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ON EXTENSIONS OF COMMUTING TUPLES OF SYMMETRIC AND ISOMETRIC OPERATORS

Sergey M. Zagorodnyuk

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ABSTRACT. In this paper we study extensions of commuting tuples of symmetric and isometric operators to commuting tuples of self-adjoint and unitary operators. Some conditions which ensure the existence of such extensions are presented. A multidimensional analog of the Godič–Lucenko Theorem is proved. An application to a multidimensional power-trigonometric moment problem is given.

1. Introduction. Extension problems for commuting tuples of symmetric and isometric operators have a long history, see, e.g., [2, Chapter VIII], [4], [12], [9], [14], [10], [11], [13], [3], [15], [1], [16], [20] and references therein. A set

$$(1) \quad \mathcal{T} = (A_1, \dots, A_\rho, B_1, \dots, B_\tau), \quad \rho, \tau \in \mathbb{Z}_+, \quad \rho + \tau \geq 2,$$

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where all A_j are symmetric operators and all B_k are linear isometric operators, having a joint invariant dense domain $\mathcal{D}$ in a Hilbert space H and pairwise commuting, is said to be a *commuting tuple of symmetric and isometric operators (with a joint invariant dense domain) in H* . The case $\rho = 0$ (or $\tau = 0$) means that operators A_j (respectively B_k) are absent.

Commuting tuples of operators appear in a natural way when dealing with various moment problems, see, e.g. [2], [5]. In this context there appears a natural question:

Question 1. *Do there exist self-adjoint operators $\tilde{A}_j \supseteq A_j$ and unitary operators $\tilde{B}_k \supseteq B_k$ in a Hilbert space $\tilde{H} \supseteq H$, all pairwise commuting?*

In [18], for the case $\rho = 2$, $\tau = 0$, we present some conditions which ensure the existence of such commuting (self-adjoint) extensions in the same space H . Namely, the following theorem was obtained:

Theorem 1 ([18, Theorem 1]). *Let $\mathbf{A}$ be a symmetric operator and $\mathbf{B}$ be an essentially self-adjoint operator with a common domain $\mathcal{D} = D(\mathbf{A}) = D(\mathbf{B})$ in a Hilbert space $\mathbf{H}$, $\overline{\mathcal{D}} = \mathbf{H}$, and*

$$\mathbf{A}\mathcal{D} \subseteq \mathcal{D}, \quad \mathbf{B}\mathcal{D} \subseteq \mathcal{D};$$

$$\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A}.$$

Suppose also that for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$, the operator $\mathbf{B}$ restricted to the domain $(\mathbf{A} - z_0 E_{\mathbf{H}})\mathcal{D}$ is essentially self-adjoint in a Hilbert space $(\overline{\mathbf{A}} - z_0 E_{\mathbf{H}})D(\overline{\mathbf{A}})$.

If there exists a conjugation $\mathbf{J}$ in $\mathbf{H}$ such that $\mathbf{J}\mathcal{D} \subseteq \mathcal{D}$, and

$$\mathbf{A}\mathbf{J} = \mathbf{J}\mathbf{A}, \quad \mathbf{B}\mathbf{J} = \mathbf{J}\mathbf{B},$$

then there exists a self-adjoint operator $\tilde{\mathbf{A}} \supseteq \mathbf{A}$, which commutes with $\overline{\mathbf{B}}$.

Recall that a conjugation J in a Hilbert space H is an antilinear operator, defined on the whole space H , such that: $J^2 = E_H$, $(Jf, Jg)_H = (g, f)_H$, for all $f, g \in H$. The proof of Theorem 1 essentially used the remarkable Godić–Lucenko Theorem (see [8], [6]).

Let us briefly describe the content of the present paper. In this paper we aim to generalize our extension result to the case of commuting tuples of symmetric and isometric operators. For this purpose we shall obtain a multidimensional extension of the Godić–Lucenko Theorem (Theorem 3). The idea of its proof is similar to the proof of the Godić–Lucenko Theorem in the paper of Garcia and Putinar [6]. It is crucial for our application that one of conjugations can be chosen to be the same for all unitary operators in a tuple. Another crucial point

is that the construction of an extension in the proof of Theorem 1 used only one of two conjugations provided by the Godič–Lucenko Theorem. In a consequence, we can generalize our result to several commuting symmetric operators. Under some suitable conditions we add some isometric operators to the commuting tuple as well (Theorem 4). In the case of bounded $A_2, \dots, A_\rho$ the situation simplifies considerably (Corollary 1). We shall illustrate our results on a multidimensional moment problem (Theorem 5). The latter problem generalizes Devinatz's (or power-trigonometric) moment problem (see [17]).

Notations. We denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. By $\mathbb{Z}_{k,l}$ we mean all integers j satisfying the following inequality: $k \leq j \leq l$; ($k, l \in \mathbb{Z}$). The Cartesian product of n copies of a number set S is denoted by S^n ; $n \in \mathbb{N}$ (e.g. $\mathbb{Z}^n, \mathbb{Z}_+^n$, etc.).

Everywhere in this paper, all Hilbert spaces are assumed to be separable, and isometric operators are supposed to be linear. By $(\cdot, \cdot)_H$ and $\|\cdot\|_H$ we denote the scalar product and the norm in a Hilbert space H , respectively. The indices may be omitted in obvious cases. For a set M in H , by $\overline{M}$ we mean the closure of M in the norm $\|\cdot\|_H$. For $\{x_k\}_{k \in S}$, $x_k \in H$, by $\text{Lin}\{x_k\}_{k \in S}$ we mean the set of linear combinations of vectors $\{x_k\}_{k \in S}$ and $\text{span}\{x_k\}_{k \in S} = \overline{\text{Lin}\{x_k\}_{k \in S}}$. Here S is an arbitrary set of indices.

