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SOME DATA DEPENDENCE RESULTS ON THE SOLUTIONS OF ITERATIVE DIFFERENTIAL EQUATIONS VIA A NEW THREE-STEP ITERATION PROCESS

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Abstract. In this study, we employ a new three-step iteration method to investigate the existence and uniqueness of the solution to the iterative differential equation of fractional order in the sense of Caputo. We also deal with data dependence results, including dependence on initial data, closeness of solutions, and dependence on the parametersand functions of the solution to the iterative differential equation. Finally, we illustrated all of our conclusions using an example and compared the rate of convergence of a new three-step iteration with other iterative approaches.

1. Introduction. We consider the following fractional order implicit iterative differential equation in the sense of the Caputo of the type:

$$\left({}^{c}D^{\alpha}_{0^+}\right)u(t) = \mathcal{F}\Big(u^{[0]}(t), u^{[1]}(t), u^{[2]}(t), \ldots, u^{[m]}(t)\Big), \tag{1}$$

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Key words: existence and uniqueness, new iterative method, fractional derivative, continuous dependence, closeness, parameters.

for $t \in J = [0, b],\ m \in \mathbb{N} \cup \{0\},\ n - 1 < \alpha \leq n, n \in \mathbb{N}$, with the given initial conditions

$$u^{(j)}(0) = d_j, j = 0, 1, 2, \ldots, n - 1, \tag{2}$$

where $\mathcal{F} : [C_M(J, L)]^{m+1} \to C_M(J, L)$ is continuous and $dj \in C_M(J, L),\ 0 \leq j \leq n - 1$. The space $C_M(J, L)$ will be specified later.

The implicit fractional-order iterative differential equation (1)–(2) formulated in the Caputo sense constitutes an appropriate mathematical model for processes involving both nonlocal memory effects and recursive dependence. The presence of multiple iterated arguments $u^{[i]}(t)$ introduces a feedback structure that cannot be adequately described by standard fractional differential equations, while the implicit formulation allows for a more general interaction among successive iterations. As a result, classical analytical approaches are not directly applicable, necessitating a dedicated investigation of the problem. From an applied perspective, such models naturally arise in systems governed by memory and repeated feedback mechanisms, including control systems, viscoelastic materials, disease transmission in epidemiology, and other biological processes, economic dynamics, two-body problems in classical electrodynamics, physical models, mechanical models, and many more (refer to [5, 8, 9, 10, 11, 12, 24, 38]).

In 1965, Petuhov [27] studied the existence of solutions to the iterative differential equation

$$x''(t) = \lambda x(x(t)),$$

with boundary conditions

$$x(0) = x(T) = \alpha,\ t \in [-T, T].$$

In 1984, Eder [10] used Banach fixed point principle to study the existence and uniqueness of the solution to the iterative differential equation

$$x'(t) = x(x(t)),$$

with initial condition

$$x(t_0) = x_0,\ t_0 \in [-1, 1].$$

In 2012 and 2013, Ibrahim [14, 15] investigated the class of fractional iterative differential equations applying non-expansive operators as follows:

$$D^{\rho} y(t) = \phi(t, y(t), y(y(t))), \quad y(0) = y_0.$$

In 2013, Ibrahim and Darus [16] investigated fractional differential equations as follows:

$$D^\rho y(t) = \phi(t, y(t), y(\beta t)y(y(t))),$$

with initial condition

$$y(0) = y_0, \quad \beta \in (0, 1].$$

In 2015, Ibrahim et al. [17] investigated the existence and uniqueness of iterative fractional differential equation as follows:

$$D^\rho y(t) = \phi(t, y^{[1]}(t), y^{[2]}(t), \ldots, y^{[n]}(t)),$$

with initial condition

$$y(t_0) = c, \quad c \in [0, \infty).$$

In 2018, Kaufmann [21] used Shauder's fixed point principle to investigate the existence and uniqueness of the solution of the functional differential equation

$$x''(t) = f(t, x(t), x(x(t))).$$

Also, several researchers studied special forms of IVP (1)–(2) and its variants under a variety of hypotheses by using different techniques.
Therefore, the study of problem (1)–(2) is essential for establishing a rigorous analytical foundation that supports both theoretical development and the reliable application of fractional iterative models.

