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ON 3-DISTANCE SPHERICAL 5-DESIGNS*

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Abstract. Inspired by a recently formulated conjecture by Bannai et al. we investigate spherical codes which admit exactly three different distances and are spherical 5-designs. Computing and analyzing distance distributions we provide new proof of the fact (due to Levenshtein) that such codes are maximal and rule out certain cases towards a proof of the conjecture.

1. Introduction. Recently, Bannai et al. [3] formulated the following conjecture.

Conjecture 1.1. *If $C \subset \mathbb{S}^{n-1}$ is a spherical 3-distance 5-design, then one of the following holds.*

(1) *$n = 2$ and C is a regular hexagon or a regular heptagon.*

(2) *C is a tight spherical 5-design.*

(3) *C is derived from a tight spherical 7-design.*

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In fact, in [3] Conjecture 1.1 was referred to as "Many people have conjectured, but not written down, the classification of 3-distance 5-designs". Indeed, it probably goes back to 1977 when Delsarte, Goethals and Seidel [8] introduced spherical designs in a seminal paper. The first author was among these who considered similar questions in 1990's (see [4, 5, 6]) and is aware about the conjecture since then.

It is known that the cardinality of any spherical 3-distance 5-design $C \subset \mathbb{S}^{n-1}$ satisfies

$$n(n+1) \le |C| \le \binom{n+2}{3} + \binom{n+1}{2} = \frac{n(n+1)(n+5)}{6}.$$

Both bounds are due to Delsarte-Goethals-Seidel [8]. The lower bound follows since C is a spherical 5-design, the upper bound – since C is a spherical 3-distance set (for upper bounds for few distance sets see also [1, 2, 10, 11, 14, 15]). The main result in Levenshtein [12] (see also [13, Section 5]) implies that any spherical 3-distance 5-design is a maximal code.

In this paper we present new proof of the Levenshtein's result from [12] and derive necessary conditions which underline a possible way to the complete proof of Conjecture 1.1. To this end we compute and carefully investigate the distance distributions of spherical 3-distance 5-designs. We prove that twice the cardinality is divisible by the dimension n, thus reducing by a factor of n the number of cases under suspicion.

In Section 2 we prove that every spherical 3-distance 5-design needs to have its three inner products exactly equal to the zeros of certain polynomial used by Levenshtein for derivation of his bounds. Section 3 is devoted to computing the distance distribution of corresponding codes and different presentations of the results. In Section 4 we obtain divisibility conditions which rule out significant amount of cases. Section 5 describes derived codes as a possible continuation of our investigation. In Section 6 we explain our computer investigation which verifies Conjecture 1.1 in all dimensions $n \le 1000$.

2. Inner products. Let $C \subset \mathbb{S}^{n-1}$ be a spherical 3-distance 5-design of cardinality $|C| = M$ and inner products $a < b < c$. The case $n = 2$ is elementary. If $n \ge 3$ and C is a tight 5-design, then $a = -1$, $c = -b = 1/m$, where m is odd positive integer, such that $n = m^2 - 2$ or $m = 1/\sqrt{5}$ and C is the icosahedron in three dimensions. Examples are known for $m = 3$ and 5 only, and another folklore conjecture says that no other tight 5-designs exist. Thus, in what follows we focus in dimensions $n \ge 3$ and assume that C is not a tight spherical 5-designs. Conjecture 1.1 is now equivalent to prove that always (3) happens. In this case, C

has dimension $n = 3m^2 - 5$, cardinality $M = m^4(3m^2 - 5)/2$ and inner products $-1/(m-1)$, $-1/(m^2-1)$, and $1/(m+1)$, where $m \geq 2$ is a positive integer. Examples are known for $m = 2$ and 3 only.

The main result from [12] implies that $|a| > |c| > |b| > 0$ (see [5, Corollary 3.9] for detailed proof of these inequalities), and a and b are the roots of the quadratic equation

$$(n+2)[(n+2)c^2 + 2c - 1]t^2 + 2c(c+1)(n+2)t + 3 - (n+2)c^2 = 0,$$

where c satisfies the equation

$$M = \frac{n((n+2)(n+3)c^2 + 4(n+2)c - n + 1)(1-c)}{2c(3-(n+2)c^2)}. \tag{1}$$

In the terminology of [12], one has $a = \alpha_0$, $b = \alpha_1$, $c = \alpha_2 = s$, and α_0, α_1, and $\alpha_2 = s$ are the zeros of the third degree polynomial

$$P_3(t)P_2(s) - P_3(s)P_2(t) = 0, \tag{2}$$

where $P_i(t) = P_i^{(\frac{n-1}{2}, \frac{n-3}{2})}(t)$ is a Jacobi polynomial normalized for $P_i(1) = 1$, and s is determined as the maximal root of the equation $M = L_5(n, s)$ (see Theorem 4.1 in [12]). Here $L_5(n, c)$ is the right hand side of (1).