The identity operator in H is denoted by E_H . For an arbitrary linear operator A in H , the operators $A^*, \overline{A}, A^{-1}$ mean its adjoint operator, its closure and its inverse (if they exist). By $D(A)$ and $R(A)$ we mean the domain and the range of the operator A .

Denote $D_{r,l} = \mathbb{R}^r \times [-\pi, \pi]^l = \{(x_1, x_2, \dots, x_r, \varphi_1, \varphi_2, \dots, \varphi_l), x_j \in \mathbb{R}, \varphi_k \in [-\pi, \pi), 1 \leq j \leq r, 1 \leq k \leq l\}$, $r, l \in \mathbb{Z}_+$. For elements $u \in D_{r,l}$ we briefly write: $u = (x, \varphi)$, $x = (x_1, x_2, \dots, x_r)$, $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_l)$. We mean that $D_{r,0} = \mathbb{R}^r$; $D_{0,l} = [-\pi, \pi]^l$. By $\mathfrak{B}(D_{r,l})$ we mean the set of all Borel subsets of $D_{r,l}$.

By $\mathbb{C}_{n \times n}$ we denote the set of all $(n \times n)$ matrices with complex elements, $\mathbb{C}_n := \mathbb{C}_{1 \times n}$, $n \in \mathbb{N}$. By $\mathbb{C}_{n \times n}^{\geq}$ we denote the set of all nonnegative Hermitian matrices from $\mathbb{C}_{n \times n}$, $n \in \mathbb{N}$. Let $M(\delta) = (m_{i,j}(\delta))_{i,j=0}^{N-1}$ be a $\mathbb{C}_{N \times N}^{\geq}$ -valued measure on $\mathfrak{B}(D_{r,l})$, and $\tau = \tau_M(\delta) := \sum_{k=0}^{N-1} m_{k,k}(\delta)$; $M'_\tau = (m'_{k,l})_{k,l=0}^{N-1} = (dm_{k,l}/d\tau_M)_{k,l=0}^{N-1}$; $N \in \mathbb{N}$. We denote by $L^2(M)$ a set of all (classes of the equivalence of) vector-valued functions $f : D_{r,l} \rightarrow \mathbb{C}_N$, $f = (f_0, f_1, \dots, f_{N-1})$, such that

$$\|f\|_{L^2(M)}^2 := \int_{D_{r,l}} f(u) \Psi(u) f^*(u) d\tau_M < \infty.$$

As it is well known, the set $L^2(M)$ is a Hilbert space with the following scalar

product:

$$(f, g)_{L^2(M)} := \int_{D_{r,l}} f(u) \Psi(u) g^*(u) d\tau_M, \quad f, g \in L^2(M).$$

Set

$$W_n f(x, \varphi) = e^{i\varphi_n} f(x, \varphi), \quad f \in L^2(M); \quad 1 \leq n \leq l;$$

and

$$X_m f(x, \varphi) = x_m f(x, \varphi),$$

$$f(x, \varphi) \in L^2(M) : x_m f(x, \varphi) \in L^2(M); \quad 1 \leq m \leq r.$$

Operators W_n are unitary, while operators X_m are self-adjoint.

2. A multidimensional version of the Godič–Lucenko Theorem and extensions of tuples of operators. We shall use a model for a set of commuting self-adjoint and unitary operators with the spectrum of a finite multiplicity provided by [19]. In order to state the corresponding result, we need to recall some relevant definitions from [19]. Consider a set

$$(2) \quad \mathcal{A} = (S_1, S_2, \dots, S_{\mathbf{r}}, U_1, U_2, \dots, U_{\mathbf{l}}), \quad \mathbf{r}, \mathbf{l} \in \mathbb{Z}_+ : \mathbf{r} + \mathbf{l} \neq 0,$$

where S_j are self-adjoint operators, U_k are unitary operators in a separable Hilbert space H , $1 \leq j \leq \mathbf{r}$, $1 \leq k \leq \mathbf{l}$. In the case $\mathbf{r} = 0$ operators S_j are absent, while the case $\mathbf{l} = 0$ means that U_k disappear. The set $\mathcal{A}$ is said to be a **SU -set of order $(\mathbf{r}, \mathbf{l})$** . The set $\mathcal{A}$ is called **commuting**, if operators S_j, U_k pairwise commute. In the latter case there exists the spectral measure $E(\delta)$, $\delta \in \mathfrak{B}(D_{\mathbf{r}, \mathbf{l}})$, of $\mathcal{A}$ such that:

$$S_j = \int_{D_{\mathbf{r}, \mathbf{l}}} x_j dE, \quad 1 \leq j \leq \mathbf{r}; \quad U_k = \int_{D_{\mathbf{r}, \mathbf{l}}} e^{i\varphi_k} dE, \quad 1 \leq k \leq \mathbf{l}.$$

A commuting SU -set $\mathcal{A}$ of order $(\mathbf{r}, \mathbf{l})$ **has the spectrum of order d** , if

1) there exist vectors $h_0, h_1, \dots, h_{d-1}$ in H such that

$$(3) \quad h_i \in D(S_1^{m_1} S_2^{m_2} \dots S_{\mathbf{r}}^{m_{\mathbf{r}}}), \quad m_1, m_2, \dots, m_{\mathbf{r}} \in \mathbb{Z}_+, \quad 0 \leq i \leq d-1;$$

$$\text{span}\{U_1^{n_1} U_2^{n_2} \dots U_{\mathbf{l}}^{n_{\mathbf{l}}} S_1^{m_1} S_2^{m_2} \dots S_{\mathbf{r}}^{m_{\mathbf{r}}} h_i,$$

$$(4) \quad m_1, m_2, \dots, m_{\mathbf{r}} \in \mathbb{Z}_+; \quad n_1, n_2, \dots, n_{\mathbf{l}} \in \mathbb{Z}; \quad 0 \leq i \leq d-1\} = H;$$

- 2) (*the minimality*) For an arbitrary $\tilde{d} \in \mathbb{Z}_+ : \tilde{d} < d$, and an arbitrary $\tilde{h}_0, \tilde{h}_1, \dots, \tilde{h}_{\tilde{d}-1}$ in H , at least one of the conditions (3),(4), with $\tilde{d}$ instead of d , and $\tilde{h}_i$ instead of h_i , is not valid.

