\alpha| - 1)B[S(z)]\right\}$$

lie in $|z| < 1$.

This gives for $|z| \geq 1$, that

$$|2B[D_\alpha T(z)] + n\beta(|\alpha| - 1)B[T(z)]| \leq |2B[D_\alpha S(z)] + n\beta(|\alpha| - 1)B[S(z)]|$$

or

$$|2B[D_\alpha Q(z) + n\overline{\gamma} M] + n\beta(|\alpha| - 1)B[Q(z) + \overline{\gamma} M]|$$

$$\leq \left|2B[D_\alpha P(z) + n\gamma M \alpha z^{n-1}] + n\beta(|\alpha| - 1)B[P(z) + \gamma M z^n]\right|.$$

This implies

$$|2B[D_\alpha Q(z)] + n\beta(|\alpha| - 1)B[Q(z)] + n\overline{\gamma}\lambda_0(2 + \beta(|\alpha| - 1))M|$$

$$\leq \left|2B[D_\alpha P(z)] + n\beta(|\alpha| - 1)B[P(z)] + n\gamma\{2\alpha B[z^{n-1}] + \beta(|\alpha| - 1)B[z^n]\}M\right|.$$

Equivalently

$$\begin{aligned}
&|2B[D_\alpha Q(z)]+n\beta(|\alpha|-1)B[Q(z)]|-n|\gamma|\,|\lambda_0|\,\,|2+\beta(|\alpha|-1)|\,M\\
(3.1)\qquad &\leq |2B[D_\alpha P(z)]+n\beta(|\alpha|-1)B[P(z)]\\
&+n\gamma\{2\alpha B[z^{n-1}]+\beta(|\alpha|-1)B[z^n]\}M\big|\,.
\end{aligned}$$

With the help of Lemma 8, it is possible to choose an argument of γ such that

$$\begin{aligned}
&\left|2B[D_\alpha P(z)]+n\beta(|\alpha|-1)B[P(z)]+n\gamma\{2\alpha B[z^{n-1}]+\beta(|\alpha|-1)B[z^n]\}M\right|\\
=&n|\gamma|\left|2\alpha B[z^{n-1}]+\beta(|\alpha|-1)B[z^n]\right|M-|2B[D_\alpha P(z)]+n\beta(|\alpha|-1)B[P(z)]|\,.
\end{aligned}$$

Using this equation in (3.1) and letting $\gamma\to 1$, we obtain for $|z|=1$

$$\begin{aligned}
&|2B[D_\alpha Q(z)]+n\beta(|\alpha|-1)B[Q(z)]|-n\,|\lambda_0|\,\,|2+\beta(|\alpha|-1)|\,M\\
&\leq n\left|2\alpha B[z^{n-1}]+\beta(|\alpha|-1)B[z^n]\right|M-|2B[D_\alpha P(z)]+n\beta(|\alpha|-1)B[P(z)]|\,.
\end{aligned}$$

Consequently

$$\begin{aligned}
&|2B[D_\alpha Q(z)]+n\beta(|\alpha|-1)B[Q(z)]|\\
(3.2)\qquad &+|2B[D_\alpha P(z)]+n\beta(|\alpha|-1)B[P(z)]|\\
&\leq n\left[|\lambda_0|\,\,|2+\beta(|\alpha|-1)|+\left|2\alpha B[z^{n-1}]+\beta(|\alpha|-1)B[z^n]\right|\right]M
\end{aligned}$$

for every $\alpha,\beta\in\mathbb{C}$ with $|\alpha|>1$, $|\beta|<1$. Using continuity of α and β, it follows that inequality (3.2) is true for every $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\geq 1$, $|\beta|\leq 1$ and the proof of Lemma 9 is thus complete. □

As above, when we substitute $B[D_\alpha P(z)]$ and related terms in Lemma 9, and then we take $\lambda_1=\lambda_2=0$, we obtain a result which is due to Liman et al. [8, Lemma 5]. Also if we take $\beta\to 0$, we get a result due to Bidkham and Mezerji [4].