We present a new proof of the above facts by using a result from [7] on the structure of spherical designs. As in [7], we use the following definition for a spherical design [9]: a code $C \subset \mathbb{S}^{n-1}$ is a spherical τ-design if and only if for any point $y \in \mathbb{S}^{n-1}$ and any real polynomial $f(t)$ of degree at most τ, the equality

$$\sum_{x \in C} f(\langle x, y \rangle) = f_0|C| \tag{3}$$

holds, where f_0 is the first coefficient in the Gegenbauer expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$.

Another useful tool is the Levenshtein quadrature [12, Theorems 4.1, 4.2], a particular case of it saying that

$$f_0 = \frac{f(1)}{M} + \rho_0 f(\alpha_0) + \rho_1 f(\alpha_1) + \rho_2 f(\alpha_2) \tag{4}$$

for every polynomial f of degree at most 5. Here the weights ρ_i, $i = 0, 1, 2$, are positive.

Theorem 2.1. *If $C \subset \mathbb{S}^{n-1}$ is a spherical 3-distance 5-design, then its inner products are exactly as described above.*

P r o o f. Using (4) and suitable polynomials in (3) Boyvalenkov and Nikova [7, Corollary 2.3] proved a general result which implies in our context that all four intervals $[-1,\alpha_0]$, $[\alpha_0,\alpha_1]$, $[\alpha_1,\alpha_2]$, and $[\alpha_2,1)$ contain an inner product of C. This means that at least one of these intervals does not contain an inner product as interior point, implying as in [7] that all intervals do so, i.e. $a=\alpha_0$, $b=\alpha_1$, and $c=\alpha_2$. □

The proof of Theorem 2.1 is naturally (and obviously) extended to the the following general result.

Theorem 2.2. *If $C\subset\mathbb{S}^{n-1}$ is a spherical k-distance $(2k-1)$-design, then its inner products are exactly the roots of the Levenshtein polynomial*

$$P_k(t)P_{k-1}(s)-P_k(s)P_{k-1}(t)=0, \tag{5}$$

where[1] *$P_i(t)=P_i^{(\frac{n-1}{2},\frac{n-3}{2})}(t)$ is a Jacobi polynomial normalized for $P_i(1)=1$ and s is determined as the maximal root of the equation*

$$|C|=L_{2k-1}(n,s),$$

$L_{2k-1}(n,s)$ is the Levenshtein bound.

P r o o f. Similarly to above, if follows from [7, Corollary 2.3] that each of the intervals $[-1,\alpha_0]$, $[\alpha_0,\alpha_1]$, $\ldots$, $[\alpha_{k-1},1)$ contains an inner product of C. The pigeonhole principle now gives that one of these intervals does not have an inner product of C as its interior point. Therefore, again by [7, Corollary 2.3], we conclude that the inner products are exactly $\alpha_0,\alpha_1,\ldots,\alpha_{k-1}$ and $|C|=L_{2k-1}(n,s)$. □

3. Distance distributions. Denote by

$$A_t(x):=|\{y\in C:\langle x,y\rangle\}|$$

the number of the points of C having inner product t with x. Then the system of numbers $(A_t(x):t\in[-1,1))$ is called distance distribution of C with respect of x.

For $x\in C$ let $(X,Y,Z)=(A_a(x),A_b(x),A_c(x))$ be the distance distribution of C with respect of x. Applying the quadrature formula (4) with the polynomials $f(t)=t^i$, $i=0,1,\ldots,5$, we obtain that the seven numbers a, b, c, X, Y, Z and $M=|C|$ satisfy the following system of six equations

$$a^iX+b^iY+c^iZ=f_iM-1,\quad i=0,1,\ldots,5, \tag{6}$$

where $f_i=0$ for odd i, $f_0=1$, $f_2=1/n$ and $f_4=3/n(n+2)$.

[1] The equation (5) is the general form of (2).

Taking the equations with odd i

$$aX + bY + cZ = a^3X + b^3Y + c^3Z = a^5X + b^5Y + c^5Z = -1.$$

and assuming that $a + b$, $b + c$, $c + a$ are all nonzero, we resolve with respect to X, Y and Z (see [6, Theorem 3.4]; this is a Vandermonde-like system) to obtain

$$\begin{aligned} X &= -\frac{(1-b^2)(1-c^2)}{a(a^2-b^2)(a^2-c^2)}, \\ Y &= -\frac{(1-c^2)(1-a^2)}{b(b^2-c^2)(b^2-a^2)}, \\ Z &= -\frac{(1-a^2)(1-b^2)}{c(c^2-a^2)(c^2-b^2)}. \end{aligned}$$

This implies, in particular, that the distance distribution of C does not depend on the choice of the point x and we therefore will omit x in what follows.