In the case $\mathbf{r} = 0$, condition (3) is redundant. Condition (4) in the cases $\mathbf{r} = 0, \mathbf{l} = 0$, does not have U_k or S_j , respectively. Set

$$\vec{e}_i = (\delta_{0,i}, \delta_{1,i}, \dots, \delta_{N-1,i}), \quad 0 \leq i \leq N-1.$$

Theorem 2 ([19, Theorem 1]). *Let $\mathcal{A}$ be a commuting SU -set of order $(\mathbf{r}, \mathbf{l})$ in a separable Hilbert space H , having the spectrum of multiplicity d . Let $x_0, x_1, \dots, x_{N-1}$, $N \geq d$, be elements of H such that*

$$(5) \quad x_i \in D(S_1^{m_1} S_2^{m_2} \dots S_{\mathbf{r}}^{m_{\mathbf{r}}}), \quad m_1, m_2, \dots, m_{\mathbf{r}} \in \mathbb{Z}_+, \quad 0 \leq i \leq N-1;$$

$$\text{span}\{U_1^{n_1} U_2^{n_2} \dots U_{\mathbf{l}}^{n_{\mathbf{l}}} S_1^{m_1} S_2^{m_2} \dots S_{\mathbf{r}}^{m_{\mathbf{r}}} x_i,$$

$$(6) \quad m_1, m_2, \dots, m_{\mathbf{r}} \in \mathbb{Z}_+; \quad n_1, n_2, \dots, n_{\mathbf{l}} \in \mathbb{Z}; \quad 0 \leq i \leq N-1\} = H.$$

Set

$$(7) \quad M(\delta) = ((E(\delta)x_i, x_j)_H)_{i,j=0}^{N-1}, \quad \delta \in \mathfrak{B}(D_{\mathbf{r}, \mathbf{l}}),$$

where E is the spectral measure of $\mathcal{A}$.

Then there exists a unitary transformation V , which maps $L^2(M)$ onto H such that:

$$(8) \quad V^{-1} S_j V = X_j, \quad 1 \leq j \leq \mathbf{r};$$

$$(9) \quad V^{-1} U_k V = W_k, \quad 1 \leq k \leq \mathbf{l}.$$

Moreover, it holds:

$$(10) \quad V \vec{e}_s = x_s, \quad 0 \leq s \leq N-1.$$

In the case $\mathbf{r} = 0$ relation (5),(8) should be removed, and in (6) operators S_j disappear. In the case $\mathbf{l} = 0$ relation (9) should be deleted and in (6) operators U_k disappear. We are now ready to state and prove the following theorem.

Theorem 3. *Let $U_1, \dots, U_n$ be pairwise commuting unitary operators in a separable Hilbert space H ($n \in \mathbb{N}$). Then there exist conjugations $J_1, \dots, J_n, C$ in H , such that*

$$(11) \quad U_k = J_k C, \quad k = 1, \dots, n.$$

Moreover, there exist conjugations $L_1, \dots, L_n, K$ in H , such that

$$(12) \quad U_k = KL_k, \quad k = 1, \dots, n.$$

Proof. On the first step we shall represent the Hilbert space H as a direct sum of some Hilbert spaces H_j . Each H_j will reduce the operators U_k , and $(U_1, \dots, U_n)$ will be a commuting SU -set of order $(0, n)$ in H_j , having the spectrum of multiplicity 1. This can be done in a classical manner, like it was done for a single self-adjoint operator, see [7, Chapter 1, pp. 166-168]. Namely, choose an arbitrary orthonormal basis $\mathfrak{F} = \{f_j\}_{j=0}^\infty$ in H . Consider the following subspace:

$$H_0 := \text{span}\{U_1^{k_1} \dots U_n^{k_n} f_0, \ k_1, \dots, k_n \in \mathbb{Z}\}.$$

On step m ($m \in \mathbb{N}$) we choose the first element u (if such exists, otherwise we stop the process) of the basis $\mathfrak{F}$ which does not lie in $G_m := \oplus_{l=0}^{m-1} H_l$. We set $u' := u - P_{G_m}^H u$, and

$$H_m := \text{span}\{U_1^{k_1} \dots U_n^{k_n} u', \ k_1, \dots, k_n \in \mathbb{Z}\}.$$

We shall obtain the required decomposition:

$$(13) \quad H = \oplus_{l=0}^r H_l, \quad r \leq +\infty.$$

Fix an arbitrary l : $0 \leq l \leq r$. Applying Theorem 2 for $(U_1, \dots, U_n)$ in the Hilbert space H_l , we conclude that there exists a unitary transformation V_l , which maps $L^2(M_l)$ (with some scalar measure M_l on $\mathfrak{B}(D_{0,n})$) onto H_l such that:

$$(14) \quad V_l^{-1} U_{k;l} V_l = W_{k;l}, \quad 1 \leq k \leq n,$$

where

$$W_{k;l} f(\varphi) = e^{i\varphi_k} f(\varphi), \quad f \in L^2(M_l); \ 1 \leq k \leq n,$$

and by $U_{k;l}$ we denoted the operator U_k restricted to H_l . Let us represent the operator $W_{k;l}$ as a product of two conjugations:

$$(15) \quad W_{k;l} = J_{k;l} C_l, \quad 1 \leq k \leq n,$$

where

$$(16) \quad J_{k;l} f(\varphi) = e^{i\varphi_k} \overline{f(\varphi)}, \quad C_l f(\varphi) = \overline{f(\varphi)}, \quad f \in L^2(M_l).$$

