

Provided for non-commercial research and educational use. Not for reproduction, distribution or commercial use.
--

Serdica

Mathematical Journal

Сердика

Математическо списание

The attached copy is furnished for non-commercial research and education use only.
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal <http://www.math.bas.bg/~serdica>
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

CHARACTERISTIC POLYNOMIAL AND WIENER INDEX OF THE COMPRESSED ZERO DIVISOR GRAPH $\Gamma_E[\mathbb{Z}_{p^n}]$

B. Surendranath Reddy, Rupali S. Jain, N. Laxmikanth

Communicated by V. Drensky

ABSTRACT. The zero divisor graph of a commutative ring R , denoted by $\Gamma[R]$, is a graph whose vertices are non-zero zero divisors of R and two vertices are adjacent if their product is zero. The relation on R given by $r \sim s$ if and only if $\text{ann}_R(r) = \text{ann}_R(s)$ is an equivalence relation. The compressed zero divisor graph $\Gamma_E(R)$ is the (undirected) graph whose vertices are the equivalence classes induced by $\sim$ other than $[0]$ and $[1]$, such that distinct vertices $[r]$ and $[s]$ are adjacent in $\Gamma_E(R)$ if and only if $rs = 0$. We show that the coefficients of the compressed zero divisor graph $\Gamma_E[\mathbb{Z}_{p^n}]$ form a Pascal like triangle and thus derive the characteristic polynomial in terms of binomial coefficients. We also calculate the Wiener index of $\Gamma_E[\mathbb{Z}_{p^n}]$.

2020 *Mathematics Subject Classification*: Primary: 13A70; Secondary: 05C12, 05C25, 13A15.

Key words: compressed zero divisor graph, characteristic polynomial, binomial coefficients, Wiener index.

1. Introduction. The concept of the zero divisor graph of a ring R was first introduced by I. Beck [4] in 1988 and later on Anderson and Livingston [2], Akbari and Mohammadian [1] continued the study of the zero divisor graph by considering only the non-zero zero divisors. After that, a zero divisor graph determined by equivalence classes of zero divisors was introduced by Mulay [6] and by S. Spiroff and C. Wickham [7], and was named as compressed zero divisor graph by David F. Anderson and John D. LaGrange [3]. In this article, we discuss about the characteristic polynomial of the compressed zero divisor graph $\Gamma_E[\mathbb{Z}_{p^n}]$ and observe that the coefficients form a nice triangle which looks like a Pascal triangle. This observation leads us to derive the characteristic polynomial in terms of binomial coefficients. Section 2 is about the preliminaries and notations related to zero divisor graph of a commutative ring R and compressed zero divisor graph. In Section 3, we derive the characteristic polynomial of the compressed zero divisor graph $\Gamma_E[\mathbb{Z}_{p^n}]$, and in Section 4, we calculate the Wiener index of the compressed zero divisor $\Gamma_E[\mathbb{Z}_{p^n}]$.

2. Preliminaries and notations.

Definition 2.1 (Zero divisor graph [2]). Let R be a commutative ring with unity and $Z[R]$ be the set of its zero divisors. Then the zero divisor graph of R denoted by $\Gamma[R]$, is the graph (undirected) with vertex set $Z^*[R] = Z[R] - \{0\}$, the non-zero zero divisors of R , such that two vertices $v, w \in Z^*[R]$ are adjacent if $vw = 0$.

Definition 2.2 (Compressed zero divisor graph [3]). Let R be a commutative ring with unity. The relation on R given by $r \sim s$ if and only if $\text{ann}_R(r) = \text{ann}_R(s)$ is an equivalence relation. The compressed zero divisor graph $\Gamma_E(R)$ is the (undirected) graph whose vertices are the equivalence classes induced by $\sim$ other than $[0]$ and $[1]$, such that distinct vertices $[r]$ and $[s]$ are adjacent in $\Gamma_E(R)$ if and only if $rs = 0$. We denote the vertices $[r]$ and $[s]$ simply by r and s .