For the implicit fractional iterative differential equation (1)–(2), the establishment of existence and uniqueness is essential to validate the mathematical formulation of the model. The implicit nature of the Caputo fractional derivative combined with multiple iterated terms $u^{[i]}(t)$ prevents the direct use of standard fractional existence theories. The existence of a solution ensures the consistency of the model under the given initial conditions, while uniqueness guarantees that the iterative dependence does not lead to ambiguity in the solution. These results confirm that the associated fractional operator is well defined on $C_M(J, L)$ and support the rigorous use of fixed-point methods and iterative constructions for the proposed problem.

The three-step iteration method has become an important tool in fixed-point theory and nonlinear analysis due to its improved convergence characteristics and computational efficiency. Iterative fixed-point techniques have been widely applied to the analysis of integral, differential, and integrodifferential equations. In contrast to several existing methods that primarily address the existence of solutions, the three-step iteration method provides sufficient conditions for both existence and uniqueness. In addition, the method facilitates the analysis of solution behavior within a unified iterative framework. When compared with classical schemes such as Picard, Mann, Ishikawa, Noor, and S-iteration methods, the three-step iteration method exhibits faster convergence for contractive, nonexpansive, and pseudocontractive operators. Its structured formulation allows greater control over the iterative process and enables applicability under weaker assumptions. According to their convergence, equivalency of convergence, rate of convergence, and other characteristics, several researchers have suggested and applied various iteration techniques for specific classes of operators (see [1, 2, 6, 13, 18, 19, 22, 25, 30, 31, 32, 33, 34, 35, 36, 37]).

Owing to these advantages, the three-step iteration method has attracted increasing attention in recent studies and is therefore employed in the present work.

In this context, we study the iterative fractional differential equation (1) with initial condition (2) using the applications of the new three-step iteration method. The main purpose of this study is to establish the existence, uniqueness and other qualitative properties of the solution to the IVP (1)–(2).

2. Preliminaries. Let $C(J, \mathbb{R})$ be the Banach space of all real-valued continuous functions defined on the compact interval J, equipped with the norm

$$\|u\| = \sup_{t \in J} |u(t)|.$$

For $0 < L \leq b$ and $M > 0$, define the sets

$$C(J, L) = \{u \in C(J, \mathbb{R}) : 0 \leq u(t) \leq L, \forall\ t \in J\}$$

and

$$C_M(J, L) = \{u \in C(J, L) : |u(t_2) - u(t_1)| \leq M|t_2 - t_1|, \forall\ t_1, t_2 \in J\}$$

Then $C_M(J, L)$ is closed convex and bounded subset of $C(J, \mathbb{R})$.

Definition 1 ([29]). *The Left-sided Riemann Liouville fractional integral (left-sided) of a function $f \in C[a, b]$ of order $\alpha \in \mathbf{R}_+ = (0, \infty)$ is defined by*

$$I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} f(s)\, ds,$$

where Γ is the Euler gamma function.

Definition 2 ([29]). *Let $n-1<\alpha\leq n,\ n\in\mathbb{N}$. Then the expression*

$$D^{\alpha}_{a^+}f(t)=\frac{d^n}{dt^n}\left[I^{n-\alpha}_{a^+}f(t)\right],\ t\in[a,b]$$

is called the left-sided Riemann Liouville fractional derivative of f of order α whenever the expression on the right-hand side is defined.

Definition 3 ([23]). *Let $f\in C^n[a,b]$ and $n-1<\alpha\leq n,\ n\in\mathbb{N}$. Then the expression*

$$\left({}^cD^{\alpha}_{a^+}\right)f(t)=I^{n-\alpha}_{a}f^{(n)}(t),\ t\in[a,b]$$

is called the (left-sided) Caputo derivative of f of order α.