Define the following conjugations in H_l :

$$\tilde{J}_{k;l} = V_l J_{k;l} V_l^{-1}, \quad 1 \leq k \leq n, \quad \tilde{C}_l = V_l C_l V_l^{-1}.$$

By (14),(16) we conclude that

$$(17) \quad U_{k;l} = \tilde{J}_{k;l} \tilde{C}_l, \quad 1 \leq k \leq n.$$

For an arbitrary $h = \sum_{l=0}^r h_l$, $h_l \in H_l$, $\sum_0^r \|h_l\|^2 < \infty$, we define the following conjugations:

$$\tilde{J}_k h = \sum_{l=0}^r \tilde{J}_{k;l} h_l, \quad 1 \leq k \leq n; \quad \tilde{C} h = \sum_{l=0}^r \tilde{C}_l h_l.$$

Then

$$U_k = \tilde{J}_k \tilde{C}, \quad 1 \leq k \leq n,$$

and representation (11) is verified. Representation (12) can be obtained by applying (11) for $(U_1^*, \dots, U_n^*)$. $\square$

A linear isometric operator V in a Hilbert space H is said to be *essentially unitary*, if $\overline{V}$ is a unitary operator in H .

Theorem 4. *Let $\mathcal{T} = (A_1, \dots, A_\rho, B_1, \dots, B_\tau)$ ($\rho \in \mathbb{N}, \tau \in \mathbb{Z}_+, \rho + \tau \geq 2$) be a commuting tuple of symmetric and isometric operators (with a joint invariant dense domain $\mathcal{D}$) in a separable Hilbert space H . Suppose that the following conditions hold:*

- (a) *The operators $A_2, \dots, A_\rho$ are essentially selfadjoint. The operators $B_1, \dots, B_\tau$ are essentially unitary. The closures of all of the above operators pairwise commute;*
- (b) *$B_k \mathcal{D} = \mathcal{D}$, $k = 1, \dots, \tau$;*
- (c) *For some $z_0 \in \mathbb{C} \setminus \mathbb{R}$, the operator A_j ($2 \leq j \leq \rho$) restricted to the domain $(A_1 - z_0 E_H) \mathcal{D}$ is essentially self-adjoint in a Hilbert space $(\overline{A_1} - z_0 E_H) D(\overline{A_1})$;*
- (d) *There exists a conjugation J in H such that $J\mathcal{D} \subseteq \mathcal{D}$, and*

$$(18) \quad \begin{aligned} A_j J &= J A_j, & j &= 1, \dots, \rho; \\ B_k J &= J B_k^{-1}, & k &= 1, \dots, \tau. \end{aligned}$$

Then there exists a self-adjoint operator $\widehat{A_1} \supseteq A_1$ in H , which commutes with $\overline{A_j}$ ($j = 2, \dots, \rho$) and with $\overline{B_k}$ ($k = 1, \dots, \tau$).

Remark 1. In the case $\rho = 1$ (or $\tau = 0$) the meaningless statements about operators A_j , $j \geq 2$, in (a), (c) (respectively about operators B_k in (a), (b), (18)) should be ignored in the conditions of the last theorem. In the same way, the commutativity with non-existing operators should be ignored in the conclusion.

Proof. We shall use the constructions in a proof of Theorem 1 (see [18]) with necessary modifications and additions. Consider the Cayley transformation of the operator $\overline{A_1}$:

$$V_1 := (\overline{A_1} - \overline{z_0}E_H)(\overline{A_1} - z_0E_H)^{-1} = E_H + (z_0 - \overline{z_0})(\overline{A_1} - z_0E_H)^{-1},$$

and denote

$$(19) \quad \begin{aligned} H_1 &= (\overline{A_1} - z_0E_H)D(\overline{A_1}), \quad H_2 = H \ominus H_1, \\ H_3 &= (\overline{A_1} - \overline{z_0}E_H)D(\overline{A_1}), \quad H_4 = H \ominus H_3. \end{aligned}$$

If $\rho \geq 2$, then conditions of Theorem 1 are satisfied with $\mathbf{A} = A_1$, $\mathbf{B} = A_j$, $\mathbf{H} = H$, $\mathbf{J} = J$, for each $j \in \mathbb{Z}_{2,\rho}$. Let us recall the construction in [18] of a selfadjoint operator $\tilde{A}_1 \supseteq A_1$, which commutes with $\overline{A_j}$ in H (for a fixed $j \in \mathbb{Z}_{2,\rho}$ at this moment). Consider the Cayley transformation of the operator $\overline{A_j}$:

$$U_j := (\overline{A_j} + iE_H)(\overline{A_j} - iE_H)^{-1} = E_H + 2i(\overline{A_j} - iE_H)^{-1}.$$

Denote by $U_{j,2}$ the restriction of U_j to the subspace H_2 , which is a unitary operator in H_2 . By the Godič–Lucenko Theorem it can be decomposed as a product of two conjugations:

$$(20) \quad U_{j,2} = K_j L_j.$$

Then $\tilde{A}_1$ is the inverse Cayley transformation of $V_1 \oplus JK_j$. If K_j could be chosen independent of j , then $\tilde{A}_1$ would commute with all A_j . In the case $\tau = 0$, we can apply Theorem 3 with $U_{j,2}$ ($2 \leq j \leq \rho$) which ensures that such a choice is indeed possible. Thus, for the case $\rho \geq 2, \tau = 0$ the proof is complete. In other cases we shall choose operators K_j later, in order to take into account the operators B_k .

