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UNIQUENESS OF ENTIRE FUNCTION AND ITS LINEAR DIFFERENTIAL POLYNOMIAL

Manab Biswas, Dilip Ch. Pramanik

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Abstract. In this paper we investigate the uniqueness problem of entire function f and its linear differential polynomial

$$a_k(z) f^{(k)} + a_{k-1}(z) f^{(k-1)} + \cdots + a_1(z) f'$$

sharing an entire function $a \equiv a(z)$ counting multiplicities(CM) with

$$\sigma(a) < \sigma(f)$$

under some restrictions imposed on the coefficients $a_j(z)\,(j = 1, 2, \ldots, k)$. Our result improves and generalizes some earlier results.

1. Introduction and main results. Let f be an entire function in the open complex plane $\mathbb{C}$. Then the order $\sigma(f)$ and the lower order $\lambda(f)$ of f are respectively defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}$$

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and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Also, the hyper order $\sigma_2(f)$ and the hyper lower order $\lambda_2(f)$ are defined respectively by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}$$

and

$$\lambda_2(f) = \liminf_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}.$$

For an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the central index $\nu(r,f)$ is the greatest exponent m such that $|a_m| r^m = \mu(r,f)$, where $\mu(r,f) = \max_{n\geq 0} |a_n| r^n$ denotes the maximum term of f on $|z| = r$ (see [8, p. 50]). It is well known (see [8, p. 51]) that

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log\nu(r,f)}{\log r}.$$

Similarly, one can verify that

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log\nu(r,f)}{\log r}.$$

By Lemma 2 in [4] we see that

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log\nu(r,f)}{\log r}$$

and in a similar way one can prove that

$$\lambda_2(f) = \liminf_{r\to\infty} \frac{\log\log\nu(r,f)}{\log r}.$$

Let $u(z)$ be a non-constant sub-harmonic function in the open complex plane. We put $B(r,u) = \sup_{|z|=r} u(z)$. The order $\sigma(u)$ and the lower order $\lambda(u)$ of u are defined by

$$\sigma(u) = \limsup_{r\to\infty} \frac{\log B(r,u)}{\log r}$$

and

$$\lambda(u) = \liminf_{r\to\infty} \frac{\log B(r,u)}{\log r},$$

respectively (see [1]).

Let $E \subset [1, \infty)$ and χ_E be the characteristic function of E. The upper and lower logarithmic densities of E are respectively defined as follows

$$\overline{\log \operatorname{dens}}\,(E) = \limsup_{r\to\infty} \frac{\int_1^r \frac{\chi_E}{t}dt}{\log r}$$

and

$$\underline{\log \operatorname{dens}}\,(E) = \liminf_{r\to\infty} \frac{\int_1^r \frac{\chi_E}{t}dt}{\log r}.$$

The quantity $\lim_{r\to\infty} \int_1^r \frac{\chi_E}{t}dt$ is called the logarithmic measure of E. It is easy to note that $\overline{\log \operatorname{dens}}\,(E) > 0$, then E has infinite logarithmic measure.

The following definition is well known:

Definition 1. *Let $f(z)$ and $g(z)$ be two entire functions and $a \equiv a(z)$ be also an entire function, which, in particular, may be a constant, if the zeros of $f-a$ and $g-a$ coincide in locations and multiplicities we say that f and g share the value a* ***CM (counting multiplicities)*** *and if coincide in locations only we say that f and g share a* ***IM (ignoring multiplicities).***

Rubel and Yang ([11]) first studied the uniqueness problem of an entire function sharing two values with its derivative. They proved the following result:

Theorem 1 ([11])**.** *If a non-constant entire function f and its derivative f' share two distinct finite complex numbers CM, then $f \equiv f'$.*

What can be said for a non-constant entire function f that shares one finite value with its derivative f'? In this connection Brück [2] gave the following conjecture:

Conjecture. Let f be a non-constant entire function with $\sigma_2(f) < \infty$ and $\sigma_2(f)$ is not a positive integer. If f and f' share a finite value a CM, then $f' - a = c(f-a)$, where c is a nonzero constant.