Definition 2.3 (Adjacency matrix of $\Gamma_E[R]$). The adjacency matrix of the compressed zero divisor graph $\Gamma_E[R]$ is the matrix $[v_{ij}]$ with rows and columns labeled by the vertices and is given by

$$v_{ij} = \begin{cases} 1, & v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise.} \end{cases}$$

The adjacency matrix of $\Gamma[R]$ is denoted by $M(\Gamma_E[R])$. Clearly for an undirected graph, the adjacency matrix is symmetric.

Definition 2.4 (Wiener index of $\Gamma_E[R]$). Let $\Gamma_E[R]$ be a compressed zero divisor graph with vertex set V . We denote the length of the shortest path between every pair of vertices $x, y \in V$ with $d(x, y)$. Then the Wiener index of $\Gamma_E[R]$ is the sum of the distances between all pairs of vertices of $\Gamma_E[R]$, i.e.,

$$W(\Gamma_E[R]) = \sum_{x, y \in V} d(x, y).$$

3. Characteristic polynomial of $\Gamma_E[\mathbb{Z}_{p^n}]$. In this section we find the relation among the coefficients of the characteristic polynomial of $\Gamma_E[\mathbb{Z}_{p^n}]$ and derive the corresponding characteristic polynomial.

Example 3.1. The characteristic polynomial of $\Gamma_E[\mathbb{Z}_{p^6}]$ is

$$(-1)(\lambda^5 - 3\lambda^4 - 3\lambda^3 + 4\lambda^2 + \lambda - 1).$$

Proof. The vertex set of $\Gamma_E[\mathbb{Z}_{p^6}]$ is given by

$$V(\Gamma_E[\mathbb{Z}_{p^6}]) = \{a = m[p], b = m[p^2], c = m[p^3], d = m[p^4], e = m[p^5]\},$$

where $m[p^i]$ denotes the equivalence class of multiples of p^i . Then the adjacency matrix is

$$M(\Gamma_E[\mathbb{Z}_{p^6}]) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Hence the characteristic polynomial is

$$|M - \lambda I| = \begin{vmatrix} -\lambda & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 0 & 1 & 1 \\ 0 & 0 & 1 - \lambda & 1 & 1 \\ 0 & 1 & 1 & 1 - \lambda & 1 \\ 1 & 1 & 1 & 1 & 1 - \lambda \end{vmatrix}.$$

Expanding the determinant we get the characteristic polynomial as

$$(-1)(\lambda^5 - 3\lambda^4 - 3\lambda^3 + 4\lambda^2 + \lambda - 1).$$

Instead of the direct evaluation of the determinant we can start with the well known matrix equality

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} \begin{vmatrix} I_2 & -A^{-1}B \\ 0 & I_3 \end{vmatrix} = \begin{vmatrix} A & 0 \\ C & D - CA^{-1}B \end{vmatrix},$$

where

$$A = \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}, \quad B = C^t = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}$$

and I_2 and I_3 are, respectively, the 2×2 and 3×3 identity matrices. Then

$$|M - \lambda I| = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D - CA^{-1}B|.$$

We shall use such kind of presentation in the proof of Theorem 3.3. $\square$

Example 3.2. The characteristic polynomial of $\Gamma_E[\mathbb{Z}_{p^7}]$ is

$$\lambda^6 - 3\lambda^5 - 6\lambda^4 + 4\lambda^3 + 5\lambda^2 - \lambda + 1.$$

Proof. The vertex set of $\Gamma_E[\mathbb{Z}_{p^6}]$ is given by

$$V(\Gamma_E[\mathbb{Z}_{p^7}]) = \{a = m[p], b = m[p^2], c = m[p^3], d = m[p^4], e = m[p^5], f = m[p^6]\}$$

where $m[p^i]$ denotes the equivalence class of multiples of p^i . Then the adjacency matrix is

$$M(\Gamma_E[\mathbb{Z}_{p^7}]) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Hence the characteristic polynomial is

$$|M - \lambda I| = \begin{vmatrix} -\lambda & 0 & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 0 & 0 & 1 & 1 \\ 0 & 0 & -\lambda & 1 & 1 & 1 \\ 0 & 0 & 1 & 1-\lambda & 1 & 1 \\ 0 & 1 & 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1 & 1 & 1-\lambda \end{vmatrix}.$$