Suppose now that $\tau \geq 1$, and $\rho \in \mathbb{N}$. For an arbitrary element $g = (A_1 - z_0E_H)f$, $f \in \mathcal{D}$, we may write:

$$B_k V_1 g = B_k (A_1 - \overline{z_0}E_H)f = (A_1 - \overline{z_0}E_H)B_k f;$$

$$V_1 B_k g = V_1 (A_1 - z_0E_H)B_k f = (A_1 - \overline{z_0}E_H)B_k f, \quad k \in \mathbb{Z}_{1,\tau}.$$

By the continuity we get

$$(21) \quad \overline{B_k} V_1 g = V_1 \overline{B_k} g, \quad g \in H_1, \quad k \in \mathbb{Z}_{1,\tau}.$$

By condition (b) and the commutativity of A_1 and B_k we obtain that

$$B_k^{-1}A_1 = A_1B_k^{-1}, \quad k \in \mathbb{Z}_{1,\tau}.$$

For an arbitrary element $g = (A_1 - z_0E_H)f$, $f \in \mathcal{D}$, we may write:

$$B_k^{-1}g = B_k^{-1}(A_1 - z_0E_H)f = (A_1 - z_0E_H)B_k^{-1}f \in H_1.$$

Therefore

$$(\overline{B_k})^{-1}H_1 \subseteq H_1.$$

In the same manner, we can see that

$$\overline{B_k}H_1 \subseteq H_1, \quad (\overline{B_k})^{-1}H_3 \subseteq H_3, \quad \overline{B_k}H_3 \subseteq H_3.$$

Therefore

$$\overline{B_k}H_1 = H_1, \quad \overline{B_k}H_3 = H_3.$$

Thus, the operators $\overline{B_k}$ ($k \in \mathbb{Z}_{1,\tau}$) are unitary when restricted to each of the Hilbert spaces H_i , $i = 1, 2, 3, 4$. Denote by $B_{k,2}$ the operator $\overline{B_k}$, restricted to H_2 , $k \in \mathbb{Z}_{1,\tau}$.

In the case $\rho = 1$, we may apply Theorem 4 to the operators $B_{k,2}$, $k = 1, \dots, \tau$, and obtain that

$$(22) \quad B_{k,2} = \tilde{K}R_k, \quad k = 1, \dots, \tau,$$

with some conjugations $\tilde{K}, R_k$ in H_2 . In the case $\rho \geq 2$, we apply Theorem 4 to the operators $U_{j,2}$, $j \in \mathbb{Z}_{2,\rho}$; $B_{k,2}$, $k \in \mathbb{Z}_{1,\tau}$, and obtain that

$$(23) \quad U_{j,2} = \hat{K}G_j, \quad j \in \mathbb{Z}_{2,\rho}, \quad B_{k,2} = \hat{K}S_k, \quad k \in \mathbb{Z}_{1,\tau},$$

with some conjugations $G_j, \hat{K}, S_k$ in H_2 . We shall use the conjugation

$$K := \begin{cases} \tilde{K}, & \text{if } \rho = 1 \\ \hat{K}, & \text{if } \rho \geq 2 \end{cases},$$

in order to construct the desired extension of A_1 . As in the proof of Theorem 1 (see formula (4) in [18]) we can check that

$$(24) \quad \overline{A_1}^* Jx = J\overline{A_1}^* x, \quad x \in D(\overline{A_1}^*),$$

and therefore

$$JH_2 = H_4.$$

Set

$$(25) \quad U_{2,4} = JK.$$

The operator $U_{2,4}$ is linear and isometric, it maps H_2 onto H_4 . We choose $\widehat{A}_1$ to be the inverse Cayley's transform of $V_1 \oplus U_{2,4}$. Fix an arbitrary $k \in \mathbb{Z}_{1,\tau}$. Let us check that $\widehat{A}_1$ commutes with the operator $\overline{B_k}$. Taking into account relation (21) we see, that it is enough to verify to the following condition:

$$(26) \quad \overline{B_k} U_{2,4} x = U_{2,4} \overline{B_k} x, \quad x \in H_2.$$

By the continuity and condition (d) of the theorem we conclude that

$$(27) \quad \overline{B_k} J = J(\overline{B_k})^{-1}, \quad k \in \mathbb{Z}_{1,\tau}.$$

Then

$$\begin{aligned} U_{2,4} B_{k,2} U_{2,4}^{-1} y &= J(B_{k,2})^{-1} J y = \\ &= J \overline{B_k}^{-1} J y = \overline{B_k} y, \quad y \in H_4, \end{aligned}$$

and relation (26) follows. If $\rho = 1$ this completes the proof. If $\rho \geq 2$, the considerations after (20) show that $\widehat{A}_1$ commutes with A_j ($j \in \mathbb{Z}_{2,\rho}$) as well. $\square$

Corollary 1. *Let $\mathcal{T} = (A_1, \dots, A_\rho, B_1, \dots, B_\tau)$ ($\rho \in \mathbb{N}, \tau \in \mathbb{Z}_+, \rho + \tau \geq 2$) be a commuting tuple of symmetric and isometric operators (with a joint invariant dense domain $\mathcal{D}$) in a separable Hilbert space H . Suppose that conditions (b) and (d) of Theorem 4 hold, as well as the following condition holds:*

(a') *The operators $A_2, \dots, A_\rho$ are bounded. The operators $B_1, \dots, B_\tau$ are essentially unitary.*

Then there exists a self-adjoint operator $\widehat{A}_1 \supseteq A_1$ in H , which commutes with $\overline{A_j}$ ($j = 2, \dots, \rho$) and with $\overline{B_k}$ ($k = 1, \dots, \tau$).

