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RESULT ON UNIQUENESS AND WEIGHTED SHARING OF ENTIRE FUNCTION OF DIFFERENTIAL POLYNOMIAL WITH THEIR DERIVATIVES

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ABSTRACT. In this article, we worked on the uniqueness of more generalised form of a function namely $f^n P(f)$ and $[f^n P(f)]^{(k)}$ sharing a small function. The Theorem proved in this paper improves the results of Kit-wing [5].

1. Introduction and main results. In this paper, a meromorphic function will mean meromorphic in the whole complex plane. Here we use the standard notations in the Nevanlinna value distribution theorem of meromorphic functions such as $T(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $m(r, f)$, (see [2, 3]). The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, provided that $T(r, a) = S(r, f)$.

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Let f and g be two non-constant meromorphic functions. Let a be a finite complex number. We denote by $E(a, f)$ the set of zeros of $f - a$ (counting multiplicity), by $\overline{E}(a, f)$ the set of zeros of $f - a$ (ignoring multiplicity). We say f and g share a CM (IM), if $E(a, f) = E(a, g)$ ($\overline{E}(a, f) = \overline{E}(a, g)$). Similarly, we define that f and g share a small function $a(z)$ CM (IM), if $E(a(z), f) = E(a(z), g)$ ($\overline{E}(a(z), f) = \overline{E}(a(z), g)$). Moreover, $GCD(n_1, n_2, \dots, n_k)$ denotes the greatest common divisor of positive integers $n_1, n_2, \dots, n_k$.

For any constant a , we define

$$\Theta(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r, f)}, \quad \delta(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.$$

We also use the notation $\Theta(a; f, g)$ and $\delta(a; f, g)$, where

$$\Theta(a; f, g) = \min\{\Theta(a, f), \Theta(a, g)\} \quad \text{and} \quad \delta(a; f, g) = \min\{\delta(a, f), \delta(a, g)\}.$$

Let a be any value in the extended complex plane. Let k be an arbitrary non negative integer. We define

$$(1.1) \quad \Theta_k(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}_k(r, \frac{1}{f-a})}{T(r, f)}.$$

Definition 1.1 (see [1]). Suppose $f(z), g(z)$ satisfy $\overline{E}(a, f) = \overline{E}(a, g)$. Let z_0 be a common 1-point of $f(z)$ and $g(z)$ with the multiplicity r and q . We denote by $\overline{N}_L(r, \frac{1}{f-1})$ the reduced counting function of those 1-points of $f(z)$ and $g(z)$ where $r > q$; $\overline{N}_E^{(2)}(r, \frac{1}{f-1})$ the reduced counting function of those 1-points of $f(z)$ and $g(z)$ where $r = q \geq 2$. In addition, we denote $N_E^1(r, \frac{1}{f-1})$ ($N_E^1(r, \frac{1}{g-1})$) the counting function of those common simple 1-points of $f(z)$ and $g(z)$.

Remark 1.1. From (1.1) we have

$$0 \leq \Theta(a, f) \leq \Theta_k(a, f) \leq \Theta_{k-1}(a, f) \leq \Theta_1(a, f) \leq 1.$$

In 1998, Q. C. Zhang [6] proved the following theorem about a meromorphic function and its k -th order derivative.

Theorem A. Let f be a non-constant meromorphic function and let k be a positive integer. Suppose that f and $f^{(k)}$ share 1CM and $2\overline{N}(r, f) + \overline{N}(r, \frac{1}{f'}) + N(r, \frac{1}{f^{(k)}}) \leq (\lambda + o(1))T(r, f^{(k)})$ for $r \in I$, where I is a set of infinite linear measure and λ satisfies $0 < \lambda < 1$ then $\frac{f^{(k)}-1}{f-1} \equiv c$ for some constant $c \in C - \{0\}$.

In 2003, Kit-wing [5] discussed the problem of a meromorphic function and its k -th derivative sharing one small function and proved the following result.

Theorem B. *Let $k \geq 1$. Let f be a non-constant non-entire meromorphic function, $a \in s(f)$ and $a(\neq 0, \infty)$ and f do not have any common pole. If $f, f^{(k)}$ share a CM and $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$, then $f = f^{(k)}$.*

In this paper, we concentrate on the uniqueness of more generalised form of a function namely $f^n P(f)$ and $[f^n P(f)]^{(k)}$ sharing a small function.