Expanding the determinant we get the characteristic polynomial as

$$\lambda^6 - 3\lambda^5 - 6\lambda^4 + 4\lambda^3 + 5\lambda^2 - \lambda + 1.$$

As in the previous example, instead of the direct evaluation of the determinant we can use the presentation

$$|M - \lambda I| = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D - CA^{-1}B|,$$

where

$$A = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}, \quad B = C^t = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and}$$

$$D = \begin{bmatrix} 1 - \lambda & 1 & 1 \\ 1 & 1 - \lambda & 1 \\ 1 & 1 & 1 - \lambda \end{bmatrix}. \quad \square$$

In a similar way we have calculated the characteristic polynomial of $\Gamma_E[\mathbb{Z}_{p^n}]$ for different values of n and observed that the coefficients of the characteristic polynomial form a Pascal like triangle which is shown below.

$$\begin{array}{cccccccccccc} & & & & & & & 1 & & & & & \\ & & & & & & & & 1 & & & & \\ & & & & & & 1 & & 1 & & & & \\ & & & & 1 & & 2 & & 1 & & 1 & & \\ & & & 1 & & 3 & & 3 & & 1 & & 1 & \\ & & 1 & & 3 & & 6 & & 4 & & 5 & & 1 & & 1 \\ & 1 & & 4 & & 6 & & 10 & & 5 & & 6 & & 1 & & 1 \\ 1 & & 4 & & 10 & & 10 & & 15 & & 6 & & 7 & & 1 & & 1 \end{array}$$

This motivated us to derive the characteristic polynomial for $\Gamma[\mathbb{Z}_{p^n}]$ in terms of binomial coefficients.

Theorem 3.3. *Let $m = p^n$ with p prime. Then the characteristic polynomial of the compressed zero divisor graph $\Gamma[\mathbb{Z}_m]$ is*

$$\begin{aligned} (-1)^{n-1}(\lambda^{n-1} - b_1\lambda^{n-2} - b_2\lambda^{n-3} + b_3\lambda^{n-4} + \dots - (-1)^{\lfloor \frac{k+1}{2} \rfloor} b_k\lambda^{n-(k+1)} + \dots \\ + (-1)^{b_1} b_{n-1}\lambda^{n-n}), \end{aligned}$$

where $b_i = \binom{p_i}{i}$ and $p_i = \left\lfloor \frac{n-1+i}{2} \right\rfloor$ (the floor function), for $i = 1, 2, 3, \dots, n-1$.

Proof. Let $m = p^n$. Then the set of non-zero zero divisors of $\mathbb{Z}_m$ is

$$Z^*[\mathbb{Z}_m] = \{[p], [p^2], [p^3], \dots, [p^{n-1}]\}$$

with cardinality $n - 1$ and the adjacency matrix is given by

$$M = M(\Gamma[\mathbb{Z}_{p^n}]) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix},$$

$$|M - \lambda I| = \begin{vmatrix} -\lambda & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & -\lambda & 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 - \lambda & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 - \lambda \end{vmatrix}.$$

Then $|M - \lambda I| = |A||D - CA^{-1}B|$, where the matrices A, B, C, D are defined in a similar way as in above two examples. Clearly zero is not an eigenvalue of M , so we have $\lambda \neq 0$. Since A is a scalar matrix of order $\lfloor \frac{n-1}{2} \rfloor$, we get $|A| = (-\lambda)^{\lfloor \frac{n-1}{2} \rfloor}$ and

$$|CA^{-1}B| = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{-1}{\lambda} & \frac{-1}{\lambda} & \cdots & \frac{-1}{\lambda} \\ 0 & \frac{-1}{\lambda} & \frac{-2}{\lambda} & \cdots & \frac{-2}{\lambda} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{-1}{\lambda} & \frac{-2}{\lambda} & \cdots & \frac{-\lfloor \frac{n-1}{2} \rfloor}{\lambda} \end{vmatrix}.$$