Proof. It is clear that by the continuity condition (a') implies condition (a) of Theorem 4. Moreover, in our case condition (c) of Theorem 4 is trivially satisfied. Thus, it remains to apply Theorem 4. $\square$

We shall now illustrate our extension result by a concrete example.

The multidimensional power-trigonometric moment problem.

Fix arbitrary $r, l \in \mathbb{N}$. For elements $(x, \varphi) \in D_{r,l}$ and $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}_+^r$, $\mathbf{n} = (n_1, \dots, n_l) \in \mathbb{Z}^l$ ($r, l \in \mathbb{N}$) we shall use the following multi-index notation:

$$x^{\mathbf{m}} e^{i\mathbf{n}\varphi} = x_1^{m_1} \dots x_r^{m_r} e^{in_1\varphi_1} \dots e^{in_l\varphi_l}.$$

We consider a moment problem which consists of finding a non-negative measure μ on $\mathfrak{B}(D_{r,l})$ such that

$$(28) \quad \int_{D_{r,l}} x^{\mathbf{m}} e^{i\mathbf{n}\varphi} d\mu = s_{\mathbf{m},\mathbf{n}}, \quad \mathbf{m} \in \mathbb{Z}_+^r, \quad \mathbf{n} \in \mathbb{Z}^l,$$

where $\mathcal{S} := \{s_{\mathbf{m},\mathbf{n}}\}_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l}$ is a prescribed set of complex numbers (moments).

The moment problem (28) is said to be *the multidimensional Devinatz moment problem* or *the multidimensional power-trigonometric moment problem*. For the case $r = l = 1$ see [17] and references therein.

Suppose that the moment problem (28) has a solution μ . Consider the following polynomial:

$$(29) \quad p(x, \varphi) = \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \alpha_{\mathbf{m},\mathbf{n}} x^{\mathbf{m}} e^{i\mathbf{n}\varphi}, \quad \alpha_{\mathbf{m},\mathbf{n}} \in \mathbb{C}, (x, \varphi) \in D_{r,l},$$

where all but finite number of $\alpha_{\mathbf{m},\mathbf{n}}$ s are zeros (such sequences of $\alpha_{\mathbf{m},\mathbf{n}}$ s we shall call *finite*). Then

$$\begin{aligned} 0 \leq \int |p|^2 d\mu &= \sum_{\mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^r, \mathbf{n}, \mathbf{l} \in \mathbb{Z}^l} \alpha_{\mathbf{m},\mathbf{n}} \overline{\alpha_{\mathbf{k},\mathbf{l}}} \int x^{\mathbf{m}+\mathbf{k}} e^{i(\mathbf{n}-\mathbf{l})\varphi} d\mu = \\ &= \sum_{\mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^r, \mathbf{n}, \mathbf{l} \in \mathbb{Z}^l} \alpha_{\mathbf{m},\mathbf{n}} \overline{\alpha_{\mathbf{k},\mathbf{l}}} s_{\mathbf{m}+\mathbf{k}, \mathbf{n}-\mathbf{l}}. \end{aligned}$$

Therefore the following condition is necessary for the solvability of the moment problem (28):

$$(30) \quad \sum_{\mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^r, \mathbf{n}, \mathbf{l} \in \mathbb{Z}^l} \alpha_{\mathbf{m},\mathbf{n}} \overline{\alpha_{\mathbf{k},\mathbf{l}}} s_{\mathbf{m}+\mathbf{k}, \mathbf{n}-\mathbf{l}} \geq 0,$$

for all finite sequences $\alpha_{\mathbf{m},\mathbf{n}} \in \mathbb{C}$.

Suppose now that the moment problem (28) is given and condition (30) holds. Denote by $\mathfrak{L}$ a set of all polynomials of the form (29). Notice that $\mathfrak{L}$ is a linear vector space with usual operations of addition and multiplication by a complex scalar (one needs to add the corresponding coefficients by $x^{\mathbf{m}} e^{i\mathbf{n}\varphi}$ or multiply $\alpha_{\mathbf{m},\mathbf{n}}$ by a scalar). Let p be given by (29) and

$$(31) \quad q(x, \varphi) = \sum_{\mathbf{k} \in \mathbb{Z}_+^r, \mathbf{l} \in \mathbb{Z}^l} \beta_{\mathbf{k},\mathbf{l}} x^{\mathbf{k}} e^{i\mathbf{l}\varphi}, \quad \beta_{\mathbf{k},\mathbf{l}} \in \mathbb{C}, (x, \varphi) \in D_{r,l},$$

where $\beta_{\mathbf{k},\mathbf{l}}$ is a finite sequence. Consider the following functional:

$$(32) \quad B(p, q) = \sum_{\mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^r, \mathbf{n}, \mathbf{l} \in \mathbb{Z}^l} \alpha_{\mathbf{m},\mathbf{n}} \overline{\beta_{\mathbf{k},\mathbf{l}}} s_{\mathbf{m}+\mathbf{k}, \mathbf{n}-\mathbf{l}}.$$

The functional $B(p, q)$ is sesquilinear. By (30) we see that $B(p, p) \geq 0$. Taking the imaginary parts of $B(p + q, p + q)$ and $B(p + iq, p + iq)$, we conclude that