The conjecture for the case $a = 0$ was resolved by Brück [2]. In 1998, Gunderson and Yang [7] resolved the case where f is of finite order. In 2004, Chen and Shon [5] resolved it when f is of hyper order $\sigma_2(f) < \frac{1}{2}$. Later, Cao [3] resolved it under the assumption $\sigma_2(f) = \frac{1}{2}$.

Considering arbitrary k-th derivatives $f^{(k)}$ instead of $f^{(1)}$, L. Z. Yang [13] proved the following result:

Theorem 2 ([13]). *Let f be a non-constant entire function of finite order. If f and $f^{(k)}$ share one finite value a CM, then $f^{(k)} - a = c(f-a)$ for some nonzero constant c.*

In 2004, J. P. Wang [12] improved Theorem 2 in the following manner:

Theorem 3 ([12]). *Let f be a non-constant entire function of finite order and a be a non-constant polynomial. If f and $f^{(k)}$ share a CM, then $f^{(k)} - a = c(f-a)$ for some nonzero constant c.*

But, the conjecture still remains open for the case $\sigma_2(f) > \frac{1}{2}$. However, Lahiri and Das [9] obtain the following result, which improves and generalizes some results mentioned above.

Theorem 4 ([9]). *Let f be a non-constant entire function with $\lambda_2(f) < \frac{1}{2}$ and $\sigma_2(f) < \infty$. Suppose that $a \equiv a(z)$ is a polynomial. If f and $f^{(k)}$ share a CM, then $f^{(k)} - a = c(f-a)$, where c is a nonzero constant.*

For the entire function f, the expression

$$L[f] = a_k(z) f^{(k)} + a_{k-1}(z) f^{(k-1)} + \cdots + a_1(z) f' + a_0(z) f,$$

is called a linear differential polynomial generated by f of degree k, where k is a positive integer and $a_0(z), a_1(z), \ldots, a_k(z)$ are entire coefficients with $a_k(z) \not\equiv 0$.

In this paper we improve and extend the result of Lahiri and Das [9] in which we replace $f^{(k)}$ by $L_1[f] = L[f] - a_0(z) f$ and under some restrictions imposed on $a_j(z)\,(j = 1, 2, \ldots, k)$, we obtain the following result:

Theorem 5. *Let f be a non-constant entire function with $\lambda_2(f) < \frac{1}{2}$ and $\sigma_2(f) < \infty$. Suppose that $a \equiv a(z)$ is an entire function with $\sigma(a) < \sigma(f)$. Let $a_0(z)$ and $a_j(z)$ $(j = 1, \ldots, k)$ be entire functions such that for some constant $\alpha > 0$, we have for all $\varepsilon > 0$ sufficiently small, $|a_j(z)| \le r^{\alpha+\epsilon}$, $j = 1, 2, ..., k$, as $r = |z| \to \infty$. If f and $L_1[f] = L[f] - a_0(z) f$ share a CM, then $L_1[f] - a = c(f-a)$, where c is a nonzero constant.*

2. Lemmas. In this section we present some necessary lemmas.

Lemma 1 ([8, p. 9]). *Let $P(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0$ be a polynomial of degree n. Then for every $\epsilon\,(>0)$ there exists $R\,(>0)$ such that for all $|z| = r > R$, we get $(1-\epsilon)\,|b_n|\,r^n \le |P(z)| \le (1+\epsilon)\,|b_n|\,r^n$.*

Lemma 2 ([8, p. 51]). *Let f be a transcendental entire function. Then there exists a set $E \subset (1,\infty)$ with finite logarithmic measure such that for $|z| = r \notin [0,1] \cup E$ and $|f(z)| = M(r,f)$ we get*

$$\frac{f^{(k)}(z)}{f(z)} = (1+o(1))\left(\frac{\nu(r,f)}{z}\right)^k.$$

Lemma 3 ([8, p. 51]). *Let $g : (0,+\infty) \to \mathbb{R}$ and $h : (0,+\infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \le h(r)$ outside an exceptional set E of finite logarithmic measure. Then for any $\alpha > 1$, there exists $R > 0$ such that $g(r) \le h(r^\alpha)$ holds for $r > R$.*