So, in general when n is odd $|D - CA^{-1}B|$ is given by

$$|D - CA^{-1}B| = \begin{vmatrix} a_3 - \lambda & a_3 & a_3 & \cdots & a_3 \\ a_3 & a_5 - \lambda & a_5 & \cdots & a_5 \\ a_3 & a_5 & a_7 - \lambda & \cdots & a_7 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_3 & a_5 & a_7 & \cdots & a_n - \lambda \end{vmatrix}.$$

And when n is even, $|D - CA^{-1}B|$ is given by

$$|D - CA^{-1}B| = \begin{vmatrix} a_2 - \lambda & a_2 & a_2 & \cdots & a_2 \\ a_2 & a_4 - \lambda & a_4 & \cdots & a_4 \\ a_2 & a_4 & a_6 - \lambda & \cdots & a_6 \\ \vdots & \vdots & \ddots & \vdots & \\ a_2 & a_4 & a_6 & \cdots & a_n - \lambda \end{vmatrix},$$

where $a_n = 1 + \frac{\lfloor \frac{n-1}{2} \rfloor}{\lambda}$ in both the cases for $n = 2, 3, 4, \dots$ on solving the above determinant and calculating $|A||D - CA^{-1}B|$. The characteristic polynomial of the compressed zero divisor graph $\Gamma[\mathbb{Z}_m]$ is

$$(-1)^{n-1}(\lambda^{n-1} - b_1\lambda^{n-2} - b_2\lambda^{n-3} + b_3\lambda^{n-4} + \cdots - (-1)^{\lfloor \frac{k+1}{2} \rfloor} b_k\lambda^{n-(k+1)} + \cdots + (-1)^{b_1} b_{n-1}\lambda^{n-n}),$$

where $b_i = \binom{p_i}{i}$ and $p_i = \left\lfloor \frac{n-1+i}{2} \right\rfloor$, for $i = 1, 2, 3, \dots, n-1$. $\square$

4. Wiener index of the compressed zero divisor graph $\Gamma_E[\mathbb{Z}_n]$. In this section, we derive the standard form of the Wiener index of the compressed zero divisor graph $\Gamma_E[\mathbb{Z}_m]$ for $m = p^n$. To start with, we consider two cases n being even and odd.

Example 4.1. Let $m = p^6$ with p prime. Then the Wiener index of the compressed zero divisor graph $\Gamma_E[\mathbb{Z}_m]$ is $W(\Gamma_E[\mathbb{Z}_m]) = 14$.

Proof. Let $m = p^6$. Then the set of non-zero zero divisors of $\mathbb{Z}_m$ is

$$Z^*[\mathbb{Z}_m] = \{[p], [p^2], [p^3], [p^4], [p^5]\}$$

with cardinality $n-1$. Let $a = [p]$, $b = [p^2]$, $c = [p^3]$, $d = [p^4]$, $e = [p^5]$. Since the element a is not adjacent with b, c and d , we have $d(a, b) = d(a, c) = d(a, d) = 2$ and $d(a, e) = 1$. The vertex b is adjacent with d, e and not adjacent with c . Hence $d(b, c) = 2$, $d(b, d) = d(b, e) = 1$. Also, c is adjacent with d and e . Therefore

$d(c, d) = d(c, e) = 1$. Similarly, d is adjacent with e and $d(d, e) = 1$. Hence

$$\begin{aligned}
 \mathbf{W}(\Gamma[\mathbb{Z}_{p^4}]) &= \sum d(x, y) \\
 &= (d(a, b) + d(a, c) + d(a, d) + d(a, e)) \\
 &\quad + (d(b, c) + d(b, d) + d(b, e)) \\
 &\quad + (d(c, d) + d(c, e)) + d(d, e) \\
 &= (2 + 2 + 2 + 1) + (2 + 1 + 1) + (1 + 1) + 1 = 14. \quad \square
 \end{aligned}$$

Example 4.2. Let $m = p^7$ with p prime. Then the Wiener index of the compressed zero divisor graph $\Gamma_E[\mathbb{Z}_m]$ is $W(\Gamma_E[\mathbb{Z}_m]) = 21$.