$\overline{B(p, q)} = B(q, p)$. We say that two polynomials $p, q \in \mathfrak{L}$ belong to the same equivalence class, if $B(p - q, p - q) = 0$. Making the completion in a set of the equivalence classes $\{[p], p \in \mathfrak{L}\}$ we obtain a Hilbert space H . It is clear that H is separable (since one can consider polynomials with coefficients having rational real and imaginary parts). Notice that

$$([p], [q])_H = B(p, q), \quad p, q \in \mathfrak{L}.$$

Denote

$$(33) \quad y_{\mathbf{m}, \mathbf{n}} = [x^{\mathbf{m}} e^{i\mathbf{n}\varphi}], \quad \mathbf{m} \in \mathbb{Z}_+^r, \quad \mathbf{n} \in \mathbb{Z}^l.$$

The following condition holds:

$$(34) \quad (y_{\mathbf{m}, \mathbf{n}}, y_{\mathbf{k}, \mathbf{l}})_H = s_{\mathbf{m}+\mathbf{k}, \mathbf{n}+\mathbf{l}}, \quad \mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^r, \quad \mathbf{n}, \mathbf{l} \in \mathbb{Z}^l.$$

Denote $\mathcal{D} = \text{Lin}\{y_{\mathbf{m}, \mathbf{n}}\}_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l}$. Observe that $\overline{\mathcal{D}} = H$. Set

$$\vec{e}_{j;t} = (\delta_{1,j}, \delta_{2,j}, \dots, \delta_{t,j}), \quad 1 \leq j \leq t; \quad t \in \mathbb{N}.$$

Define the following operators:

$$(35) \quad A_j x = \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \alpha_{\mathbf{m}, \mathbf{n}} y_{\mathbf{m} + \vec{e}_{j;r}, \mathbf{n}}, \quad j \in \mathbb{Z}_{1,r};$$

$$(36) \quad B_k x = \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \alpha_{\mathbf{m}, \mathbf{n}} y_{\mathbf{m}, \mathbf{n} + \vec{e}_{k;l}}, \quad k \in \mathbb{Z}_{1,l};$$

where

$$(37) \quad x = \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \alpha_{\mathbf{m}, \mathbf{n}} y_{\mathbf{m}, \mathbf{n}} \in \mathcal{D},$$

and $\alpha_{\mathbf{m}, \mathbf{n}}$ is a finite sequence of complex numbers. Let us check that these operators are well-defined. Suppose that for an element $x \in \mathcal{D}$ with the representation (37) there exists another representation:

$$(38) \quad x = \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \beta_{\mathbf{m}, \mathbf{n}} y_{\mathbf{m}, \mathbf{n}},$$

where $\beta_{\mathbf{m}, \mathbf{n}}$ is a finite sequence of complex numbers. We need to show that

$$(39) \quad \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \alpha_{\mathbf{m}, \mathbf{n}} y_{\mathbf{m} + \vec{e}_{j;r}, \mathbf{n}} = \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \beta_{\mathbf{m}, \mathbf{n}} y_{\mathbf{m} + \vec{e}_{j;r}, \mathbf{n}}, \quad j \in \mathbb{Z}_{1,r};$$

and

$$(40) \quad \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \alpha_{\mathbf{m}, \mathbf{n}} y_{\mathbf{m}, \mathbf{n} + \vec{e}_{k;l}} = \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \beta_{\mathbf{m}, \mathbf{n}} y_{\mathbf{m}, \mathbf{n} + \vec{e}_{k;l}}, \quad k \in \mathbb{Z}_{1,l}.$$

Observe that for arbitrary $\mathbf{k} \in \mathbb{Z}_+^r, \mathbf{l} \in \mathbb{Z}^l$ we may write:

$$\begin{aligned} \left(\sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \alpha_{\mathbf{m}, \mathbf{n}} y_{\mathbf{m} + \vec{e}_{j;r}, \mathbf{n}} y_{\mathbf{k}, \mathbf{l}} \right)_H &= \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \alpha_{\mathbf{m}, \mathbf{n}} s_{\mathbf{m} + \vec{e}_{j;r} + \mathbf{k}, \mathbf{n} - \mathbf{l}} = \\ &= \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \alpha_{\mathbf{m}, \mathbf{n}} (y_{\mathbf{m}, \mathbf{n}}, y_{\mathbf{k} + \vec{e}_{j;r}, \mathbf{l}})_H = (x, y_{\mathbf{k} + \vec{e}_{j;r}, \mathbf{l}})_H. \end{aligned}$$

The same result we shall obtain if we take $\beta_{\mathbf{m}, \mathbf{n}}$ s instead of $\alpha_{\mathbf{m}, \mathbf{n}}$ s, and relation (39) follows. On the other hand, we may write:

$$\begin{aligned} &\left\| \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} (\alpha_{\mathbf{m}, \mathbf{n}} - \beta_{\mathbf{m}, \mathbf{n}}) y_{\mathbf{m}, \mathbf{n} + \vec{e}_{k;l}} \right\|^2 = \\ &= \sum_{\mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^r, \mathbf{n}, \mathbf{l} \in \mathbb{Z}^l} (\alpha_{\mathbf{m}, \mathbf{n}} - \beta_{\mathbf{m}, \mathbf{n}}) \overline{(\alpha_{\mathbf{k}, \mathbf{l}} - \beta_{\mathbf{k}, \mathbf{l}})} (y_{\mathbf{m}, \mathbf{n} + \vec{e}_{k;l}}, y_{\mathbf{k}, \mathbf{l} + \vec{e}_{k;l}}) = \\ &= \sum_{\mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^r, \mathbf{n}, \mathbf{l} \in \mathbb{Z}^l} (\alpha_{\mathbf{m}, \mathbf{n}} - \beta_{\mathbf{m}, \mathbf{n}}) \overline{(\alpha_{\mathbf{k}, \mathbf{l}} - \beta_{\mathbf{k}, \mathbf{l}})} (y_{\mathbf{m}, \mathbf{n}}, y_{\mathbf{k}, \mathbf{l}}) = \\ &= (x - x, x - x) = 0. \end{aligned}$$