Lemma 4 ([10]). *Let f be a transcendental entire function and $E \subset (1,+\infty)$ be of finite logarithmic measure. Then there exists $\{z_m = r_m e^{i\theta_m}\}$ such that $|f(z_m)| = M(r_m,f)$, $\theta_m \notin [0,2\pi)$, $\lim\limits_{m\to\infty} \theta_m = \theta_0 \in [0,2\pi)$, $r_m \notin E, r_m \to \infty$,*

(i) if $0 < \sigma(f) < \infty$, then for given $\varepsilon > 0$ and sufficiently large r_m, $r_m^{\sigma(f)-\epsilon} < \nu(r_m,f) < r_m^{\sigma(f)+\epsilon}$;

(ii) If $\sigma(f) = \infty$, then for any given $K > 0$ and sufficiently large r_m, $\nu(r_m,f) > r_m^K$.

Lemma 5 ([1]). *Let $u(z)$ be a non-constant sub-harmonic function in the open complex plane $\mathbb{C}$ of lower order $\lambda, 0 \le \lambda < 1$. If $\lambda < \alpha < 1$, then*

$$\overline{\log\operatorname{dens}}\,\{r : A(r) > (\cos\alpha\pi)\,B(r)\} \ge 1 - \frac{\lambda}{\alpha},$$

where $A(r) = \inf\limits_{|z|=r} u(z)$ and $B(r) = \sup\limits_{|z|=r} u(z)$.

Remark 1. Since for an entire function Q, $\log|Q(z)|$ is a sub-harmonic function in $\mathbb{C}$ [6, p. 394], we can apply Lemma 5 to the function $u(z) = \log|Q(z)|$.

3. Proof of Theorem 5.

Proof. Since $L_1[f] - a$ and $f - a$ share 0 CM, there exists an entire function Q such that

$$\frac{L_1[f] - a}{f - a} = e^Q. \tag{1}$$

If Q is a constant, then we are done. So we suppose that Q is non-constant and consider the following cases:

Case 1. Let $\sigma(f) < \infty$. Then from (1) we see that Q is a polynomial. Further $\sigma(f) \geq 1$, because if $\sigma(f) < 1$, then (1) implies that Q is a constant. Therefore, f is transcendental. Since $a(z)$ is an entire function with $\sigma(a) < \sigma(f)$, we get

$$\left|\frac{a(z)}{f(z)}\right| < \frac{M(r,a)}{M(r,f)} \to 0 \tag{2}$$

as $r \to \infty$.

From (1) we get

$$e^{Q} = \frac{\frac{L_1[f]}{f} - \frac{a}{f}}{1 - \frac{a}{f}}. \tag{3}$$

Now by Lemma 2, there exists a set $E \subset (1, +\infty)$ with finite logarithmic measure such that for all large $|z| = r \notin [0,1] \cup E$ and $|f(z)| = M(r,f)$, we get

$$\frac{f^{(j)}(z)}{f(z)} = \left\{\frac{\nu(r,f)}{z}\right\}^{j} (1 + o(1)), \qquad j = 1, 2, \ldots, k. \tag{4}$$

as $r \to \infty$,

By the hypothesis of Theorem 5, for any given $\epsilon > 0$ there exists a value $r_1 > 0$ such that

$$|a_j(z)| \leq r^{\alpha+\epsilon}, \; j = 1, 2, \ldots, k; \; r \geq r_1, \quad \text{as} \quad r = |z| \to \infty. \tag{5}$$

Now from (2), (3), (4) and (5) for all large $|z| = r \notin [0,1] \cup E$ with $|f(z)| = M(r,f)$we get

$$\begin{aligned} |Q(z)| &= \left|\log e^{Q(z)}\right| \\ &\leq \left|\log\left(\left|\frac{\nu(r,f)}{z}\right|^{\mu}\right)\right| + (\alpha+\epsilon)\log r + o(1) \end{aligned}$$

where $\mu = 1$ or k according as $\nu(r,f) < r$ or $\nu(r,f) > r$. This implies

$$\begin{aligned} |Q(z)| &< \mu\log\nu(r,f) + (\mu+\alpha+\epsilon)\log r + o(1) \\ &< 2(\mu+\alpha+\epsilon)(\sigma(f)+1)\log r + o(1). \end{aligned} \tag{6}$$

Also, by Lemma 1 for all large $|z| = r$ we obtain

$$\frac{1}{2}|d|\, r^{\deg Q} \leq |Q(z)|, \tag{7}$$

where d is the leading coefficient of Q.