Theorem 1.1. *Let k, m and n are three positive integers with $k \leq m$ and f be a non-constant entire function. Also let $a(z)(\neq 0, \infty)$ be a small function with respect to f . If $\overline{E}_l(a, f^n P(f)) = \overline{E}_l(a, (f^n P(f))^{(k)})$, where l is a positive integer and*

$$\begin{aligned} 2N_2\left(r, \frac{1}{f^n P(f)}\right) + 2N_2\left(r, \frac{1}{(f^n P(f))^{(k)}}\right) + \overline{N}\left(r, \frac{1}{(f^n P(f))^{(k)}}\right) \\ \leq (\lambda + O(1))T\left(r, (f^n P(f))^{(k)}\right), \end{aligned}$$

for $0 < \lambda < 1$, where $r \in I$ and I is a set of infinite linear measure, then $\frac{(f^n P(f))^{(k)} - a}{f^n P(f) - a} = c$ for some constant $a \in C - \{0\}$ where C is the set of complex numbers.

Here we mention some existing lemmas of the literature which will be frequently used to prove the aforementioned theorems.

Lemma 1.1 (see [4]). *Let $f(z)$ be a non-constant meromorphic function and k, p be two positive integers. Then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Lemma 1.2 (see [2]). *Let f be a non-constant meromorphic function and let n be a positive integer. $P(f) = a_n f^n + a_{n-1} f^{n-1} + \dots + a_1 f$ where a_i is a meromorphic function such that $T(r, a_i) = S(r, f)$ ($i = 1, 2, \dots, n$). Then $T(r, P(f)) = nT(r, f) + S(r, f)$.*

2. Proof of the theorem.

Proof of Theorem 1. Let $F = \frac{f^n P(f)}{a}$ and $G = \frac{(f^n P(f))^{(k)}}{a}$. Therefore, $F - 1 = \frac{f^n P(f) - a}{a}$ and $G - 1 = \frac{(f^n P(f))^{(k)} - a}{a}$. Now, $\overline{E}_l(a, f^n P(f)) = \overline{E}_l\left(a, (f^n P(f))^{(k)}\right)$ except the zeros and poles of $a(z)$. Define,

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right).$$

We now consider two cases:

Case 1. Suppose $H \not\equiv 0$. Then $m(r, H) = s(r, f)$. Now if z_0 is a common simple zero of $F - 1$ and $G - 1$ (except the zeros and poles of $a(z)$), then after simple calculation, we get $H(z_0) = 0$. So,

$$N_E^{(1)}\left(r, \frac{1}{G-1}\right) \leq N\left(r, \frac{1}{H}\right) + S(r, f) \leq T(r, H) + S(r, f) \leq N(r, H) + S(r, f).$$

Again by analysis, we can deduce that,

$$\begin{aligned} N(r, H) &\leq \overline{N}(r, f) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) \\ &\quad + \overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}_*\left(r, \frac{1}{F-1}\right) + \overline{N}_*\left(r, \frac{1}{G-1}\right) \\ &\quad + \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) + S(r, f). \end{aligned}$$

Also

$$\begin{aligned} \overline{N}\left(r, \frac{1}{G-1}\right) &= N_E^{(1)}\left(r, \frac{1}{G-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{G-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) \\ &\quad + \overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_*\left(r, \frac{1}{G-1}\right) + S(r, f). \end{aligned}$$

Therefore,

$$\begin{aligned}
 \overline{N}\left(r, \frac{1}{G-1}\right) &\leq \overline{N}(r, f) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) + 2\overline{N}_L\left(r, \frac{1}{F-1}\right) \\
 (2.1) \quad &+ 2\overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{G-1}\right) + 2\overline{N}_*\left(r, \frac{1}{G-1}\right) \\
 &+ \overline{N}_*\left(r, \frac{1}{F-1}\right) + \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) + S(r, f)
 \end{aligned}$$

Since, $\overline{E}_{l)}(1, F) = \overline{E}_{l)}(1, G)$.

Therefore,

$$2\overline{N}_L\left(r, \frac{1}{G-1}\right) + 2\overline{N}_*\left(r, \frac{1}{G-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{G-1}\right) \leq 2\overline{N}_{(2)}\left(r, \frac{1}{G-1}\right).$$

From (3.1), we have,

$$\begin{aligned}
 \overline{N}\left(r, \frac{1}{G-1}\right) &\leq \overline{N}(r, f) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) + 2\overline{N}_{(2)}\left(r, \frac{1}{G-1}\right) \\
 (2.2) \quad &+ 2\overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_*\left(r, \frac{1}{F-1}\right) + \overline{N}_0\left(r, \frac{1}{F'}\right) \\
 &+ \overline{N}_0\left(r, \frac{1}{G'}\right) + S(r, f)
 \end{aligned}$$