Proof. Let $m = p^7$. Then the set of non-zero zero divisors of $\mathbb{Z}_m$ is

$$Z^*[\mathbb{Z}_m] = \{[p], [p^2], [p^3], [p^4], [p^5], [p^6]\}$$

with cardinality $n - 1$. Let $a = [p], b = [p^2], c = [p^3], d = [p^4], e = [p^5], f = [p^6]$. The element a is not adjacent with b, c, d, e and is adjacent only with f . Therefore $d(a, b) = d(a, c) = d(a, d) = d(a, e) = 2$ and $d(a, f) = 1$. Similarly b is also not adjacent with c, d but is adjacent with e, f and $d(b, c) = d(b, d) = 2$ and $d(b, e) = d(b, f) = 1$. The vertex c is adjacent with d, e and f . Hence $d(c, d) = d(c, e) = d(c, f) = 1$. Since d adjacent with e, f we obtain that $d(d, e) = d(d, f) = 1$. Finally, e is adjacent with all other elements except a and $d(e, f) = 1$. Therefore

$$\begin{aligned}
 W(\Gamma_E[\mathbb{Z}_{p^7}]) &= \sum d(x, y) \\
 &= (d(a, b) + d(a, c) + d(a, d) + d(a, e) + d(a, f)) \\
 &\quad + (d(b, c) + d(b, d) + d(b, e) + d(b, f)) \\
 &\quad + (d(c, d) + d(c, e) + d(c, f)) \\
 &\quad + (d(d, e) + d(d, f)) + d(e, f) \\
 &= (2 + 2 + 2 + 2 + 1) + (2 + 2 + 1 + 1) \\
 &\quad + (1 + 1 + 1) + (1 + 1) + 1 = 21. \quad \square
 \end{aligned}$$

With the above examples, we now head towards the general case

Theorem 4.3. *Let $m = p^n$ with p prime. Then the Wiener index of the compressed zero divisor graph $\Gamma_E[\mathbb{Z}_m]$ is*

$$W(\Gamma_E[\mathbb{Z}_m]) = \begin{cases} \frac{(n-2)(3n-4)}{4}, & \text{if } n \text{ is odd;} \\ \frac{(n-1)(3n-7)}{4}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $m = p^n$. Then the set of non-zero zero divisors of $\mathbb{Z}_m$ is

$$Z^*[\mathbb{Z}_m] = \{[p], [p^2], [p^3], \dots, [p^{n-1}]\}$$

with cardinality $n-1$. Let $a_1 = [p], a_2 = [p^2], a_3 = [p^3], \dots, a_{n-1} = [p^{n-1}]$. Since the element a_1 is not adjacent with $a_2, a_3, \dots, a_{n-2}$ and is adjacent only with a_{n-1} , we obtain

$$d(a_1, a_2) = d(a_1, a_3) = \dots = d(a_1, a_{n-2}) = 2 \text{ and } d(a_1, a_{n-1}) = 1.$$

With a similar argument, we get

$$d(a_2, a_3) = d(a_2, a_4) = \dots = d(a_2, a_{n-3}) = 2 \text{ and } d(a_2, a_{n-2}) = d(a_2, a_{n-1}) = 1.$$