Definition 4 ([4]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists*

$$l=\lim_{n\to\infty}\frac{|a_n-a|}{|b_n-b|}.$$

(*a*) *If $l=0$, then it can be said that $\{a_n\}$ converges to a faster than $\{b_n\}$ converges to b.*

(*b*) *If $0<l<1$, then it can be said that $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.*

Suppose that for two fixed point iteration procedures $\{u_n\}$ and $\{v_n\}$, both converging to the same fixed point p, the error estimates

$$\|u_n-p\|\leq a_n,\ \forall\ n\in\mathbb{N}\cup\{0\}, \tag{3}$$

$$\|v_n-p\|\leq b_n,\ \forall\ n\in\mathbb{N}\cup\{0\}, \tag{4}$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers (converging to zero). Following Definition 4, we will adopt the following notion.

Definition 5 ([4]). *Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy* (3) *and* (4)*, respectively. If $\{a_n\}$ converges faster than $\{b_n\}$, then it can be said that $\{u_n\}$ converges faster than $\{v_n\}$ to p.*

Lemma 1 ([7]). *If the function $f=(f_1,\ldots,f_n)\in C^1[0,b]$, then the initial value problems*

$$\left(D^{\alpha_i}_*\right)x_i(t)=f_i(t,x_1,\ldots,x_n),\ x_i^{(k)}(0)=c^i_k,\ i=1,2,\ldots,n,\ k=1,2,\ldots,m_i$$

where $m_i < \alpha_i \leq m_i + 1$ *is equivalent to Volterra integral equations:*

$$x_i(t) = \sum_{k=0}^{m_i} c_k^i \frac{t^k}{k!} + I_a^{\alpha_i} f_i(t, x_1, \ldots, x_n),\ 1 \leq i \leq n.$$

As a consequence of the Lemma 1, it is easy to observe that if $u \in C_M(J, L)$ and $\mathcal{F} \in C^1[0, b]$, then $u(t)$ satisfies the following integral equation which is equivalent to (1)–(2).

(5)
$$u(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!}\ t^j + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}\Big(u^{[0]}(s), u^{[1]}(s), u^{[2]}(s), \ldots, u^{[m]}(s)\Big) ds.$$

Lemma 2 ([39])**.** *If* $u, v \in C_M(J, L)$ *then*

$$\|u^{[n]} - v^{[n]}\| \leq \sum_{j=0}^{n-1} M^j \|u - v\|, n = 1, 2, \ldots$$

Definition 6 ([20, p. 627])**.** *The self-map* $T : C \to C$ *is called weak-contraction if there exist* $\delta \in (0, 1)$ *and* $L \geq 0$ *such that*

$$\|Tu - Tv\| \leq \delta \|u - v\| + L\|v - Tv\|.$$

V. Karakaya, Y. Atalan, K. Dogan, and NH. Bouzara [20] proposed a new three-step iteration process as follows:

$$\left\{ \begin{array}{l} u_{k+1} = Tv_k, \\ v_k = (1 - \xi_k) w_k + \xi_k T w_k, \\ w_k = Tu_k,\ k \in \mathbb{N} \cup \{0\}, \end{array} \right. \tag{6}$$

with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0, 1]$ satisfying $\sum_{k=0}^{\infty} \xi_k = \infty$.

Some researchers in [18, 26, 28, 30] introduced following iterations methods for different classes of operators as follows:

Let X be a Banach space and $T : X \to X$ be a self-mapping and $x_0 \in X$, be the initial solution then

Picard Iteration: $x_{k+1} = Tx_k, \qquad k \in \mathbb{N} \cup \{0\}$, Mann iteration: $x_{k+1} = (1-\xi_k)x_k + \xi_k T x_k, \qquad k \in \mathbb{N} \cup \{0\},$

Ishikawa Iteration: $\begin{cases} y_k = (1-\eta_k)x_k + \eta_k T x_k, \\ x_{k+1} = (1-\xi_k)x_k + \xi_k T y_k, \qquad k \in \mathbb{N} \cup \{0\}, \end{cases}$

Normal S-iteration: $\begin{cases} y_k = T x_k, \\ x_{k+1} = (1-\alpha_k)T x_k + \alpha_k T y_k, \qquad k \in \mathbb{N} \cup \{0\}, \end{cases}$

with the real control sequence $\{\xi_k\}_{k=0}^{\infty}, \{\eta_k\}_{k=0}^{\infty}$ in $[0,1]$.

