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UNIQUENESS OF CERTAIN POLYNOMIALS OF MEROMORPHIC FUNCTIONS SHARING A SET WITH FINITE WEIGHT

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Abstract. We investigated the Brück type conjecture and generalized the existing result by extending them up to a difference-differential polynomial $\mathscr{P}[\xi]$ sharing a small function with a certain differential polynomial $\mathscr{L}[\xi]$ of a meromorphic function. The class of all meromorphic solutions of the differential equation $\mathscr{P}[\xi] \equiv \mathscr{L}[\xi]$ has been explored. Our result will generalize and extend the result due to A. Banerjee and B. Chakraborty [3]. For the generalization of our main result, some relevant questions have finally been posed for further study in this direction.

1. Introduction and main results. In this article, "meromorphic function" refers to a function that is analytic except for poles in $\mathbb{C}$, whereas "entire function" refers to a function that is analytic everywhere in $\mathbb{C}$. The Nevanlinna theory describes the asymptotic distribution of the solutions to the equation $\xi(z) = \mathscr{V}$ as $\mathscr{V}$ changes. Initially, we anticipate that readers will have a basic

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understanding of the fundamentals of the Nevanlinna theory, which can be found in [8, 11, 19]. Next, we address the general concept of sharing. We denote set $E = \{a : a \in \mathbb{R}^+\}$. Let $\mathscr{F} = \{\xi : \xi$ is non-constant meromorphic function in $\mathbb{C}\}$. For $\xi, \zeta \in \mathscr{F}$ and $b \in \mathbb{C} \cup \{\infty\}$, if $\xi - b$ and $\zeta - b$ have identical zeros, including multiplicities, then ξ and ζ share b CM (counting multiplicities); if the multiplicities are ignored, then ξ and ζ share b IM (ignoring multiplicities), and if $\frac{1}{\xi}$ and $\frac{1}{\zeta}$ share 0 CM, then ξ and ζ share ∞ CM [23]. We call b the b points of ξ and ζ or the value points of ξ and ζ, respectively. For $\kappa(z) \in \mathscr{F}$, if $T(r, \kappa) = \mathscr{S}(r, \xi)$ then κ is called the small function of ξ where $T(r, \kappa)$ is the Nevanlinna characteristic function and $\mathscr{S}(r, \xi) = o(T(r, \xi)), r \notin E, r \to \infty$.

The problem of meromorphic functions sharing values with their derivatives is a special subclass in the literature of uniqueness theory. The subject of sharing values between entire functions and their derivatives was first studied by Rubel and Yang ([18]). In 1977, they proved that if a non-constant entire function ξ and ξ' share two distinct finite numbers a, b CM, then $\xi = \xi'$.

Theorem 1.1 ([16]). *Let ξ be a non-constant entire function. If ξ and ξ' share two distinct values a, b IM then $\xi \equiv \xi'$.*

Example 1.2. Let

$$\xi(z) = e^{e^z} \int_0^z e^{-e^t}\left(1 - e^t\right) dt.$$

It is clear that $\xi' - 1 = c^2(\xi - 1)$. So ξ and ξ' share 1 CM but $\xi \not\equiv \xi'$.

The example below illustrates that the same result occurs when the entire function becomes a non-constant meromorphic function.

Example 1.3. Let

$$\xi(z) = \frac{2Ae^{2z}}{e^{2z} - B}.$$

It is clear that ξ and ξ' share A IM but $\xi \not\equiv \xi'$.

To advance the discussion, we adopted a widely recognized definition of set sharing.

Let $\mathscr{S}$ be a set of complex numbers and let

$$E_\xi(\mathscr{S}) = \bigcup_{a \in \mathscr{S}} \{z : \xi(z) = a\},$$

where each zero is counted according to its multiplicity. If we do not count the multiplicity, then the set

$$\bigcup_{a\in\mathcal{S}}\{z : \xi(z) = a\}$$

is denoted by $\overline{E}_{\xi}(\mathcal{S})$. If $E_{\xi}(\mathcal{S}) = E_{\zeta}(\mathcal{S})$ we see that ξ and ζ share the set $\mathcal{S}$ CM. On the other hand, if $\overline{E}_{\xi}(\mathcal{S}) = \overline{E}_{\zeta}(\mathcal{S})$, we say that ξ and ζ share the set $\mathcal{S}$ IM. If $\mathcal{S}$ contains only one element, then it coincides with the usual definition of CM (resp. IM) sharing of values.

Consideringthe previous notion, it is fascinating to study the relationship between ξ and its derivative, ξ', when they share a set. When we apply Rubel-Yang or Mues-Steinmetz logic to a set of two elements rather than values, we see that the results are not usually true.

Example 1.4. Let $\mathcal{S} = \{a, b\}$, where a and b are any two distinct complex numbers. Let $\xi(z) = e^{-z} + a + b$, then $E_{\xi}(\mathcal{S}) = E_{\xi'}(\mathcal{S})$ but $\xi \not\equiv \xi'$

To ensure the uniqueness of the meromorphic function and its derivative, the cardinality of sharing sets of meromorphic functions must be at least three. In 2003, Fang and Zalcman proved the following results by using normal families:

Theorem 1.5 ([7]). *Let $\mathcal{S} = \{0, a, b\}$, where where a, b are two non-zero distinct complex numbers satisfying $a^2 \neq b^2$, $a \neq 2b$ and $a^2 - ab + b^2 \neq 0$. If for a non-constant entire function ξ, $E_{\xi}(\mathcal{S}) = E_{\xi'}(\mathcal{S})$, then $\xi \equiv \xi'$.*

In 2007, Chang, Fang, and Zalcman [5] extended Theorem 1.5 to permit an arbitrary set of three elements.

Theorem 1.6 ([5]). *Let ξ be a non-constant entire function and let $\mathcal{S} = \{a, b, c\}$, where a, b and c are distinct complex numbers. If $E_{\xi}(\mathcal{S}) = E_{\xi'}(\mathcal{S})$, then one of the following assertion holds, where C is a non-zero complex number:*

(1) $\xi(z) = Ce^{z}$ *or*

(2) $\xi(z) = Ce^{-z} + \dfrac{2}{3}(a + b + c)$, *where* $(2a - b - c)(2b - c - a)(2c - a - b) = 0$ *or*

(3) $\xi(z) = Ce^{\frac{-1+i\sqrt{3}}{2}z} + e^{\frac{3+i\sqrt{3}}{6}z}(a + b + c)$, *where* $a^2 + b^2 + c^2 - ab - bc - ca = 0$ *where C is a non-zero complex number.*

In the same year, Chang, Fang, and Zalcman [5] demonstrate that, by Theorem 1.6, one cannot relax the CM sharing to the IM sharing of the set $\mathcal{S}$ by analyzing the following example.When multiplicity is neglected, the uniqueness result no longer holds true.

Example 1.7. Let $\mathscr{S} = \{-1, 0, 1\}$ and $\xi(z) = \sin(z)$. Then ξ and ξ' share $\mathscr{S}$ IM, but $\xi \not\equiv \xi'$.

In the following year, Chang and Zalcman [6] used the entire function instead of a meromorphic function with many simple poles in Theorems 1.5 and 1.6 and established analogous conclusions as follows:

Theorem 1.8 ([6]). *Let $\mathscr{S} = \{0, a, b\}$ where a, b are two non-zero distinct complex numbers. If ξ is a meromorphic function with at most nitely many poles and $E_\xi(\mathscr{S}) = E_{\xi'}(\mathscr{S})$ then $\xi \equiv \xi'$.*

Theorem 1.9 ([6]). *Let ξ be a nonconstant transcendental meromorphic function with at most finitely many simple poles and $\mathscr{S} = \{0, a, b\}$, where a, b are distinct complex numbers. If $E_\xi(\mathscr{S}) = E_{\xi'}(\mathscr{S})$, then either*

(1) $\xi(z) = Ce^z$ *or*

(2) $\xi(z) = Ce^{-z} + \dfrac{2}{3}(a + b + c)$, *and* $(2a^2 - 5ab + 2b^2)(a + b) = 0$ *or*

(3) $\xi(z) = Ce^{\frac{-1 \pm \iota\sqrt{3}}{2}z} + e^{\frac{3 \pm \iota\sqrt{3}}{6}z}(a + b)$ *and* $a^2 - ab + b^2 = 0$, *where C is a non zero complex constant.*

In 2011, Feng Lü [14] showed that Theorem 1.6 remained valid when the entire function was replaced with a meromorphic function.