4. Proofs of Theorems.

Proof of Theorem 1. Let

$$C=\min_{z\in\mathbb{T}_1}|P(z)|$$

then $|P(z)|\geq C$ on $\mathbb{T}_1$. Therefore, for every λ with $|\lambda|<1$

$$|P(z)| > |\lambda C| \text{ on } \mathbb{T}_1.$$

If $P(z)$ has a zero on $\mathbb{T}_1$ then $C = 0$ and the result is trivial. Therefore, from now onwards, we will assume that $P(z)$ has all its zeros in $\mathbb{E}_1^-$. By Rouche's theorem the polynomial

$$F(z) = P(z) - \lambda C z^n$$

also has all its zeros in $\mathbb{E}_1^-$. Thus, on applying Lemma 5 to $F(z)$ we obtain for $|\alpha_1| \geq 1, |\alpha_2| \geq 1, \ldots, |\alpha_t| \geq 1$

$$|D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} F(z)| \geq \frac{N_t}{2^t} A_{\alpha_t} |F(z)|.$$

Since $F(z)$ has all its zeros in $\mathbb{E}_1^-$, again applying Rouche's theorem for every real or complex number β with $|\beta| \leq 1$, we get all the zeros of

$$D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} F(z) + \frac{\beta N_t}{2^t} A_{\alpha_t} F(z)$$

lie in $\mathbb{E}_1^-$. Thus by Lemma 4 all the zeros of

$$X(z) = B[D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} F(z) + \beta N_t 2^{-t} A_{\alpha_t} F(z)]$$

also lie in $\mathbb{E}_1^-$. Now operate $D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1}$ and B on $F(z) = P(z) - \lambda C z^n$, we get

$$\begin{aligned} X(z) &= B[D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} P(z) - C\lambda N_t z^{n-t} \Lambda_t + \beta N_t 2^{-t} A_{\alpha_t} (P(z) - \lambda C z^n)] \\ &= B[D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} P(z) + \beta N_t 2^{-t} A_{\alpha_t} P(z) \\ &\quad - \lambda\{C N_t z^{n-t} \Lambda_t + \beta N_t 2^{-t} A_{\alpha_t} C z^n\}]. \end{aligned}$$

As B is linear operator, we obtain

$$\begin{aligned} X(z) = B[D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} P(z)] &+ \beta N_t 2^{-t} A_{\alpha_t} B[P(z)] \\ &- \lambda\{C\ N_t \Lambda_t B[z^{n-t}] + \beta N_t 2^{-t} A_{\alpha_t} C\ B[z^n]\}. \end{aligned} \tag{4.1}$$

This gives

$$\begin{aligned} &|B[D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} P(z)] + \beta N_t 2^{-t} A_{\alpha_t} B[P(z)]| \\ &\qquad \geq |C\ N_t \Lambda_t B[z^{n-t}] + \beta N_t 2^{-t} A_{\alpha_t} C\ B[z^n]| \text{ for } |z| \geq 1, \end{aligned}$$

that is

$$\begin{aligned}&|B[D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}P(z)]+\beta N_t 2^{-t}A_{\alpha_t}B[P(z)]|\\&\qquad\geq C\ N_t|\Lambda_t B[z^{n-t}]+\beta 2^{-t}A_{\alpha_t}B[z^n]| \text{ for } |z|\geq 1.\end{aligned}$$

If this is not the case, then we have

$$\begin{aligned}&|B[D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}P(z_0)]+\beta N_t 2^{-t}A_{\alpha_t}B[P(z_0)]|\\&\qquad< C\ N_t|\Lambda_t B[z_0^{n-t}]+\beta 2^{-t}A_{\alpha_t}B[z_0^n]|.\end{aligned}$$

for any point $z=z_0$ with $|z_0|\geq 1$. If we take

$$\lambda=\frac{B[D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}P(z_0)]+\beta N_t 2^{-t}A_{\alpha_t}B[P(z_0)]}{C\ N_t\left(\Lambda_t B[z_0^{n-t}]+\beta 2^{-t}A_{\alpha_t}B[z_0^n]\right)}$$

then we see that λ is well defined and with this choice of λ, we get from (4.1) that

$$X(z_0)=0$$

which is a contradiction that all the zeros of $X(z)$ lie in $\mathbb{E}_1^-$. Hence we conclude that

$$\begin{aligned}&\left|B[D_{\alpha_t}\cdots D_{\alpha_2}D_{\alpha_1}P(z)]+\beta N_t 2^{-t}A_{\alpha_t}B[P(z)]\right|\\&\qquad\geq C\ N_t\left|\Lambda_t B[z^{n-t}]+\beta 2^{-t}A_{\alpha_t}B[z^n]\right|.\end{aligned}$$

This completes the proof of Theorem 1. □

P r o o f o f T h e o r e m 2. For every $\gamma\in\mathbb{C}$ with $|\gamma|>1$, we get by Rouche's theorem all the zeros of $P(z)-\gamma Q(z)$ lie in $|z|\leq 1$ and for $t>1$ the polynomial $P(tz)-\gamma Q(tz)$ vanishes in $|z|\leq \frac{1}{t}<1$. Now by Lemma 6, we obtain for $|\alpha|>1$ and $|z|=1$

$$|D_\alpha(P(tz)-\gamma Q(tz))|\geq\frac{n(|\alpha|-1)}{2}|P(tz)-\gamma Q(tz)|,\ |z|=1.$$