We also have,

$$\begin{aligned}
 \overline{N}_2\left(r, \frac{1}{F}\right) + 2\overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_*\left(r, \frac{1}{F-1}\right) + \overline{N}_0\left(r, \frac{1}{F'}\right) \\
 (2.3) \quad &\leq 2\overline{N}\left(r, \frac{1}{F'}\right)
 \end{aligned}$$

Now by the second fundamental theorem we get,

$$(2.4) \quad T(r, G) \leq \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) - \overline{N}_0\left(r, \frac{1}{G'}\right) + S(r, G)$$

From (2.4) using (2.2) and (2.3) we get,

$$(2.5) \quad T(r, G) \leq 2\overline{N}(r, f) + 2\overline{N}\left(r, \frac{1}{F'}\right) + 2\overline{N}\left(r, \frac{1}{G'}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r, f)$$

By Lemma 1.1 we have,

$$\begin{aligned} T\left(r, (f^n P(f))^{(k)}\right) &\leq 2N_2\left(r, \frac{1}{f^n P(f)}\right) + 2N_2\left(r, \frac{1}{(f^n P(f))^{(k)}}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{(f^n P(f))^{(k)}}\right) + S(r, f) \end{aligned}$$

which contradicts the given conditions of the theorem.

Case II. Suppose $H(z) \equiv 0$ i.e., $\frac{F''}{F'} - \frac{2F'}{F-1} = \frac{G''}{G'} - \frac{2G'}{G-1}$. Integrating we get, $\log F' - 2\log(F-1) = \log G' - 2\log(G-1) + \log A$. Where A is a constant $\neq 0$. That is $\log \frac{F'}{(F-1)^2} = \log \frac{AG'}{(G-1)^2}$. Again integrating we get,

$$(2.6) \quad \frac{1}{F-1} = \frac{A}{G-1} + B$$

Now if z_0 is a pole of f with multiplicity p which is not the poles and the zeros of $a(z)$, then z_0 is the pole of F with multiplicity $(n+m)p$ and the pole of G with multiplicity $(n+m)p+k$ ($\neq (n+m)p$). This contradicts (2.6). This implies f has no pole, that is f is an entire function.

So, $\overline{N}(r, F) = S(r, f)$ and $\overline{N}(r, G) = S(r, f)$. Now we prove that $B = 0$.

We first assume that $B \neq 0$, then $\frac{1}{F-1} = \frac{B\left(G-1 + \frac{A}{B}\right)}{G-1}$.

Therefore, $\overline{N}\left(r, \frac{1}{G-1 + \frac{A}{B}}\right) = \overline{N}(r, F) = S(r, f)$.

Now we assume $\frac{A}{B} \neq 1$. By the Second fundamental theorem,

$$\begin{aligned} T(r, G) &\leq \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G-1 + \frac{A}{B}}\right) + S(r, G) \\ &\leq \overline{N}\left(r, \frac{1}{G}\right) + S(r, f) \\ &\leq T(r, G) + S(r, f) \end{aligned}$$

Hence, $T(r, G) = \overline{N}\left(r, \frac{1}{G}\right) + S(r, f)$ i.e.,

$$T\left(r, (f^n P(f))^{(k)}\right) = \overline{N}\left(r, \frac{1}{(f^n P(f))^{(k)}}\right) + S(r, f)$$

This contradicts the given condition of the theorem. Next, we assume $\frac{A}{B} = 1$.

Then, $(AF - A - 1)G = -1$. So, $\frac{a^2}{f^n P(f)(Af^n P(f) - Aa - A)} = -\frac{(f^n P(f))^{(k)}}{f^n P(f)}$.

Now by Lemma 1.1 and 1.2, we get,

$$\begin{aligned} 2(n+m)T(r, f) &= T\left(r, \frac{(f^n P(f))^{(k)}}{f^n P(f)}\right) + S(r, f) \\ &\leq N\left(r, \frac{(f^n P(f))^{(k)}}{f^n P(f)}\right) + S(r, f) \\ &\leq N_k\left(r, \frac{1}{f^n P(f)}\right) + k\overline{N}(r, f) + S(r, f) \\ &\leq k\overline{N}\left(r, \frac{1}{f}\right) + mN\left(r, \frac{1}{f}\right) + S(r, f) \end{aligned}$$

i.e., $(2n+m-k)T(r, f) = S(r, f)$. Which is impossible since $m \geq k$. Hence our assumption is not true and therefore $B = 0$. So, $\frac{G-1}{F-1} = A$. This proves the theorem. $\square$

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