Theorem 1.10 ([14]). *Let ξ be a transcendental meromorphic function with atmost finitely many poles and $\mathscr{S} = \{a, b, c\}$, where a, b and c are distinct complex numbers. If ξ and its derivative ξ' satisifies $E_\xi(\mathscr{S}) = E_{\xi'}(\mathscr{S})$, then the conclusion of Theorem 1.6 holds.*

Now, for the sake of discussion, we look at a few more definitions:

Definition 1.11 ([10]). *Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.*

(*i*) *$N(r, a; \xi| \geq p)(\overline{N}(r, a; \xi| \geq p))$ denotes the counting function (reduced counting function) of those a-points of ξ whose multiplicities are not less than p.*

(*ii*) *$N(r, a; \xi| \leq p)(\overline{N}(r, a; \xi| \leq p)$ denotes the counting function (reduced counting function) of those a-points of ξ whose multiplicities are not greater than p.*

Definition 1.12 ([22]). *For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by*

$N_p(r,a;\xi) = \overline{N}(r,a;\xi) + \overline{N}(r,a;\xi| \geq 2) + \cdots + \overline{N}(r,a;\xi| \geq p)$. *It is clear that* $N_1(r,a;\xi) = \overline{N}(r,a;\xi)$.

Throughout this study, we utilized the following uniqueness polynomial introduced by Lin-Yi [12]:

$$\mathscr{P}(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2, \tag{1.1}$$

where $n \geq 3$ is an integer and $a, b \in \mathbb{C}^*$ satisfying $ab^{n-2} \neq 2$. It is easy to verify that the polynomial $\mathscr{P}(w)$ has only simple zeros.

Let us now review the idea of weighted sharing, which is a scaling between CM and IM sharing of sets or values that was initially introduced in the literature in 2001 [9].

Definition 1.13 ([9]). *Let k be a nonnegative integer or innity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;\xi)$ the set of all a-points of ξ, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;\xi) = E_k(a;g)$ we say that ξ, ζ share the value a with weight k.*

We write ξ, ζ share (a,k) to mean that ξ, ζ share the value a with weight k. Clearly if ξ, ζ share (a,k), then ξ, ζ share (a,p) for any integer p, $0 \leq p < k$. Also we note that ξ, ζ share a value a IM or CM if and only if ξ, ζ share $(a,0)$ or (a,∞), respectively.

Definition 1.14 ([9]). *Let $\mathscr{S}$ be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_\xi(\mathscr{S},k)$ the set of $\bigcup_{a \in \mathscr{S}} E_k(a;\xi)$. If $E_\xi(\mathscr{S},k) = E_\zeta(\mathscr{S},k)$, then we say ξ, ζ share the set $\mathscr{S}$ with weight k.*

From the preceding discussion, we recognize that research has increasingly focused on determining the uniqueness of an entire or meromorphic function with its first derivative sharing a set at the expense of allowing several constraints. Because the logical extension of a derivative is a differential monomial, it is natural to expect the results to be extended and improved up to a relationship between the power of a meromorphic function and a general differential monomial sharing a set of small functions.

Very recently, in this direction, with the help of weighted sharing of sets in 2016, A. Banerjee and B. Chakraborty [3] proved with the motivation that there exists any set that, when shared by a meromorphic function together with its higher-order derivative or even a power of a meromorphic function together with its differential polynomial, leads to its uniqueness.

Theorem 1.15 ([3]). *Let $n(\geq 1), m(\geq 1)$ be positive integers and ξ be a non-constant meromorphic function. Suppose that $\mathcal{S} = \{w : \mathscr{P}(w) = 0\}$ and $E_{\xi^m}(\mathcal{S}, l) = E_{\mathscr{L}(\xi)}(\mathcal{S}, l)$. If one of the following conditions hold:*

(1) $2 \leq l < \infty$ *and* $n > 6 + 6\dfrac{\mu + 1}{\delta - 2\mu}$,

(2) $l = 1$ *and* $n > \dfrac{13}{2} + 7\dfrac{\mu + 1}{\delta - 2\mu}$,

(3) $l = 0$ *and* $n > 6 + 3\mu + 6\dfrac{(\mu + 1)^2}{\delta - 2\mu}$;

then $\xi^m \equiv \mathscr{L}[\xi]$, *where* $\delta = \min\{m(n-2) - 1, (1+k)l(n-2) - 1\}$ *and* $\mu = \min\left\{\dfrac{1}{p}\right\}$.

To make our investigation easier, we now provide an important definition that will be used throughout this paper.

Definition 1.16 ([17]). *Let n_{ij}, m_{ij} with $(i = 0, 1, \ldots, k)$ and $(j = 1, 2, \ldots, t)$ be non-negative integers and $\xi(z)$ be a non-constant meromorphic function. We shall define a*

(1) *general difference-differential monomial as follows*

$$M_j[\xi] = \prod_{i=0}^{k} [\xi^{(i)}(z)]^{n_{ij}} [\xi^{(i)}(z + c_i)]^{m_{ij}},$$

where $c_i's (i = 0, 1, \ldots, k)$ are complex constants. Let $d_{M_j} = \sum_{i=0}^{k} n_{ij} + m_{ij}$ denote the degree of $M_j[\xi]$ and $W_{M_j} = \sum_{i=0}^{k}(i+1)(n_{ij} + m_{ij})$ denote the weight of $M_j[\xi]$.

(2) *Then the expression*

$$\mathscr{P}[\xi] = \sum_{j=1}^{t} a_j M_j[\xi], \tag{1.2}$$

where $T(r, a_j) = \mathcal{S}(r, \xi)$ for $j = 1, 2, \ldots, t$ is called the difference-differential polynomial generated by ξ of upper degree $U_d(\mathscr{P}) = \max_{1 \leq j \leq t}\{d_{M_j}\}$, lower degree

$L_d(\mathscr{P}) = \min_{1\le j\le t}\{d_{M_j}\}$, *weight* $W_{\mathscr{P}} = \max_{1\le j\le t}\{W_{M_j}\}$ *and the order* k *(where* k *is the highest order of the derivative of* ξ *in* $\mathscr{P}[\xi]$*). Let* ϑ *denote* $\max_{1\le j\le t}\{W_{M_j} - d_{M_j}\}$*, i.e.,*

$$\begin{aligned}\vartheta &= \max_{1\le j\le t}\sum_{i=0}^{k}[(i+1)-1](n_{ij}+m_{ij})\\ &= \max_{1\le j\le t}(n_{1j}+m_{1j}+2n_{2j}+2m_{2j}+\cdots+kn_{kj}+km_{kj}).\end{aligned}$$

(3) *In particular, if* $U_d(\mathscr{P}) = L_d(\mathscr{P}) = d$*, then* $\mathscr{P}[\xi]$ *is called a homogeneous difference-differential polynomial.*

We now define

$$\pounds[z] = \sum_{j=1}^{t} b_j z^{d_{M_j}} = b_1 z^{d_{M_1}} + b_2 z^{d_{M_2}} + \cdots + b_t z^{d_{M_t}},$$

where b_j $(j = 1, 2, \cdots, t)$ are all constants.