Theorem 1 ([20, p. 626]). *Let (X,d) be a complete metric space and $T : X \to X$ be a weak contraction for which there exist $\delta \in (0,1)$ and some $L_1 \geq 0$ such that*

$$\|Tu - Tv\| \leq \delta\|u - v\| + L_1\|u - Tu\|. \tag{7}$$

Then, T has a unique fixed point.

Theorem 2 ([20, p. 627]). *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a weak contraction map satisfying condition* (7). *Let $\{u_k\}_{k=0}^{\infty}$ be an iterative sequence generated by the scheme* (6) *with a real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\sum_{k=0}^{\infty} \xi_k = \infty$. Then $\{u_k\}_{k=0}^{\infty}$ converges to a unique fixed point x^* of T.*

Lemma 3 ([31, p. 4]). *Let $\{\beta_k\}_{k=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $k_0 \in \mathbb{N} \cup \{0\}$, such that for all $k \geq k_0$ one has satisfied the inequality*

$$\beta_{k+1} \leq (1-\mu_k)\beta_k + \mu_k\gamma_k, \tag{8}$$

where $\mu_k \in (0,1)$, for all $k \in \mathbb{N} \cup \{0\}$, $\sum_{k=0}^{\infty} \mu_k = \infty$ and $\gamma_k \geq 0$, $\forall k \in \mathbb{N} \cup \{0\}$. Then the following inequality holds

$$0 \leq \limsup_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \gamma_k. \tag{9}$$

3. Existence and uniqueness of solutions via new three-step iteration. We are able to state and prove the main theorem, which addresses the existence of solutions to the equations (1)–(2).

Theorem 3. *Assume that, there exist real constants c_i, $i = 1, 2, 3, \ldots, m$ such that*

$$\|\mathcal{F}(u^{[0]}(t), u^{[1]}(t), u^{[2]}(t), \ldots, u^{[m]}(t)) - \mathcal{F}(v^{[0]}(t), v^{[1]}(t), v^{[2]}(t), \ldots, v^{[m]}(t))\| \le \sum_{i=1}^{m} c_i \|u^{[i]}(t) - v^{[i]}(t)\|, \ t \in J. \tag{10}$$

If $\Delta = \frac{b^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{m} c_i \sum_{j=0}^{i-1} M^j < 1$, *then the IVP* (1)–(2) *has a unique solution* $u \in C_M(J, L)$, *and the sequence* $\{u_k\}_{k=0}^{\infty}$ *generated by a new three-step iterative method* (6) *converges to* $u \in C_M(J, L)$ *with the following error estimate at the* k^{th} *successive approximation,*

$$\|u_{k+1} - u\| \le \frac{\Delta^{2k+2}}{e^{(1-\Delta)\sum_{i=0}^{k} \xi_i}} \|u_0 - u\|. \tag{11}$$

Proof. For $u(t) \in C_M(J, L)$, define the operator

$$(Tu)(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!}\, t^j + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}\Big(u^{[0]}(s), u^{[1]}(s), u^{[2]}(s), \ldots, u^{[m]}(s)\Big) ds. \tag{12}$$

Let $\{u_k\}_{k=0}^{\infty}$ be iterative sequence generated by the new three-step iteration method (6) for the operator given in (12) with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0, 1]$.
We will show that $u_k \to u$ as $k \to \infty$. From (6), (12), assumptions and Lemma 2, we obtain

$$\begin{aligned}
&\|w_k(t) - u(t)\| \\
&= \|(Tu_k)(t) - (Tu)(t)\| \\
&= \Big\| \sum_{j=0}^{n-1} \frac{d_j}{j!}\, t^j + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}\Big(u_k^{[0]}(s), u_k^{[1]}(s), u_k^{[2]}(s), \ldots, u_k^{[m]}(s)\Big) ds \\
&\quad - \sum_{j=0}^{n-1} \frac{d_j}{j!}\, t^j - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}\Big(u^{[0]}(s), u^{[1]}(s), u^{[2]}(s), \ldots, u^{[m]}(s)\Big) ds \Big\| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big\| \mathcal{F}\Big(u_k^{[0]}(s), u_k^{[1]}(s), u_k^{[2]}(s), \ldots, u_k^{[m]}(s)\Big)
\end{aligned}$$

$$
\begin{aligned}
& - \mathcal{F}\Big(u^{[0]}(s), u^{[1]}(s), u^{[2]}(s), \ldots, u^{[m]}(s)\Big)\Big\| ds \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{m} c_i \|u_k^{[i]} - u^{[i]}\| ds \\
(13) \quad & \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{m} c_i \sum_{j=0}^{i-1} M^j \|u_k - u\| ds.
\end{aligned}
$$