Thus, relation (40) holds as well. It is easily checked that all A_j are symmetric, all B_k are linear isometric, and all the above operators pairwise commute on the joint invariant dense domain $\mathcal{D}$. We shall need the following operator:

$$(41) \quad J_0 x = \sum_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l} \overline{\alpha_{\mathbf{m}, \mathbf{n}}} y_{\mathbf{m}, -\mathbf{n}},$$

where x is given by relation (37). It is easily checked that J_0 is well-defined and

$$(J_0 x, J_0 y) = (y, x), \quad x, y \in \mathcal{D}.$$

By the continuity we extend J_0 to a conjugation J on H .

If $r \geq 2$ we additionally suppose that the following condition holds:

(B): For some $j_0 \in \mathbb{Z}_{1,r}$, there exist positive constants C_j ($j \in \mathbb{Z}_{1,r} \setminus \{j_0\}$) such that

$$(42) \quad \sum_{\mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^r, \mathbf{n}, \mathbf{l} \in \mathbb{Z}^l} \alpha_{\mathbf{m}, \mathbf{n}} \overline{\alpha_{\mathbf{k}, \mathbf{l}}} s_{\mathbf{m} + \mathbf{k} + 2\mathbf{e}_{j_0}, \mathbf{n} - \mathbf{l}} \leq C_j \sum_{\mathbf{m}, \mathbf{k} \in \mathbb{Z}_+^r, \mathbf{n}, \mathbf{l} \in \mathbb{Z}^l} \alpha_{\mathbf{m}, \mathbf{n}} \overline{\alpha_{\mathbf{k}, \mathbf{l}}} s_{\mathbf{m} + \mathbf{k}, \mathbf{n} - \mathbf{l}},$$

for all finite sequences of complex numbers $\alpha_{\mathbf{m}, \mathbf{n}}$ and all $j \in \mathbb{Z}_{1,r} \setminus \{j_0\}$.

Condition (B) ensures that operators A_j ($j \in \mathbb{Z}_{1,r} \setminus \{j_0\}$) are bounded. By Corollary 1 we conclude that there exists a self-adjoint extension $\tilde{A}_{j_0} \supseteq A_{j_0}$ in H , which commutes with the closures of the rest of A_j s and B_k s.

By the induction one can check that

$$(43) \quad y_{(m_1, \dots, m_r), (n_1, \dots, n_l)} = A_1^{m_1} \cdots A_r^{m_r} B_1^{n_1} \cdots B_l^{n_l} y_{\mathbf{0}_r, \mathbf{0}_l},$$

where $\mathbf{0}_t$ means the null vector of size t . Then

$$s_{\mathbf{m}, \mathbf{n}} = (y_{(m_1, \dots, m_r), (n_1, \dots, n_l)}, y_{\mathbf{0}_r, \mathbf{0}_l})_H = (A_1^{m_1} \cdots A_r^{m_r} B_1^{n_1} \cdots B_l^{n_l} y_{\mathbf{0}_r, \mathbf{0}_l}, y_{\mathbf{0}_r, \mathbf{0}_l})_H = \int_{D_{r,l}} x^{\mathbf{m}} e^{i\mathbf{n}\varphi} d\tilde{\mu}, \quad \mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}_+^r, \quad \mathbf{n} = (n_1, \dots, n_l) \in \mathbb{Z}^l,$$

where $\tilde{\mu}(\delta) = (E(\delta) y_{\mathbf{0}_r, \mathbf{0}_l}, y_{\mathbf{0}_r, \mathbf{0}_l})_H$, $\delta \in \mathfrak{B}(D_{r,l})$, and $E(\delta)$ is the spectral measure for the commuting SU-set of $\overline{A_j}$ ($j \in \mathbb{Z}_{1,r} \setminus \{j_0\}$), $\tilde{A}_{j_0}$ and $\overline{B_k}$ ($k \in \mathbb{Z}_{1,l}$). Thus, $\tilde{\mu}$ is a solution of the moment problem (28).

Theorem 5. *Let the moment problem (28) be given with some moments $\{s_{\mathbf{m}, \mathbf{n}}\}_{\mathbf{m} \in \mathbb{Z}_+^r, \mathbf{n} \in \mathbb{Z}^l}$. If $r = 1$, then condition (30) is necessary and sufficient for the solvability of the moment problem (28).*

If $r \geq 2$, then condition (30) is still necessary, while condition (30) together with condition (B) are sufficient for the solvability of the moment problem (28).

Proof. All statements of the theorem were already proved in the preceding considerations. $\square$

Finally, we remark that conditions of Theorem 4 can be applied to the multidimensional power-trigonometric moment problem as well. In fact, there are various criteria for the commutativity of the closures of self-adjoint operators, as

well as various conditions which ensure the essential self-adjointness (e.g. in terms of quasi-analytic vectors which lead to the Carleman type criteria). In details, this will be studied elsewhere.

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Sergey M. Zagorodnyuk

V. N. Karazin Kharkiv National University

School of Mathematics and Computer Sciences

Department of Higher Mathematics and Informatics

Svobody Square 4, 61022, Kharkiv, Ukraine

e-mail: Sergey.M.Zagorodnyuk@gmail.com;

Sergey.M.Zagorodnyuk@univer.kharkov.ua

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