The above justification and definition motivate us to investigate whether we get the uniqueness relation between differential polynomials and difference-differential polynomials of meromorphic functions when they share a set $\mathscr{S}$ or not. In light of this, we obtain the following main result:

Theorem 1.17. *Let* $n, k \in \mathbb{N}$ *and* ξ *be a non-constant meromorphic function. Suppose that* $\mathscr{S} = \{w : \mathscr{P}(w) = 0\}$ *and* $E_{\pounds[\xi]}(\mathscr{S}, l) = E_{\mathscr{P}[\xi]}(\mathscr{S}, l)$*. If one of the following conditions hold:*

(1) $2 \le l < \infty$ *and* $n > 5 + \dfrac{(3k+9)(2l^2 + l\gamma_m - 2)}{(\delta l - (k+3))(\gamma_m l - 2) - l(l+k+3)}$,

(2) $l = 1$ *and* $n > \dfrac{11}{2} + \dfrac{\left(\frac{7k+21}{2}\right)(2l^2 + l\gamma_m - 2)}{(l\delta - (k+3))(l\gamma_m - 2) - l(k+l+3)}$,

(3) $l = 0$ *and* $n > 11 + \dfrac{(5k+15)\gamma_m}{(\delta - (k+3))(\gamma_m - 2) - (k+4)}$,

then $\pounds[\xi] \equiv \mathscr{P}[\xi]$*, where* $\delta = (n-2)\overline{d}(P) - 1$ *and* $\gamma_m = \min_{1\le j\le t}\{2d_{M_j} - W_{M_j}\}n - 1$.

Remark 1.18. We may observe that Theorem 1.17 generalizes and extends Theorem 1.15 by enhancing it from homogeneous differential polynomials to non-homogeneous difference-differential polynomials.

2. Preliminary lemmas. The following lemmas are helpful in establishing our main results.

We define

$$\mathscr{R}(w)=\frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)},$$

where $\alpha_i, (i=1,2)$ are the distinct roots of the equation

$$n(n-1)w^2-2n(n-2)bw+(n-1)(n-2)b^2=0.$$

Then

$$\mathscr{R}(w)-1=\frac{\mathscr{P}(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}.$$

Let

$$\mathscr{F}=\mathscr{R}(\mathscr{L}[\xi]) \quad \text{and} \quad \mathscr{G}=\mathscr{R}(\mathscr{P}[\xi]),$$

where ξ is a non-constant meromorphic function associated with $\mathscr{F}$ and $\mathscr{G}$, we define

$$\mathscr{H}=\left(\frac{\mathscr{F}''}{\mathscr{F}'}-\frac{2\mathscr{F}'}{\mathscr{F}-1}\right)-\left(\frac{\mathscr{G}''}{\mathscr{G}'}-\frac{2\mathscr{G}'}{\mathscr{G}-1}\right).$$

Lemma 2.1 ([21]). *Let ξ be a non-constant meromorphic function and $\mathscr{Q}(\xi)=a_m\xi^q+a_{m-1}\xi^{m-1}+\cdots+a_0$, where $a_0,a_1,\ldots,a_m$ are constants with $a_m\neq 0$. Then*

$$T(r,\mathscr{Q}(\xi))=mT(r,\xi)+S(r,\xi).$$

Lemma 2.2 ([20]). *Let h be a non-constant meromorphic function and let a_j be distinct finite complex numbers such that $a_j\neq 0$ for $j=1,2,\cdots,q$. Then*

$$\sum_{j=1}^{q}\left\{N\left(r,\frac{1}{h-a_j}\right)-\overline{N}\left(r,\frac{1}{h-a_j}\right)\right\}\leq \overline{N}(r,0;h)+\overline{N}(r,\infty;h)+S(r,h).$$

Lemma 2.3 ([22]). *Let ξ be a non-constant meromorphic function, then*

$$N\left(r,0;\xi^{(k)}\right)\leq N\left(r,0;\xi\right)+k\overline{N}\left(r,\infty;\xi\right)+S(r,\xi).$$

Lemma 2.4. *Let $\mathscr{F}$ and $\mathscr{G}$ share $(1,l)$, where $\mathscr{F}$ and $\mathscr{G}$ defined as earlier. Then*

(1) *for $l=0$,* $\overline{N}_L(r,1;\mathscr{F})\leq \overline{N}(r,0;\mathscr{L}[\xi])+\overline{N}(r,\infty;\xi)+S(r,\xi)$.

(2) *for $l=0$,* $\overline{N}_L(r,1;\mathscr{G})\leq \overline{N}(r,0;\mathscr{P}[\xi])+(k+2)\overline{N}(r,\infty;\xi)+S(r,\xi)$.

(3) *for* $l \geq 1$, $\overline{N}_L(r,1;\mathscr{F}) \leq \frac{1}{l}\{\overline{N}(r,0;\pounds[\xi]) + \overline{N}(r,\infty;\xi)\} + S(r,\xi)$.

(4) *for* $l \geq 1$, $\overline{N}_L(r,1;\mathscr{G}) \leq \frac{1}{l}\{\overline{N}(r,0;\mathscr{P}[\xi]) + (k+2)\overline{N}(r,\infty;\xi)\} + S(r,\xi)$.

Proof. First we note that in view of Lemma 2.1, we get $S(r,\pounds[\xi]) = S(r,\xi)$.

By using Lemma 2.2, we obtain when $l = 0$,

$$\begin{aligned}\overline{N}_L(r,1;\mathscr{F}) &\leq N(r,1;\mathscr{F}) - \overline{N}(r,1;\mathscr{F})\\ &\leq \overline{N}(r,0;\pounds[\xi]) + \overline{N}(r,\infty;\pounds[\xi]) + S(r,\xi)\\ &\leq \overline{N}(r,0;\pounds[\xi]) + \overline{N}(r,\infty;\xi) + S(r,\xi).\end{aligned}$$

When $l \geq 1$, we get by using Lemma 2.2

$$\begin{aligned}\overline{N}_L(r,1;\mathscr{F}) &\leq \overline{N}(r,1;\mathscr{F}| \geq l+1)\\ &\leq \frac{1}{l}\{N(r,1;\mathscr{F}) - \overline{N}(r,1;\mathscr{F})\}\\ &\leq \frac{1}{l}\{\overline{N}(r,0;\pounds[\xi]) + \overline{N}(r,\infty;\pounds[\xi])\} + S(r,\xi)\\ &\leq \frac{1}{l}\{\overline{N}(r,0;\pounds[\xi]) + \overline{N}(r,\infty;\xi)\} + S(r,\xi).\end{aligned}$$

Combining the two cases we get the proof.

In the same way, we can prove for $\mathscr{G}$.

i.e. when $l = 0$,

$$\overline{N}_L(r,1;\mathscr{G}) \leq \overline{N}(r,0;\mathscr{P}[\xi]) + (k+2)\overline{N}(r,\infty;\xi) + S(r,\xi),$$

when $l \geq 1$,

$$\overline{N}_L(r,1;\mathscr{F}) \leq \frac{1}{l}\{\overline{N}(r,0;\mathscr{P}[\xi]) + (k+2)\overline{N}(r,\infty;\xi\} + S(r,\xi). \qquad \square$$

Lemma 2.5. *Suppose that* $\mathscr{F}$ *and* $\mathscr{G}$ *share* $(1,l)$, *where* $0 \leq l < \infty$. *If* $\mathscr{F} \not\equiv \mathscr{G}$, *then*

(1) *for* $l \geq 1$,

$$\overline{N}(r,0;\xi) \leq \left(\frac{k+l+3}{l\gamma_m - 2}\right)\overline{N}(r,\infty;\xi) + \left(\frac{2l}{l\gamma_m - 2}\right)\{T(r,\pounds[\xi]) + T(r,\mathscr{P}[\xi])\},$$