In the same way

$$d(a_i, a_{i+1}) = d(a_i, a_{i+2}) = \dots = d(a_i, a_{n-i-1}) = 2$$

and

$$d(a_i, a_{n-i}) = d(a_i, a_{n-i+1}) = \dots = d(a_i, a_{n-1}) = 1.$$

Finally $d(a_{n-1}, a_n) = 1$. The combinations of the elements with distance 2 from a_1 are

$$d(a_1, a_2) = d(a_1, a_3) = \dots = d(a_1, a_{n-2}) = 2.$$

Hence these are $(n-3)$ combinations as a_1 and a_{n-1} are to be neglected. Similarly

$$d(a_2, a_3) = d(a_2, a_4) = \dots = d(a_2, a_{n-3}) = 2$$

and these are $(n-5)$ combinations as a_1, a_2, a_{n-2} and a_{n-1} are to be neglected. Continuing in this way we reach the case $d(a_{\lfloor \frac{n}{2} \rfloor - 1}, a_{\lfloor \frac{n}{2} \rfloor}) = 2$ and this is only one combination as the suffixes are consecutive terms, where states $\lfloor \frac{n}{2} \rfloor$ for the floor function. Now the combinations of the elements with distance is 1 are

$$d(a_1, a_{n-1}) = 1; \text{ this is only one term,}$$

$d(a_2, a_{n-2}) = d(a_2, a_{n-1}) = 1$; these are two terms,

$d(a_3, a_{n-3}) = d(a_3, a_{n-2}) = d(a_3, a_{n-1}) = 1$; these are three terms,

and so on until

$$d(a_{\lfloor \frac{n}{2} \rfloor - 1}, a_{\lfloor \frac{n}{2} \rfloor + 1}) = d(a_{\lfloor \frac{n}{2} \rfloor - 1}, a_{\lfloor \frac{n}{2} \rfloor + 2}) = \cdots = d(a_{\lfloor \frac{n}{2} \rfloor - 1}, a_{n-1}) = 1;$$

these are $(\lfloor \frac{n}{2} \rfloor - 1)$ in all. Similarly

$$d(a_{\lfloor \frac{n}{2} \rfloor}, a_{\lfloor \frac{n}{2} \rfloor + 1}) = d(a_{\lfloor \frac{n}{2} \rfloor}, a_{\lfloor \frac{n}{2} \rfloor + 2}) = \cdots = d(a_{\lfloor \frac{n}{2} \rfloor}, a_{n-1}) = 1;$$

these are also $(\lfloor \frac{n}{2} \rfloor - 1)$ in all, and so on. Finally $d(a_{n-2}, a_{n-1}) = 1$; this is only one term.

Case 1: $m = p^n$, where n is even.

$$\begin{aligned} W(\Gamma_E[\mathbb{Z}_m]) &= \sum_{x, y \in V} d(x, y) \\ &= \sum_{\text{dist}=2} d(a_i, a_j) + \sum_{\text{dist}=1} d(a_i, a_j) \\ &= 2 \times [(n-3) + (n-5) + \cdots + (n - (n-1))] \\ &\quad + 2 \times \left[1 + 2 + \cdots + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] \\ &= 2 \times \left[(n-1) - 2 + (n-1) - 4 + \cdots + (n-1) - \left(2 \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] \\ &\quad + 2 \times \left[1 + 2 + \cdots + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] \\ &= 2 \times \left[(n-1) \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] - \left[2 + 4 + \cdots + 2 \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] \\ &\quad + 2 \times \left[1 + 2 + \cdots + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] \\ &= 2 \times \left[(n-1) \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] - 2 \left[1 + 2 + \cdots + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] \\ &\quad + 2 \times \left[1 + 2 + \cdots + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] \\ &= 2 \left[(n-1) \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] - 4 \left[1 + 2 + \cdots + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] \\ &\quad + 2 \times \left[1 + 2 + \cdots + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] \\ &= 2 \left[(n-1) \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] - 2 \left[1 + 2 + \cdots + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] \\ &= 2 \times \left[(n-1) \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] - \left[\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right]. \end{aligned}$$

$$\begin{aligned}
W(\Gamma_E[\mathbb{Z}_m]) &= 2 \times \left[(n-1) \binom{n-2}{2} \right] - \left[\frac{n}{2} \binom{n-2}{2} \right] \\
&= (n^2 - 3n + 2) - \left(\frac{n^2 - 2n}{4} \right) \\
&= \left(\frac{n-2}{4} \right) [4n - 4 - n] \\
&= \frac{(n-2)(3n-4)}{4}.
\end{aligned}$$

Case 2: $m = p^n$, where n is odd.