Again applying Rouche's theorem, we have for every $\beta\in\mathbb{C}$ with $|\beta|\leq 1$, the polynomial

$$S(z) = D_\alpha(P(tz) - \gamma Q(tz)) + \beta\frac{n(|\alpha|-1)}{2}\{P(tz) - \gamma Q(tz)\}$$

has all its zeros in $|z| \leq 1$, so by Lemma 3 all the zeros of

$$\begin{aligned}(4.2)\quad B[S(z)] = B[D_\alpha P(tz)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(tz)] \\ - \gamma\left\{B[D_\alpha Q(tz)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(tz)]\right\}\end{aligned}$$

also lie in $|z| \leq 1$. This gives for $|z| \geq 1$

$$\begin{aligned}\left|B[D_\alpha P(tz)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(tz)]\right| \\ \leq \left|B[D_\alpha Q(tz)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(tz)]\right|.\end{aligned}$$

Otherwise there is a contradiction that is, if there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$\begin{aligned}\left|B[D_\alpha P(tz_0)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(tz_0)]\right| \\ > \left|B[D_\alpha Q(tz_0)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(tz_0)]\right|.\end{aligned}$$

If we take

$$\gamma = \frac{B[D_\alpha P(tz_0)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(tz_0)]}{B[D_\alpha Q(tz_0)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(tz_0)]}.$$

It is easy to see that γ is well defined and with this choice of γ, (4.2) becomes zero for $|z_0| \geq 1$, i.e., $B[S(z_0)] = 0$, which is contradiction that all the zeros of $B[S(z)]$ lie in $|z| \leq 1$. Thus making $t \to 1$ and using continuity, we get for every $|\alpha| \geq 1$, $|\beta| \leq 1$ and $|z| \geq 1$

$$\left|B[D_\alpha P(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right| \leq \left|B[D_\alpha Q(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(z)]\right|.$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. On combining Lemma 7 and Lemma 9, we get

$$
\begin{aligned}
&2\left|B[D_\alpha P(z)]+n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right|\\
&\leq\left|B[D_\alpha P(z)]+n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right|+\left|B[D_\alpha Q(z)]+n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(z)]\right|\\
&\leq n\left[|\lambda_0|\ \left|1+\beta\left(\frac{|\alpha|-1}{2}\right)\right|+\left|\alpha B[z^{n-1}]+\beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right|\right]\max_{z\in\mathbb{T}_1}|P(z)|
\end{aligned}
$$

and the proof of Theorem 3 follows. □

Proof of Theorem 4. Let $m=\min_{z\in\mathbb{T}_1}|P(z)|$. In case $P(z)=0$ for any z on $\mathbb{T}_1$, then $m=0$ and Theorem 4 reduces to Theorem 3. Therefore, the case when $P(z)$ has all its zeros in $|z|>1$, so that $m>0$. Now by Rouche's theorem the polynomial $W_1(z)=P(z)-\delta m$ has no zeros in $|z|<1$. Therefore the polynomial $V_1(z)=z^n\ \overline{W_1(1/\overline{z})}=Q(z)-\overline{\delta}mz^n$, where $Q(z)=z^n\ \overline{P(1/\overline{z})}$ will have all its zeros in $|z|\leq 1$. Also $|W_1(z)|=|V_1(z)|$ on $\mathbb{T}_1$. Applying Theorem 2, we get for all $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\geq 1,|\beta|\leq 1$ and $|z|=1$

$$
\begin{aligned}
&\left|B[D_\alpha W_1(z)]+n\beta\left(\frac{|\alpha|-1}{2}\right)B[W_1(z)]\right|\\
&\qquad\leq\left|B[D_\alpha V_1(z)]+n\beta\left(\frac{|\alpha|-1}{2}\right)B[V_1(z)]\right|.
\end{aligned}
$$

Since B and D_α are linear operators, we have

$$
\begin{aligned}
&\left|B[D_\alpha P(z)]-nm\delta\lambda_0+n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]-n\beta\left(\frac{|\alpha|-1}{2}\right)\delta m\lambda_0\right|\\
&\leq\left|B[D_\alpha Q(z)]-\overline{\delta}nm\alpha B[z^{n-1}]+n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(z)]\right.\\
&\qquad\left.-n\beta\left(\frac{|\alpha|-1}{2}\right)\overline{\delta}mB[z^n]\right| \qquad (4.3)\\
&=\left|B[D_\alpha Q(z)]+n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(z)]\right.\\
&\qquad\left.-\overline{\delta}mn\left\{\alpha B[z^{n-1}]+\beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right\}\right|.
\end{aligned}
$$