(2) *for* $l = 0$,

$$\overline{N}(r,0;\xi) \leq \left(\frac{k+4}{\gamma_m - 2}\right)\overline{N}(r,\infty;\xi) + \left(\frac{2}{\gamma_m - 2}\right)\{T(r,\pounds[\xi]) + T(r,\mathscr{P}[\xi])\},$$

where $\gamma_m = \min_{1\le j\le t}\{2d_{M_j} - W_{M_j}\}n - 1.$

P r o o f. By using similar techniques as Lemma 3.3 of [4], we may easily obtain proof of Lemma 2.5. □

Lemma 2.6. *Suppose that* $\mathscr{F}$ *and* $\mathscr{G}$ *share* $(1, l)$, *where* $\mathscr{F}$ *and* $\mathscr{G}$ *are defined as earlier. If* $\mathscr{F} \not\equiv \mathscr{G}$, *then*

(1) *for* $l = 0$,

$$\overline{N}(r,\infty;\xi) \le \frac{\gamma_m}{(\delta-(k+3))(\gamma_m-2)-(k+4)}\{T(r,\pounds[\xi]) + T(r,\mathscr{P}[\xi])\} + S(r,\mathscr{P}[\xi]) + S(r,\xi),$$

(2) *for* $l \ge 1$,

$$\overline{N}(r,\infty;\xi) \le \frac{2l^2+l\gamma_m-2}{(l\delta-(k+3))(l\gamma_m-2)-l(k+l+3)}\{T(r,\pounds[\xi]) + T(r,\mathscr{P}[\xi])\} + S(r,\mathscr{P}[\xi]) + S(r,\xi),$$

where $\delta = (n-2)\overline{d}(\mathscr{P}) - 1$ *and* $\gamma_m = \min_{1\le j\le t}\{2d_{M_j} - W_{M_j}\}n - 1.$

P r o o f. Let us define $\Psi = \dfrac{\mathscr{F}'}{\mathscr{F}(\mathscr{F}-1)} - \dfrac{\mathscr{G}'}{\mathscr{G}(\mathscr{G}-1)}$. Now let us look at two cases of the problem:

Case I. Suppose $\Psi \equiv 0$.

By integration, we get $\left(1-\dfrac{1}{\mathscr{F}}\right) = A\left(1-\dfrac{1}{\mathscr{G}}\right)$.

As $\pounds[\xi]$ and $\mathscr{P}[\xi]$ share $(\infty, 0)$, so if $\overline{N}(r,\infty;\xi) \ne S(r,\xi)$ then $A = 1$, that is, $\mathscr{F} \equiv \mathscr{G}$, which is not possible. So $\overline{N}(r,\infty;\xi) = S(r,\xi)$. Thus the lemma holds.

Case II. Let $\Psi \not\equiv 0$.

Let us suppose that z_0 be a pole of ξ of order r, then it is a pole of $\pounds[\xi]$ of order $r\overline{d}(\mathscr{P})$ and of $\mathscr{P}[\xi]$ of order $rU_d(\mathscr{P}) + W_{\mathscr{P}}$ and that of $\mathscr{F}$ and $\mathscr{G}$ are $r\overline{d}(\mathscr{P})(n-2)$ and $(rU_d(\mathscr{P}) + W_{\mathscr{P}})(n-2)$, respectively.

Clearly, z_0 is a zero of $\dfrac{\mathscr{F}'}{\mathscr{F}-1} - \dfrac{\mathscr{F}'}{\mathscr{F}}$ and $\dfrac{\mathscr{G}'}{\mathscr{G}-1} - \dfrac{\mathscr{G}'}{\mathscr{G}}$ of order at least $\overline{d}(\mathscr{P})(n-2) - 1$ and $(U_d(\mathscr{P}) + W_{\mathscr{P}})(n-2) - 1$, respectively and hence a zero of Ψ of order at least,

$$\min\{(n-2)\overline{d}(\mathscr{P}) - 1, (U_d(\mathscr{P}) + W_{\mathscr{P}})(n-2) - 1\} = (n-2)\overline{d}(\mathscr{P}) - 1 = \delta(say).$$

Thus using Lemma 2.4, we get for $l \ge 1$,

$$\begin{aligned}\overline{N}(r,\infty;\xi) &\leq \frac{1}{\delta}N(r,o;\Psi)\\ &\leq \frac{1}{\delta}N(r,\infty;\Psi)+S(r,\mathscr{P}[\xi])+S(r,\xi)\\ &\leq \frac{1}{\delta}\{\overline{N}_L(r,1;\mathscr{F})+\overline{N}_L(r,1;\mathscr{G})+\overline{N}(r,0;\mathscr{P}[\xi])\}+S(r,\mathscr{P}[\xi])+S(r,\xi)\\ &\leq \frac{1}{\delta}\left[\frac{1}{l}\{\overline{N}(r,0;\pounds[\xi])+(k+3)\overline{N}(r,\infty;\xi)+\overline{N}(r,0;\mathscr{P}[\xi])\}+\overline{N}(r,0;\mathscr{P}[\xi])\right]\\ &\quad + S(r,\mathscr{P}[\xi])+S(r,\xi).\end{aligned}$$

Thus using Lemma 2.5, we get

$$\begin{aligned}\left(\delta-\frac{k+3}{l}\right)\overline{N}(r,\infty;\xi) \leq& \overline{N}(r,0;\mathscr{P}[\xi])+\frac{1}{l}\{T(r,\pounds[\xi])+T(r,\mathscr{P}[\xi])\}\\ &+S(r,\mathscr{P}[\xi])+S(r,\xi),\end{aligned}$$

i.e.,

$$\begin{aligned}\overline{N}(r,\infty;\xi) \leq& \frac{2l^2+l\gamma_m-2}{(l\delta-(k+3))(l\gamma_m-2)-l(l+k+3)}\{T(r,\pounds[\xi])+T(r,\mathscr{P}[\xi])\}\\ &+S(r,\mathscr{P}[\xi])+S(r,\xi).\end{aligned}$$

Next for $l=0$, using Lemma 2.4, 2.5 and proceeding exactly as above, we get

$$\begin{aligned}\overline{N}(r,\infty;\xi)) \leq& \frac{\gamma_m}{(\delta-(k+3))(\gamma_m-2)-(k+4)}\{T(r,\pounds[\xi])+T(r,\mathscr{P}[\xi])\}\\ &+S(r,\mathscr{P}[\xi])+S(r,\xi).\end{aligned}$$

This completes the proof. □

Lemma 2.7. *If* $\mathscr{H}\not\equiv 0$ *and* $\mathscr{F}$ *and* $\mathscr{G}$ *share* $(1,l)$, *then*

$$\begin{aligned}N(r,\infty;\mathscr{H}) \leq& \overline{N}(r,\infty;\xi)+\overline{N}(r,0;\pounds[\xi])+\overline{N}(r,0;\mathscr{P}[\xi])+\overline{N}(r,b;\pounds[\xi])\\ &+\overline{N}(r,b;\mathscr{P}[\xi])+\overline{N}_L(r,1;\mathscr{F})+\overline{N}_L(r,1;\mathscr{G})+\overline{N}_0(r,0;(\pounds[\xi])')\\ &+\overline{N}_0(r,0;(\mathscr{P}[\xi])')+S(r,\xi),\end{aligned}$$

where $\overline{N}_0(r,0;(\pounds[\xi])')$ *denotes the counting function of the zeros of* $(\pounds[\xi])'$ *which are not the zeros of* $\pounds[\xi](\pounds[\xi]-b)$ *and* $\mathscr{F}-1$. *Similar expressions holds for* $\mathscr{P}[\xi]$.