Now from (6) and (7) we get $\deg Q = 0$, which is a contradiction.

Case 2. Let $\sigma(f) = \infty$. We note from (1) that $\lambda(Q) \leq \lambda_2(f) < \frac{1}{2}$. We now consider the following subcases :

Subcase 2.1. Let Q be a polynomial. Then f is transcendental. By Lemma 4, there exists $\{z_m = r_m e^{i\theta_m}\}$ with $|f(z_k)| = M(r_m, f)$, $\theta_m \notin [0, 2\pi)$, $\lim\limits_{m\to\infty} \theta_m = \theta_0 \in [0, 2\pi)$, $r_m \notin [0,1] \cup E$, such that for sufficiently large $K > 0$ and sufficiently large r_m, we have

$$\nu(r_m, f) > r_m^K. \tag{8}$$

Now from (2), (3), (4) and (8) for sufficiently large $|z_m| = r_m \notin [0,1] \cup E$ with $|f(z_m)| = M(r_m, f)$ we get

$$e^{Q(z_m)} = a_k \left(\frac{\nu(r_m, f)}{z_m}\right)^k (1 + o(1)).$$

Using (5) we obtain

$$\begin{aligned} |Q(z_m)| &= \left|\log e^{Q(z_m)}\right| \\ &\leq \left|\log\left(\left|\frac{\nu(r_m,f)}{z_m}\right|^{k}\right)\right| + (\alpha+\epsilon)\log r_m + o(1) \\ &< k\log\nu(r_m,f) + (k+\alpha+\epsilon)\log r_m + o(1). \end{aligned} \tag{9}$$

So from (7) and (9) for sufficiently large $|z_m| = r_m \notin [0,1] \cup E$ we get

$$\frac{1}{2}\,|d|\,r_m^{\deg Q} < k\log\nu(r_m, f) + (k+\alpha+\epsilon)\log r_m + o(1).$$

Hence by Lemma 3 for given β, $1 < \beta < \frac{3}{2}$, for all large values of r_m we get

$$\frac{1}{2}\,|d|\,r_m^{\deg Q} < k\log\nu(r_m^{\beta}, f) + \beta\,(k+\alpha+\epsilon)\log r_m + o(1)$$

and so

$$r_m^{\deg Q}\left(\frac{1}{2}\,|d| - \frac{\beta\,(k+\alpha+\epsilon)\log r}{r_m^{\deg Q}}\right) \le k\log\nu(r_m^{\beta}, f) + o\,(1)\,.$$

This implies $\deg Q \le \beta\lambda_2\,(f) < \frac{\beta}{2} < 1$, which is a contradiction.

Subcase 2.2. Let Q be a transcendental entire function. We see by Remark 1 that $u\,(z) = \log|Q\,(z)|$ is a sub harmonic function and also $\lambda\,(u) = \lambda\,(Q) < \frac{1}{2}$. Suppose that $H = \{r : A\,(r) > (\cos\alpha\pi)\,B\,(r)\}$, where $A\,(r) = \inf\limits_{|z|=r}\log|Q\,(z)|$, $B\,(r) = \sup\limits_{|z|=r}\log|Q\,(z)|$ and $\lambda\,(Q) < \alpha < \frac{1}{2}$.

Then by Lemma 5, H has infinite logarithmic measure. Also, by Lemma 2 for $|z| = r \in H\backslash\{[0,1]\cup E\}$ with $|f\,(z)| = M\,(r, f)$ we get (4).