The computation of the distance distribution of the codes in Conjecture 1.1 (3) gives

$$\begin{aligned} X &= \frac{m(m^2-2)(m-1)^3}{4}, \\ Y &= (m^2-1)^3, \\ Z &= \frac{m(m^2-2)(m+1)^3}{4}. \end{aligned}$$

Further algebraic manipulations with the system (6) lead to another proof of Theorem 2.1. In particular, we obtain that a, b, and c are the roots of the equation

$$(7)\quad (n+2)[n(n+3)-2M]t^3 - n(n+2)(n-1)t^2 + (6M-5n^2-7n)t + n(n-1) = 0$$

(compare to (1)). We use (7) to compute the elementary symmetric polynomials $a + b + c$, $ab + bc + ca$, and abc as rational functions of n and M. It is clear that further advances in this direction can be based only on the investigation of the integrality conditions for the distance distribution of C. We focus on this in the next section.

4. Investigation of XYZ. Since X, Y, Z are not symmetric in a, b, c, we need to produce and consider symmetric expressions in order to express then as functions of n and M. We present the investigation of the product XYZ. It

follows from the explicit formulas from the previous section that

$$XYZ = \frac{1}{abc}\left(\frac{(1-a^2)(1-b^2)(1-c^2)}{(a^2-b^2)(b^2-c^2)(c^2-a^2)}\right)^2. \tag{8}$$

Using Vieta formulas and elementary symmetric polynomials from (7) we consecutively compute

$$\frac{(1-a^2)(1-b^2)(1-c^2)}{(a+b)(b+c)(c+a)} = \frac{M(n-1)}{n(n+2)},$$

$$abc = \frac{n(n-1)}{(n+2)(2M-n(n+3))},$$

and (the most complicated)

$$(a-b)^2(b-c)^2(c-a)^2 = \frac{R_1(n,M)}{(n+2)^3(2M-n(n+3))^4},$$

where

$$\begin{aligned} R_1(n,M) &= 432M^4 - 432M^3n(3n+5) + 36M^2n^2(n+2)(n^2+38n+57) \\ &\quad - 4Mn^3(3n+5)(5n+7)(n+5)^2 + n^4(n+1)(n+3)^2(n+5)^2. \end{aligned}$$

Therefore,

$$XYZ = \frac{M^2\,(n-1)\,(n+2)^2(2M-n\,(n+3))^5}{R_1(n,M)n^3}. \tag{9}$$

We proceed with derivation of divisibility conditions from (9). As usually, we denote by $v_p(A)$ the largest power of the prime number p which divides an integer A.

Theorem 4.1. *If $C \subset \mathbb{S}^{n-1}$ is a spherical 3-distance 5-design, then n divides $2M$.*

Proof. The cases $n = 2$ and $M = n(n+1)$ (i.e. C is a tight spherical 5-design) are trivial. So we may continue in the above context.

First we see that n divides $M^2(2M)^5$. Hence, all odd primes divisors of n divide $2M$ as well. For any odd prime divisor p of n we assume that $v_p(M) < v_p(n)$ and compute the power of p in the numerator as

$$v_p(M^2(2M-n\,(n+3))^5) = 7v_p\,(M)$$

and the denominator as

$$v_p(n^3R_1(n,M)) = 3v_p\,(n) + 4v_p\,(M)$$

(the case $p = 3$ needs separate consideration but leads to the same conclusion). Hence, $7v_p(M) \geq 3v_p(n) + 4v_p(M)$, i.e. $v_p(M) \geq v_p(n)$, a contradiction.

Further, denote $v_2(M) = y$, $v_2(n) = x$ and assume that $x > 1 + y \geq 2$ for a contradiction (the case $y = 0$ is easily ruled out). Then the powers of 2 in the denominator and the numerator are

$$v_2(4n^3 R_1(n, M)) \geq 2 + 3x + v_2\left(432M^4\right) = 3x + 4y + 6,$$

$$v_2(M^2(n-1)(n+2)^2(2M - n(n+3))^5) = 2y + 5(y+1) + 2 = 7y + 7,$$

respectively. Therefore

$$3y + 1 \geq 3x \geq 3(y+2) = 3y + 6,$$

a contradiction. This completes the proof. □

Remark 4.2. Note that the conclusion of Theorem 4.1 holds true in the cases (1) and (2) of Conjecture 1.1.