Proof. If we use the following fact, then the proof will be easy. A zero of ξ may not be a zero of $\mathscr{P}[\xi]$ but an elementary calculations shows that each zeros of ξ must be a zero of $\mathscr{P}[\xi]$, so we have $\overline{N}(r,0;\mathscr{F})\leq\overline{N}(r,0;\mathscr{G})$. Also, we note that $\overline{N}(r,\infty;\mathscr{F})\leq\overline{N}(r,\infty;\xi)+\overline{N}(r,\alpha_1;\pounds[\xi])+\overline{N}(r,\alpha_2;\pounds[\xi])$. But note that the simple zeros of $\mathscr{L}[\xi]-\alpha_i$ are not the poles of $\mathscr{H}$ and multiple zeros of $\mathscr{L}[\xi]-\alpha_i$ are zeros of $(\mathscr{L}[\xi])'$. Similar explanations hold for $\mathscr{G}$ also. □

Lemma 2.8 ([1]). *Let*

$$\phi(w)=(n-1)^2(\omega^{n-2}-1)(\omega^n-1)-n(n-2)(\omega^{n-1}-1)^2.$$

Then

$$\phi(w)=(\omega-1)^4\prod_{i=1}^{2n-6}(\omega-\beta_i),$$

where $\beta_i\in\mathbb{C}^*-\{1\},(i=1,2,\ldots,2n-6)$, *which are distinct.*

3. Proof of Theorem. We allocated the entire proof to two different cases, as follows:

Case I. In this case, we assume that $\mathscr{H}\not\equiv 0$. So, one can see that $\mathscr{F}\not\equiv\mathscr{G}$.

We note that $\overline{N}(r,1;\mathscr{F}|=1)=\overline{N}(r,1;\mathscr{G}|=1)\le N(r,\infty;\mathscr{H})$. By using the Second fundamental theorem and Lemma 2.4, we get

$$\begin{aligned}(n+1)T(r,\mathscr{L}[\xi])&\le\overline{N}(r,\infty;\xi)+\overline{N}(r,0;\mathscr{L}[\xi])+\overline{N}(r,b;\mathscr{L}[\xi])+\overline{N}(r,1;\mathscr{F})\\&\quad-N_0(r,0;(\mathscr{L}[\xi])')+S(r,\xi),\\&\le\left\{\overline{N}(r,1;\mathscr{F}|\ge2)+\overline{N}_L(r,1;\mathscr{F})+\overline{N}_L(r,1;\mathscr{G})+\overline{N}_0(r,0;(\mathscr{P}[\xi])')\right\}\\&\quad+2\{\overline{N}(r,\infty;\xi)+\overline{N}(r,b;\mathscr{L}[\xi])\}+\overline{N}(r,0;\mathscr{L}[\xi])+\overline{N}(r,0;\mathscr{P}[\xi])\\&\quad+\overline{N}(r,b;\mathscr{P}[\xi])+S(r,\xi).\end{aligned}\tag{3.1}$$

Subcase 1.1. when $l\ge 2$. Then we have the following

$$\begin{aligned}&\overline{N}(r,1;\mathscr{F}|\ge2)+\overline{N}_L(r,1;\mathscr{F})+\overline{N}_L(r,1;\mathscr{G})+\overline{N}_0(r,0;(\mathscr{P}[\xi])')\\&\le\overline{N}(r,1;\mathscr{G}|\ge2)+\overline{N}(r,1;\mathscr{G}|\ge3)+\overline{N}_0(r,0;(\mathscr{P}[\xi])')\\&\le N\left(r,0;(\mathscr{P}[\xi])'\mid\mathscr{P}[\xi]\ne0\right)+S(r,\mathscr{P}[\xi])\\&\le N\left(r,\infty;\frac{(\mathscr{P}[\xi])'}{\mathscr{P}[\xi]}\right)+S(r,\mathscr{P}[\xi])\\&\le\overline{N}(r,0;\mathscr{P}[\xi])+(k+2)\overline{N}(r,\infty;\xi)+S(r,\mathscr{P}[\xi])+S(r,\xi).\end{aligned}\tag{3.2}$$

With the help of this, note that (3.1) becomes

$$\begin{aligned}(n+1)T(r,\mathscr{L}[\xi])&\le2\left\{\overline{N}(r,\infty;\xi)+\overline{N}(r,b;\mathscr{L}[\xi])\right\}+2\overline{N}(r,0;\mathscr{P}[\xi])+\overline{N}(r,0;\mathscr{L}[\xi])\\&\quad+\overline{N}(r,b;\mathscr{P}[\xi])+(k+2)\overline{N}(r,\infty;\xi)+S(r,\mathscr{P}[\xi])+S(r,\xi).\end{aligned}\tag{3.3}$$

Similarly, for $\mathscr{P}[\xi]$, we get

$$(n+1)T(r,\mathscr{P}[\xi])\le2\left\{(k+2)\overline{N}(r,\infty;\xi)+\overline{N}(r,b;\mathscr{P}[\xi])\right\}+2\overline{N}(r,0;\mathscr{L}[\xi])$$

$$+\overline{N}(r,0;\mathscr{P}[\xi])+\overline{N}(r,b;\mathscr{P}[\xi])+\overline{N}(r,\infty;\xi)+S(r,\mathscr{P}[\xi])+S(r,\xi). \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$(n-5)(T(r,\pounds[\xi])+T(r,\mathscr{P}[\xi]))\leq(3k+9)\overline{N}(r,\infty;\xi)+S(r,\mathscr{P}[\xi])+S(r,\xi). \tag{3.5}$$

Using Lemma 2.6 in (3.5), we get

$$(n-5)\{T(r,\pounds[\xi])+T(r,\mathscr{P}[\xi])\}\leq\left\{\frac{(3k+9)(2l^2+l\gamma_m-2)}{(\delta l-(k+3))(\gamma_m l-2)-l(l+k+3)}\right\}$$
$$(T(r,\pounds[\xi])+T(r,\mathscr{P}[\xi]))+S(r,\mathscr{P}[\xi])+S(r,\xi),$$

which contradicts to

$$n>5+\frac{(3k+9)(2l^2+l\gamma_m-2)}{(\delta l-(k+3))(\gamma_m l-2)-l(l+k+3)}.$$

Subcase 1.2. When $l=1$, we see that

$$\begin{aligned}&\overline{N}(r,1;\mathscr{F}|\geq 2)+\overline{N}_L(r,1;\mathscr{F})+\overline{N}_L(r,1;\mathscr{G})+\overline{N}_0(r,0;(\mathscr{P}[\xi])')\\&\leq\overline{N}(r,1;\mathscr{F}|\geq 2)+\overline{N}(r,1;\mathscr{G}|\geq 2)+\overline{N}_0(r,0;(\mathscr{P}[\xi])')\\&\leq N\left(r,0;(\mathscr{P}[\xi])'\mid\mathscr{P}[\xi]\neq 0\right)+\frac{1}{2}N\left(r,0;(\pounds[\xi])'\mid\pounds[\xi]\neq 0\right)\\&\quad+S(r,\mathscr{P}[\xi])+S(r,\xi)\\&\leq\overline{N}(r,0;\mathscr{P}[\xi])+(k+2)\overline{N}(r,\infty;\xi)+\frac{1}{2}\left\{\overline{N}(r,0;\pounds[\xi])+\overline{N}(r,\infty;\xi)\right\}\\&\quad+S(r,\mathscr{P}[\xi])+S(r,\xi).\end{aligned} \tag{3.6}$$

Thus from (3.1), we get

$$\begin{aligned}(n+1)T(r,\pounds[\xi])\leq&\frac{5}{2}\left\{\overline{N}(r,\infty;\xi)+\overline{N}(r,0;\pounds[\xi])\right\}+2\overline{N}(r,0;\mathscr{P}[\xi])\\&+\{\overline{N}(r,0;\pounds[\xi])+\overline{N}(r,b;\mathscr{P}[\xi])+(k+2)\overline{N}(r,\infty;\xi)\}\\&+S(r,\mathscr{P}[\xi])+S(r,\xi).\end{aligned} \tag{3.7}$$