Now, since n is odd, in addition to above combinations, we get one more combination extra with distance 1 as

$$d(a_{\lfloor \frac{n}{2} \rfloor}, a_{n - \lfloor \frac{n}{2} \rfloor}) = d(a_{\lfloor \frac{n}{2} \rfloor}, a_{n - \lfloor \frac{n}{2} \rfloor - 1}) = \cdots = d(a_{\lfloor \frac{n}{2} \rfloor}, a_{n-1}) = 1.$$

These are $\lfloor \frac{n}{2} \rfloor$ in all. Hence we get

$$\begin{aligned}
W(\Gamma_E[\mathbb{Z}_m]) &= 2 \times \left[(n-1) \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] - \left[\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] + \left\lfloor \frac{n}{2} \right\rfloor \\
&= 2 \times \left[(n-1) \left(\frac{n-1}{2} - 1 \right) \right] - \left[\frac{n-1}{2} \left(\frac{n-1}{2} - 1 \right) \right] + \frac{n-1}{2} \\
&= 2 \times \left[(n-1) \left(\frac{n-1-2}{2} \right) \right] - \left[\frac{(n-1)(n-1-2)}{2} \right] + \frac{n-1}{2} \\
&= (n-3) \left[(n-1) - \left(\frac{n-1}{4} \right) \right] + \frac{n-1}{2} \\
&= \left(\frac{n-3}{4} \right) [4n - 4 - n + 1] + \frac{n-1}{2} \\
&= \frac{3(n-1)(n-3)}{4} + \frac{n-1}{2} \\
&= \frac{(n-1)(3n-7)}{4}.
\end{aligned}$$

□

Remark 4.4. The above result has also been proved very recently by S. Pirzada, M. Aijaz and M. Imran Bhat [8], independently. The paper [8] was posted at the home page of the journal on Jan. 1, 2020 and we posted our paper in [arXiv.org](https://arxiv.org) on Jan. 5, 2020.

REFERENCES

- [1] S. AKBARI, A. MOHAMMADIAN. On the zero-divisor graph of a commutative ring. *J. Algebra* **274**, 2 (2004), 847–855.
- [2] D. F. ANDERSON, P. S. LIVINGSTON. The zero-divisor graph of commutative ring. *J. Algebra* **217**, 2 (1999), 434–447.
- [3] D. F. ANDERSON, J. D. LAGRANGE. Commutative Boolean monoids, reduced rings, and the compressed zero-divisor graph. *J. Pure Appl. Algebra* **216**, 7 (2012), 1626–1636.
- [4] I. BECK. Coloring of commutative rings. *J. Algebra* **116**, 1 (1988), 208–226.
- [5] J. CLARK, D. A. HOLTON. A First Look at Graph theory. Teaneck, NJ, World Scientific Publishing Co., Inc., 1991, <https://doi.org/10.1142/1280>.
- [6] S. B. MULAY. Cycles and symmetries of zero-divisors. *Comm. Algebra* **30**, 7 (2002) 3533–3558.
- [7] S. SPIROFF, C. WICKHAM. A zero divisor graph determined by equivalence classes of zero divisors. *Comm. Algebra* **39**, 7 (2011) 2338–2348.
- [8] S. PIRZADA, M. AIJAZ, M. I. BHAT. On zero divisor graphs of the rings $\mathbb{Z}_n$. *Afr. Mat.* (2020), <https://doi.org/10.1007/s13370-019-00755-3>.

Swami Ramanand Teerth Marathwada University
Nanded, India

e-mail: surendra.phd@gmail.com (B. Surendranath Reddy)

rupalisjain@gmail.com (Rupali S. Jain)

laxmikanth.nandala@gmail.com (N. Laxmikanth)

Received January 26, 2020