Let $2M = Tn$ for some positive integer T. After cancelation, we arrive at

$$XYZ = \frac{T^2(n-1)(n+2)^2(T-n-3)^5}{4R_2(n,T)}, \tag{10}$$

where

$$\begin{aligned} R_2(n,T) &= 27T^4 - 54(3n+5)T^3 + 9(n+2)(n^2+38n+57)T^2 \\ &\quad - 2(3n+5)(5n+7)(n+5)^2T + (n+1)(n+3)^2(n+5)^3. \end{aligned}$$

We were unable to continue with significant divisibility conclusions from (10).

In the end of this section we express either in terms of m and the dimension n the above parameters in the case (3) of Conjecture 1.1. Taking the values of X, Y and Z from Section 3, we obtain

$$XYZ = \frac{m^2(m^2-2)^2(m^2-1)^3}{16} = \frac{(n+5)(n-1)^2(n+2)^3}{2^4.3^6}.$$

Theorem 4.1 gives $T = m^4 = (n+5)^2/9$.

5. Derived codes. We descibe a construction from [8] defining derived codes (sections) which produces good codes in lower dimensions provided the original codes are good.

Following [8, Section 8] we consider the three derived codes, C_a, C_b, and C_c, of C with respect to the three inner products, respectively. These codes are spherical 3-designs [8, Theorem 8.2] and admit at most three inner products.

Therefore their distance distributions satisfy a system of 4 equations, which could be investigated.

For fixed point $x \in C$, the set

$$C_a(x) := \{y \in C : \langle x, y\rangle = a\}$$

defines, after rescaling on $\mathbb{S}^{n-2}$, a spherical 3-design which we denote by C_a. The cardinality of C_a is obviously equal to X. The inner products of C_a are found by the Cosine Law to be these among

$$\frac{a-a^2}{1-a^2} = \frac{a}{1+a}, \quad \frac{b-a^2}{1-a^2}, \quad \frac{c-a^2}{1-a^2}$$

belonging to $[-1, 1]$.

The distance distribution of C_a does not depend on the choice of the point and will be denoted by (X_a, Y_a, Z_a). It satisfies the system

$$\left(\frac{a}{1+a}\right)^i X_a + \left(\frac{b-a^2}{1-a^2}\right)^i Y_a + \left(\frac{c-a^2}{1-a^2}\right)^i Z_a = f_i X - 1, \quad i = 0, 1, 2, 3, \tag{11}$$

where f_i are defined as in (6) (note that n is replaced by $n-1$). The codes C_b and C_c are defined analogously.

6. Computer investigation of the distance distributions. The conditions from Sections 2 and 3 allow easy computational investigation of particular cases of Conjecture 1.1. For fixed dimension n, we consider all feasible cardinalities

$$M \in \left(n(n+1), \frac{n(n+1)(n+5)}{6}\right],$$

satisfying the condition $2M = nT$ from Theorem 4.1. We compute X, Y and Z in each case and check if they are nonnegative integers. It is clear that enough precision ensures the correctness of the results.

We implemented this algorithm in all dimensions $n \leq 1000$. This computation confirms Conjecture 1.1 in all cases except for the pairs

$$(n, T) = (341, 3744), \ (638, 7011),$$

where analysis of the derived codes is needed. In both cases, the distance distribution of the derived codes gives a contradiction.

For $(n, T) = (341,\ 3744)$ we find from (7) that $(a, b, c) = \left(-\frac{1}{7}, -\frac{1}{35}, \frac{1}{14}\right)$. It follows that $X = 23205$. Further, resolving (11) for $i = 0, 1, 2$, we find $(X_a, Y_a, Z_a) = \left(\frac{1872}{7}, \frac{571392}{49}, \frac{552500}{49}\right)$, a contradiction. Analogously, in the

case $(n, T) = (638,\ 7011)$ we find $(a, b, c) = \left(-\frac{1}{8}, -\frac{1}{40}, \frac{1}{20}\right)$, whence $X = 40508$ and, finally, $(X_a, Y_a, Z_a) = \left(\frac{52193}{224}, \frac{1245375}{56}, \frac{577125}{32}\right)$, a contradiction.

We note that these two exceptional solutions show that the proof can not be continued by proving that all solutions satisfy $n + 5|9T$, as the expected solution suggests.

Summarizing, we have verified the following theorem.

Theorem 6.1. *Conjecture* 1.1 *is true in all dimensions* $n \leq 1000$.

7. Conclusions. We have proved that all spherical 3-distance 5-designs have inner products exactly as coming from the Levenshtein framework [12, 5] and generalize this result. We prove that in all cases twice the cardinality $2M$ is divisible by the dimension n. Using a computer we have verified Conjecture 1.1 in all dimensions $n \leq 1000$.

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