Similarly, for $\mathscr{P}[\xi]$, we get

$$\begin{aligned}(n+1)T(r,\mathscr{P}[\xi])\leq&\frac{5}{2}\left\{(k+2)\overline{N}(r,\infty;\xi)+\overline{N}(r,0;\mathscr{P}[\xi])\right\}+2\overline{N}(r,0;\pounds[\xi])\\&+\{\overline{N}(r,0;\mathscr{P}[\xi])+\overline{N}(r,b;\pounds[\xi])+\overline{N}(r,\infty;\xi)\}\\&+S(r,\mathscr{P}[\xi])+S(r,\xi).\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we get

$$\left(n-\frac{11}{2}\right)\{T(r,\pounds[\xi])+T(r,\mathscr{P}[\xi])\} \tag{3.9}$$

$$\leq \left(\frac{7k+21}{2}\right)\overline{N}(r,\infty;\xi)+S(r,\mathscr{P}[\xi])+S(r,\xi).$$

That is by, using Lemma 2.6 in (3.9), we obtain

$$\left(n-\frac{11}{2}\right)\{T(r,\pounds[\xi])+T(r,\mathscr{P}[\xi])\} \leq \frac{\left(\frac{7k+21}{2}\right)(2l^2+l\gamma_m-2)}{(l\delta-(k+3))(l\gamma_m-2)-l(k+l+3)} \{T(r,\pounds[\xi])+T(r,\mathscr{P}[\xi])\}+S(r,\mathscr{P}[\xi])+S(r,\xi),$$

which contradicts to $n>\frac{11}{2}+\frac{\left(\frac{7k+21}{2}\right)(2l^2+l\gamma_m-2)}{(l\delta-(k+3))(l\gamma_m-2)-l(k+l+3)}$.

Subcase 1.3. When $l=0$. Now, using the second fundamental theorem and Lemma 2.4, we get

$$\begin{aligned}
&(n+1)\{T(r,\pounds[\xi])+T(r,\mathscr{P}[\xi])\}\\
&\leq \overline{N}(r,\infty;\pounds[\xi])+\overline{N}(r,\infty;\mathscr{P}[\xi])+\overline{N}(r,0;\pounds[\xi])+\overline{N}(r,0;\mathscr{P}[\xi])\\
&\quad+\overline{N}(r,b;\pounds[\xi])+\overline{N}(r,b;\mathscr{P}[\xi])+\overline{N}(r,1;\mathscr{F})+\overline{N}(r,1;\mathscr{G})\\
&\quad-N_0(r,0;(\pounds[\xi])')-N_0(r,0;(\mathscr{P}[\xi])')+S(r,\mathscr{P}[\xi])+S(r,\xi),\\
&\leq (k+3)\overline{N}(r,\infty;\xi)+2\overline{N}(r,0;\pounds[\xi])+2\overline{N}(r,0;\mathscr{P}[\xi])+2\overline{N}(r,b;\pounds[\xi])\\
&\quad+2\overline{N}(r,b;\mathscr{P}[\xi])+\overline{N}(r,1;\mathscr{F})+\overline{N}(r,1;\mathscr{G})\\
&\quad+\overline{N}_L(r,1;\mathscr{F})+\overline{N}_L(r,1;\mathscr{G})-\overline{N}(r,1;\mathscr{F}|=1)+S(r,\mathscr{P}[\xi])+S(r,\xi).
\end{aligned}\tag{3.10}$$

Again,

$$\overline{N}(r,1;\mathscr{F})+\overline{N}(r,1;\mathscr{G})-\overline{N}(r,1;\mathscr{F}|=1)\leq \overline{N}_L(r,1;\mathscr{F})+\overline{N}(r,1;\mathscr{G}),$$

i.e.,

$$\begin{aligned}
&\overline{N}(r,1;\mathscr{F})+\overline{N}(r,1;\mathscr{G})-\overline{N}(r,1;\mathscr{F}|=1)\\
&\leq \frac{1}{2}\{\overline{N}_L(r,1;\mathscr{F})+\overline{N}_L(r,1;\mathscr{G})+N(r,1;\mathscr{F})+N(r,1;\mathscr{G})\}.
\end{aligned}$$

So in view of Lemma 2.4 and 2.6, from (3.10), we have

$$\begin{aligned}
&(n+1)\{T(r,\pounds[\xi])+T(r,\mathscr{P}[\xi])\}\\
&\leq (k+3)\overline{N}(r,\infty;\xi)+2\overline{N}(r,0;\pounds[\xi])+2\overline{N}(r,0;\mathscr{P}[\xi])+2\overline{N}(r,b;\pounds[\xi])\\
&\quad+2\overline{N}(r,b;\mathscr{P}[\xi])+\frac{3}{2}\left\{\overline{N}_L(r,1;\mathscr{F})+\overline{N}_L(r,1;\mathscr{G})\right\}+\frac{1}{2}\{N(r,1;\mathscr{F})+N(r,1;\mathscr{G})\}\\
&\quad+S(r,\mathscr{P}[\xi])+S(r,\xi),
\end{aligned}$$

i.e.,

$$(n+1)\{T(r,\pounds[\xi])+T(r,\mathscr{P}[\xi])\}$$

$$
\begin{aligned}
&\leq (k+3)\overline{N}(r,\infty;\xi) + 2\overline{N}(r,0;\mathcal{L}[\xi]) + 2\overline{N}(r,0;\mathscr{P}[\xi]) \\
&\quad + 2\overline{N}(r,b;\mathcal{L}[\xi]) + 2\overline{N}(r,b;\mathscr{P}[\xi]) \\
&\quad + \frac{3}{2}\left\{\overline{N}(r,0;\mathcal{L}[\xi]) + \overline{N}(r,\infty;\xi) + \overline{N}(r,0;\mathscr{P}[\xi]) + (k+2)\overline{N}(r,\infty;\xi)\right\} \\
&\quad + \frac{1}{2}\{N(r,1;\mathscr{F}) + N(r,1;\mathscr{G})\} + S(r,\mathscr{P}[\xi]) + S(r,\xi) \\
&\leq 2(k+3)\overline{N}(r,\infty;\xi) + 4\{T(r,\mathcal{L}[\xi]) + T(r,\mathscr{P}[\xi])\} \\
&\quad + 3\{\overline{N}(r,0;\mathcal{L}[\xi]) + (k+2)\overline{N}(r,\infty;\xi) + \overline{N}(r,0;\mathscr{P}[\xi])\} \\
&\quad + \frac{3}{2}\left\{\overline{N}(r,0;\mathcal{L}[\xi]) + \overline{N}(r,\infty;\xi) + \overline{N}(r,0;\mathscr{P}[\xi]) + (k+1)\overline{N}(r,\infty;\xi)\right\} \\
&\quad + \frac{1}{2}\{N(r,1;\mathscr{F}) + N(r,1;\mathscr{G})\} + S(r,\mathscr{P}[\xi]) + S(r,\xi). \qquad (3.11)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(n-8)\{T(r,\mathcal{L}[\xi]) + T(r,\mathscr{P}[\xi])\} \\
&\leq (2k+6)\overline{N}(r,\infty;\xi) + 3\big\{\overline{N}(r,0;\mathcal{L}[\xi]) + (k+3)\overline{N}(r,\infty;\xi) \\
&\quad + \overline{N}(r,0;\mathscr{P}[\xi])\big\} + S(r,\mathscr{P}[\xi]) + S(r,\xi),
\end{aligned}
$$

Thus

$$
\begin{aligned}
&(n-11)\{T(r,\mathcal{L}[\xi]) + T(r,\mathscr{P}[\xi])\} \\
&\qquad \leq \frac{(5k+15)\gamma_m}{(\delta-(k+3))(\gamma_m-2)-(k+4)}\{T(r,\mathcal{L}[\xi]) + T(r,\mathscr{P}[\xi])\} \\
&\qquad\quad + S(r,\mathscr{P}[\xi]) + S(r,\xi), \qquad (3.12)
\end{aligned}
$$

which contradicts to

$$n > 11 + \frac{(5k+15)\gamma_m}{(\delta-(k+3))(\gamma_m-2)-(k+4)}.$$

Case II. Let $\mathscr{H} \equiv 0$. In this case $\mathscr{F}$ and $\mathscr{G}$ share $(1,\infty)$.