Since $Q(z) = 0$ in $|z| \le 1$, so we have $\min\limits_{z\in\mathbb{T}_1} |P(z)| = \min\limits_{z\in\mathbb{T}_1} |Q(z)| = m$. Therefore by Theorem 1 for $t = 1$, we have

$$\left|B[D_\alpha Q(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(z)]\right| \ge mn\left|\alpha B[z^{n-1}] + \beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right|,$$

and this will allow us to rewrite (4.3) as

$$\left|B[D_\alpha P(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right| - mn|\delta|\left|\lambda_0 + \beta\left(\frac{|\alpha|-1}{2}\right)\lambda_0\right|$$
$$\le \left|B[D_\alpha Q(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(z)]\right| - mn|\delta|\left|\alpha B[z^{n-1}] + \beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right|.$$

Letting $|\delta| \to 1$, we get from above inequality

$$\left|B[D_\alpha P(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right| - \left|B[D_\alpha Q(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[Q(z)]\right|$$
$$\le -mn\left\{\left|\alpha B[z^{n-1}] + \beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right| - \left|\lambda_0 + \beta\left(\frac{|\alpha|-1}{2}\right)\lambda_0\right|\right\}.$$

Now by Lemma 9, we get for $|z| = 1$

$$2\left|B[D_\alpha P(z)] + n\beta\left(\frac{|\alpha|-1}{2}\right)B[P(z)]\right|$$
$$\le n\left[\left\{\left|\alpha B[z^{n-1}] + \beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right| + |\lambda_0|\ \left|1 + \beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\}\max_{z\in\mathbb{T}_1}|P(z)|\right.$$
$$\left. - \left\{\left|\alpha B[z^{n-1}] + \beta\left(\frac{|\alpha|-1}{2}\right)B[z^n]\right| - |\lambda_0|\ \left|1 + \beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\}\min_{z\in\mathbb{T}_1}|P(z)|\right].$$

This completes the proof of Theorem 4. □

P r o o f o f T h e o r e m 5. By hypothesis $Q(z) = 0$ in $|z| \le 1$, therefore, by Lemma 6, we have for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge 1$

$$\frac{n(|\alpha|-1)}{2}|Q(z)| \le |D_\alpha Q(z)|.$$

By Lemma 1 for $t = 1$ all the zeros of $D_\alpha Q(z)$ also lie in $|z| \le 1$. By Rouche's theorem for every real or complex number σ with $|\sigma| < 1$ all the zeros of

$$D_\alpha Q(z) - \frac{n\sigma(|\alpha|-1)}{2}Q(z)$$

lie in $|z| \leq 1$,. Thus by Lemma 2 and Lemma 3 all the zeros of

$$T(z) = B[D_\alpha Q(z)] - \frac{n\sigma(|\alpha|-1)}{2} B[Q(z)]$$

also lie in $|z| \leq 1$, and this gives for $|z| \geq 1$

$$|B[D_\alpha Q(z)]| \geq \frac{n(|\alpha|-1)}{2} |B[Q(z)]|,$$

otherwise there is a contradiction with the zeros of $T(z)$. Therefore, we have for every real or complex number β with $|\beta| \leq 1$

$$|B[D_\alpha Q(z)]| - \frac{n\beta(|\alpha|-1)}{2} |B[Q(z)]| \geq 0.$$

Now it is possible to choose an argument of β in the right hand side of Theorem 2 such that

$$\left| B[D_\alpha Q(z)] + n\beta \left(\frac{|\alpha|-1}{2} \right) B[Q(z)] \right| = |B[D_\alpha Q(z)]| - \frac{n(|\alpha|-1)}{2} |\beta| \, |B[Q(z)]|.$$

Hence from Theorem 2, we obtain for $|z| \geq 1$

$$|B[D_\alpha P(z)]| - \frac{n(|\alpha|-1)}{2} |\beta| \, |B[P(z)]| \leq |B[D_\alpha Q(z)]| - \frac{n(|\alpha|-1)}{2} |\beta| \, |B[Q(z)]|.$$

Letting $|\beta| \to 1$ and consequently

$$\left| \frac{B[D_\alpha P(z)]}{n} \right| + (|\alpha|-1) \left| \frac{B[Q(z)]}{2} \right| \leq \left| \frac{B[D_\alpha Q(z)]}{n} \right| + (|\alpha|-1) \left| \frac{B[P(z)]}{2} \right|,$$

for $|z| \geq 1$. This proves Theorem 5 completely. □

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Tawheeda Akhter
Department of Mathematics
Lovely Professional University
Punjab, India
e-mail: takhter595@gmail.com

Shabir Ahmad Malik
Department of Mathematics
University of Kashmir
Srinagar 190006, India
e-mail: shabirams2@gmail.com

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