Now by using the same reasoning as before for sufficiently large $|z_m| = r_m \in H\backslash\{[0,1]\cup E\}$ with $|f(z_m)| = M(r_m, f)$ we get

$$\begin{aligned} |Q\,(z)| &< k\log\nu(r_m, f) + (k+\alpha+\epsilon)\log r_m + o(1) \\ &< 2\,(k+\alpha+\epsilon)\,r_m^{\sigma_2(f)+1}. \end{aligned} \tag{10}$$

So by (10) and by Lemma 5 there exists a constant t, $0 \le t \le 1$, such that $(M(r_m, Q))^t \le 2\,(k+\alpha+\epsilon)\,r_m^{\sigma_2(f)+1}$ for all large values of $|z_m| = r_m \in H\backslash\{[0,1]\cup E\}$ and $|f\,(z_m)| = M(r_m, f)$. This is impossible because Q is transcendental and so $\lim\limits_{r_m\to\infty}\frac{(M(r_m, Q))^t}{r_m^{\sigma_2(f)+1}} = \infty$. This proves the Theorem. □

4. Conclusions. In the preceding discussion, several thought-provoking questions emerged naturally:

- What outcomes can we anticipate in Theorem 5 for cases where the hyper-lower order of f exceeds 1/2 or the hyper-order of f is infinite?

- Can we modify the growth constraints on f and the entire coefficients $a_j(z)(j = 0, 1, 2, \dots, k)$ in Theorem 5 while still preserving the desired sharing property?

- If we choose non-linear differential polynomials and expand our considerations to include higher-order growth conditions, such as the iterated p-order growth of f in Theorem 5, will the established conclusion remain valid?

These unresolved questions demand closer examination and continued research.

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REFERENCES

[1] P. D. Barry. On a theorem of Kjellberg. *Quart. J. Math. Oxford Ser. (2)* **15** (1964), 179–191.

[2] R. Brück. On entire functions which share one value CM with their first derivative. *Results Math.* **30**, *1–2* (1996), 21–24.

[3] T. Cao. On the Brück conjecture. *Bull. Aust. Math. Soc.* **93**, *2* (2016), 248–259.

[4] Z. X. Chen, C. C. Yang. Some further results on the zeros and growths of entire solutions of second order linear differential equations. *Kodai Math. J.* **22**, *2* (1999), 273–285.

[5] Z. X. Chen, K. H. Shon. On conjecture of R. Brück concerning the entire function sharing one value CM with its derivative. *Taiwanese J. Math.* **8**, *2* (2004), 235–244.

[6] T. W. Gamelin. Complex Analysis. Undergrad. Texts Math. New York, Springer-Verlag, 2001.

[7] G. G. Gundersen, L. Z. Yang. Entire functions that share one value with one or two of their derivatives, J. Math. Anal. Appl. **223**, *1* (1998), 88–95.

[8] I. Laine. Nevanlinna Theory and Complex Differential Equations. De Gruyter Stud. Math., vol. **15**, Berlin, Walter de Gruyter & Co., 1993.

[9] I. Lahiri, S. Das. A note on a conjecture of R. Brück. Appl. *Math. E-Notes*, **21** (2021), 152–156.

[10] Z. Mao. Uniqueness theorems on entire functions and their linear differential polynomials. *Results Math.* **55**, *3–4* (2009), 447–456.

[11] L. A. Rubel, C. C. Yang. Values shared by an entire function and its derivative. Lecture Notes in Math. vol. **599**. Berlin-New York, Springer-Verlag, 1977, 101–103.(1977).

[12] J. P. Wang. Entire functions that share a polynomial with one of their derivatives. *Kodai Math. J.* **27**, *2* (2004), 144–151.

[13] L. Z. Yang. Solution of a differential equation and its applications. *Kodai Math. J.* **22**, *3* (1999), 458–464.

Manab Biswas
Department of Mathematics
Kalimpong College
P.O. Kalimpong, Dist-Kalimpong
PIN-734301, West Bengal, India
e-mail: dr.manabbiswas@gmail.com

Dilip Ch. Pramanik
Department of Mathematics
University of North Bengal
Raja Rammohanpur, Dist-Darjeeling
PIN-734013, West Bengal, India
e-mail: dcpramanik.nbu2012@gmail.com

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