Now by integrating twice, we have

$$\mathscr{F} = \frac{u\mathscr{G}+v}{w\mathscr{G}+y} \ \text{ or } \ \mathscr{G} = \frac{v-y\mathscr{F}}{w\mathscr{F}-u}, \qquad (3.13)$$

where u, v, w, y are constants satisifying $uy - vw \neq 0$.

Thus by applying Mokhon'ko's Lemma [15], we get

$$T(r,\mathcal{L}[\xi]) = \frac{1}{n}T(r,\mathscr{F}) + S(r,\xi), \ T(r,\mathscr{P}[\xi]) = \frac{1}{n}T(r,\mathscr{G}) + S(r,\mathscr{P}[\xi]). \qquad (3.14)$$

From (3.13), we get $T(r,\mathscr{F}) = T(r,\mathscr{G}) + O(1)$, that is, $T(r,\pounds[\xi]) = T(r,\mathscr{P}[\xi]) + O(1)$.

Clearly from equation (3.13), if $w \neq 0$, we get $\overline{N}(r,\infty;\xi) = S(r,\xi)$.

As $uy - vw \neq 0$, so $u = w = 0$ is not possible. So we consider the following cases.

Subcase 2.1. Let $uw \neq 0$, this implies $u \neq 0$ and $w \neq 0$.

Subcase 2.1.1. Let $v = 0$. Then we must have $y \neq 0$ otherwise $uy - vw = 0$.

In this case, (3.13) reduces to

$$\mathscr{F} = \frac{u\mathscr{G}}{w\mathscr{G} + y}. \tag{3.15}$$

Since $\overline{N}(r,\infty;\xi) = S(r,\xi)$, it follows from (3.15) that

$$\overline{N}\left(r, \frac{-y}{w}; \mathscr{G}\right) = \overline{N}(r,\infty;\mathscr{F}) = S(r,\xi).$$

Applying second fundamental theorem, we get

$$\begin{aligned} T(r,\mathscr{G}) &\leq \overline{N}(r,\infty;\mathscr{G}) + \overline{N}(r,0;\mathscr{G}) + \overline{N}\Big(r,\frac{-y}{w};\mathscr{G}\Big) + S(r,\mathscr{G}) \\ &\leq \overline{N}(r,\infty;\mathscr{P}[\xi]) + \overline{N}(r,0;\mathscr{P}[\xi]) + S(r,\mathscr{P}[\xi]) \\ &\leq (k+2)\overline{N}(r,\infty;\xi) + \sum_{i=1}^{2}\overline{N}(r,\alpha_i;\mathscr{P}[\xi]) + \overline{N}(r,0;\mathscr{P}[\xi]) + S(r,\mathscr{P}[\xi]) \\ &\leq \left(\frac{k+4}{n}\right) T(r,\mathscr{G}) + S(r,\mathscr{P}[\xi]), \end{aligned}$$

which contradicts as $n > k+4$.

Subcase 2.1.2. Let $v \neq 0$.

In this case, we have

$$\mathscr{G} + \frac{v}{u} = \frac{\mathscr{F}(vw - uy)}{u(w\mathscr{F} - u)}.$$

Applying second fundamental theorem, we get

$$\begin{aligned} T(r,\mathscr{G}) &\leq \overline{N}(r,\infty;\mathscr{G}) + \overline{N}(r,0;\mathscr{G}) + \overline{N}\Big(r,\frac{-v}{u};\mathscr{G}\Big) + S(r,\mathscr{G}), \\ &\leq (k+2)\overline{N}(r,\infty;\xi) + \sum_{i=1}^{2}\overline{N}(r,\alpha_i;\mathscr{P}[\xi]) \\ &\qquad + \overline{N}(r,0;\mathscr{P}[\xi]) + \overline{N}(r,0;\pounds[\xi]) + S(r,\xi), \\ &\leq \left(\frac{k+5}{n}\right) T(r,\mathscr{G}) + S(r,\mathscr{P}[\xi]) + S(r,\xi), \end{aligned}$$

which is contradiction as $n > k+5$.

Subcase 2.2. Let $uw \equiv 0$.

Subcase 2.2.1. Let $u = 0$ and $w \neq 0$.

In this case $v \neq 0$ and $\mathscr{F} = \dfrac{1}{\gamma\mathscr{G} + \delta}$, where $\gamma = \dfrac{w}{v}$ and $\delta = \dfrac{y}{v}$.

If $\mathscr{F}$ has no $1-$points, then by the second fundamental theorem and (3.14), we get

$$\begin{aligned} T(r,\mathscr{F}) &\leq \overline{N}(r,\infty;\mathscr{F}) + \overline{N}(r,0;\mathscr{F}) + \overline{N}(r,1;\mathscr{F}) + S(r,\mathscr{F}), \\ &\leq \frac{3}{n} T(r,\mathscr{F}) + S(r,\mathscr{F}), \end{aligned}$$

which contradicts as $n > 5$.

Thus $\gamma + \delta = 1$ and $\gamma \neq 0$. So,

$$\mathscr{F} = \frac{1}{\gamma\mathscr{G} + 1 - \gamma},$$

from above we get $\overline{N}\left(r, \dfrac{1}{1-\gamma}; \mathscr{F}\right) = \overline{N}(r,0;\mathscr{G})$.

If $\gamma \neq 1$, then by using the Second fundamental theorem and (3.14), we get

$$\begin{aligned} T(r,\mathscr{F}) &\leq \overline{N}(r,\infty;\mathscr{F}) + \overline{N}(r,0;\mathscr{F}) + \overline{N}\left(r, \frac{1}{1-\gamma}; \mathscr{F}\right) + S(r,\mathscr{F}) \\ &\leq \frac{4}{n} T(r,\mathscr{F}) + S(r,\mathscr{F}), \end{aligned}$$

which contradicts as $n > 5$.

Thus $\gamma = 1$ and hence $\mathscr{F}\mathscr{G} \equiv 1$, which yields

$$a^2 \left(\pounds[\xi]\mathscr{P}[\xi]\right)^n = n^2(n-1)^2 \prod_{i=1}^{2} \left(\pounds[\xi] - \alpha_i\right) \prod_{i=1}^{2} \left(\mathscr{P}[\xi] - \alpha_i\right).$$

Assuming z_1 is a pole of order r, it follows that z_1 must also be a pole of $\pounds[\xi]$ and $\mathscr{P}[\xi]$ of order $\overline{d}(\mathscr{P})r$ and $\overline{d}(\mathscr{P})r + W_{\mathscr{P}}$ respectively, as shown by the previously established results. Furthermore, applying this result yields $n\overline{d}(\mathscr{P})r + n(\overline{d}(\mathscr{P})r + W_{\mathscr{P}}) = 2\overline{d}(\mathscr{P})r + 2(\overline{d}(\mathscr{P})r + W_{\mathscr{P}})$, which simplifies to $n = 2$. However, this is not possible because from the hypothesis of Theorem 1.17, we see that $n > 2$. Therefore, based on the earlier equation, it becomes clear that ξ has no poles.

Let $\pounds[\xi] - \alpha_i = b_1 \prod_{j=1}^{r_i} (\xi - \alpha_{ij})^{p_{ij}}$, where $1 \leq i \leq 2$, $1 \leq r_i \leq \overline{d}(\mathscr{P})$ and $1 \leq p_{ij} \leq \overline{d}(\mathscr{P})$, r_i, $p_{ij} \in \mathbb{N}$. Let z_0 be a α_{ij}-points of ξ of order s, for

$j = 1, 2, \ldots, r_i$, then as these types of points can be neutralized by the zeros of $\mathscr{P}[\xi]$, we must have $\left(n(\overline{d}(\mathscr{P}) - n_0)\right) sp_{ij} = nW_{\mathscr{P}}$, which implies $sp_{ij} \geq n$. Consequently, we have $\overline{N}(r, \alpha_{ij}; \xi) \leq \dfrac{p_{ij}}{n} N(r, \alpha_{ij}; \xi)$, for $j = 1, 2, \ldots, r,\ i = 1, 2$. Thus by the Second fundamental theorem, we get

$$\begin{aligned} (r_1 + r_2 - 1)T(r, \xi) &\leq \left(\sum_{i=1}^{2} r_i - 1\right) T(r, \xi) \\ &\leq \overline{N}(r, \infty; \xi) + \sum_{j=1}^{r_1} \overline{N}(r, \alpha_{1j}; \xi) + \sum_{j=1}^{r_2} \overline{N}(r, \alpha_{2j}; \xi) + S(r, \xi) \\ &\leq \sum_{j=1}^{r_1} \frac{p_{1j}}{n} N(r, \alpha_{1j}; \xi) + \sum_{j=1}^{r_2} \frac{p_{1j}}{n} N(r, \alpha_{2j}; \xi) + S(r, \xi) \\ &\leq \left(\frac{r_1 + r_2}{n}\right) T(r, \xi) + S(r, \xi), \end{aligned} \tag{3.16}$$

which is a contradiction if $r_1 + r_2 \geq 3$.

Next, suppose $r_1 + r_2 = 2$, which implies that $r_1 = 1 = r_2$. So let $\mathscr{L}[\xi] - \alpha_1 = b_1(\xi - \alpha_1^*)^{\overline{d}(\mathscr{P})}$ and $\mathscr{L}[\xi] - \alpha_2 = b_1(\xi - \alpha_2^*)^{\overline{d}(\mathscr{P})}$. Now by the same argument as above if we assume z_1 be a α_1^*-point of ξ of order s, we see that $n \leq \overline{d}(\mathscr{P})s$ and hence similar to (3.16), we get, $T(r, \xi) \leq \left(\dfrac{2\overline{d}(\mathscr{P})}{n}\right) T(r, \xi) + S(r, \xi)$, which is a contradiction as $n > 2\overline{d}(\mathscr{P})$.

Subcase 2.2.2. Let $u \neq 0$ and $w = 0$.

In this case $y \neq 0$ and $\mathscr{F} = \lambda \mathscr{G} + \mu$, where $\lambda = \dfrac{u}{y}$ and $\mu = \dfrac{v}{y}$.

If $\mathscr{F}$ has no $1-$ points, then by the similar argument as above, we arrive at a contradiction.

Thus $\lambda + \mu = 1$ with $\lambda \neq 0$, it is easy to see that $\overline{N}\left(r, 0; \mathscr{G} + \dfrac{1-\lambda}{\lambda}\right) = \overline{N}(r, 0; \mathscr{F})$.

If $\lambda \neq 1$, then by the Second fundamental Theorem and (3.14), we get

$$\begin{aligned} T(r.\mathscr{G}) &\leq \overline{N}(r, \infty; \mathscr{G}) + \overline{N}(r, 0; \mathscr{G}) + \overline{N}\left(r, 0; \mathscr{G} + \frac{1-\lambda}{\lambda}\right) + S(r, \mathscr{G}) \\ &\leq \overline{N}(r, \infty; \mathscr{P}[\xi]) + \sum_{i=1}^{2} \overline{N}(r, \alpha_i; \mathscr{P}[\xi]) + \overline{N}(r, 0; \mathscr{P}[\xi]) + \overline{N}(r, 0; L[\xi])) \\ &+ S(r, \mathscr{P}[\xi]) \\ &\leq \left(\frac{k+5}{n}\right) T(r, \mathscr{G}) + S(r, \mathscr{P}[\xi]) + S(r, \xi), \end{aligned}$$

which contradicts as $n > k+5$.

Thus $\lambda = 1$, and hence $\mathscr{F} = \mathscr{G}$. Therefore,

$$\begin{aligned} &n(n-1)(\pounds[\xi])^2(\mathscr{P}[\xi])^2\left[(\pounds[\xi])^{n-2}-(\mathscr{P}[\xi])^{n-2}\right] \\ &-2n(n-2)b\pounds[\xi]\mathscr{P}[\xi]\left[(\pounds[\xi])^{n-1}-(\mathscr{P}[\xi])^{n-1}\right] \\ &+(n-1)(n-2)b^2\left[(\pounds[\xi])^{n}-(\mathscr{P}[\xi])^{n}\right]=0. \end{aligned}$$

By substituting $h = \dfrac{\mathscr{P}[\xi]}{\pounds[\xi]}$, we get

$$\begin{aligned} (3.17)\quad n(n-1)(\pounds[\xi])^2h^2\left(h^{n-2}-1\right)-2n(n-2)bh(\pounds[\xi])\left(h^{n-1}-1\right)& \\ +(n-1)(n-2)b^2\left(h^{n}-1\right)=0.& \end{aligned}$$

If h is non-constant, then using Lemma 2.8, we obtain from (3.17)

$$\begin{aligned} &\{n(n-1)h\pounds[\xi]\left(h^{n-2}-1\right)-n(n-2)b\left(h^{n-1}-1\right)\}^2 \\ &\qquad = -n(n-2)b^2\left(h-1\right)^4\prod_{i=1}^{2n-6}\left(h-\beta_i\right). \end{aligned}$$

By applying the Second fundamental theorem, we obtain

$$\begin{aligned} (2n-6)\,T(r,h) &\le \overline{N}(r,\infty;h)+\overline{N}(r,0;h)+\sum_{i=1}^{2n-6}\overline{N}(r,\beta_i;h)+S(r,h), \\ &\le \overline{N}(r,\infty;h)+\overline{N}(r,0;h)+\frac{1}{2}\sum_{i=1}^{2n-6}\overline{N}(r,\beta_i;h)+S(r,h), \\ &\le (n-1)T(r,h)+S(r,h). \end{aligned}$$

Which is a contradiction as $n > 5$.

Thus h is a constant. Hence as ξ is non-constant and $b \neq 0$, we get from the equation (3.17), that $(h^{n-2}-1) = 0$, $(h^{n-1}-1) = 0$ and $(h^n-1) = 0$. Therefore $h^d - 1 = 0$, where $d = \gcd(n, n-1, n-2) = 1$. Consequently $\pounds[\xi] \equiv \mathscr{P}[\xi]$. $\square$

4. Conclusion. In this paper, we studied the uniqueness of differential polynomials and difference-differential polynomials of meromorphic functions that share a set with finite weight. Since the natural extensions of the differential polynomials are difference-differential polynomials, our result will generalize and extend the result due to A. Banerjee and B. Chakraborty [3]. Finally, some questions have been posed in this direction.

Open questions:

(1) Is it possible to generalize the difference-differential polynomial (1.2) to any other to deduce a generalized result?

(2) Is it possible to find out the specific form of the function ξ in Theorem 1.17 by considering a set with a cardinality less than that of $\mathcal{S} = \{w : aw^n + bw^{2m} + cw^m + 1 = 0\}$?

(3) Is it possible to additionally weaken the condition in Theorem 1.16 by considering two polynomials $P(w) = aw^n + bw^{n-m} + d$, and $Q(w) = uw^n + vw^{n-m} + t$, where n, m are two positive integers and a, b, d, u, v, t are nonzero complex numbers such that P and Q have no multiple zeros and two sets $\mathcal{S} = \{w : P(w) = 0\}$ and $T = \{w : Q(w) = 0\}$?

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