Now, we estimate

$$
\begin{aligned}
\|v_k(t) - u(t)\| &= \|(1-\xi_k)w_k(t) + \xi_k (Tw_k)(t) - u(t)\| \\
&= \|(1-\xi_k)w_k(t) + \xi_k (Tw_k)(t) - (1-\xi_k)u(t) - \xi_k u(t)\| \\
&= \|(1-\xi_k)(w_k(t) - u(t)) + \xi_k \big((Tw_k)(t) - (Tu)(t)\big)\| \\
&\leq (1-\xi_k)\|w_k(t) - u(t)\| + \xi_k \|(Tw_k)(t) - (Tu)(t)\| \\
&\leq (1-\xi_k)\|w_k(t) - u(t)\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{m} c_i \sum_{j=0}^{i-1} M^j \|w_k - u\| ds. \qquad (14)
\end{aligned}
$$

Also, we estimate

$$
\begin{aligned}
\|u_{k+1}(t) - u(t)\| &= \|(Tv_k)(t) - (Tu)(t)\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{m} c_i \sum_{j=0}^{i-1} M^j \|v_k - u\| ds. \qquad (15)
\end{aligned}
$$

Now, by taking supremum in the inequalities (13), (14) and (15), we obtain

$$
\begin{aligned}
\|w_k - u\| &\leq \frac{b^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{m} c_i \sum_{j=0}^{i-1} M^j \|u_k - u\| \\
&= \Delta \|u_k - u\|. \qquad (16)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\|v_k - u\| &\leq (1-\xi_k)\|w_k - u\| + \xi_k \Delta \|w_k - u\| \\
&= \big[1 - \xi_k(1-\Delta)\big] \|w_k - u\| \qquad (17)
\end{aligned}
$$

Similarly,

$$
\|u_{k+1} - u\| \leq \Delta \|v_k - u\|. \qquad (18)
$$

Therefore, using (16), (17), and (18), we obtain

$$\begin{aligned} \|u_{k+1} - u\| &\le \Delta \|v_k - u\| \\ &\le \Delta \big[1 - \xi_k(1-\Delta)\big] \|w_k - u\| \\ &\le \Delta^2 \big[1 - \xi_k(1-\Delta)\big] \|u_k - u\|. \end{aligned} \tag{19}$$

Thus, by induction, we get

$$\|u_{k+1} - u\| \le \Delta^{2k+2} \prod_{j=0}^{k} \Big[1 - \xi_j \Big(1 - \Delta\Big)\Big] \|u_0 - u\|. \tag{20}$$

Since $\xi_k \in [0,1]$ for all $k \in \mathbb{N} \cup \{0\}$, and $0 < \Delta < 1$ gives,

$$\Rightarrow \xi_k \Big(1 - \Delta\Big) < 1, \ \forall\, k \in \mathbb{N} \cup \{0\}. \tag{21}$$

We know that, $1 - u \le e^{-u}, \ \forall\, u \in [0,1]$. Hence by utilizing this fact with (21) in (20), we obtain

$$\begin{aligned} \|u_{k+1} - u\| &\le \Delta^{2k+2} e^{-(1-\Delta)\sum_{j=0}^{k} \xi_j} \|u_0 - u\| \\ &= \frac{\Delta^{2k+2}}{e^{(1-\Delta)\sum_{i=0}^{k} \xi_i}} \|u_0 - u\|. \end{aligned} \tag{22}$$

Since $\sum_{k=0}^{\infty} \xi_k = \infty$, then we have

$$e^{-(1-\Delta)\sum_{j=0}^{k} \xi_j} \to 0 \quad \text{as} \quad k \to \infty. \tag{23}$$

Hence, the inequality (22) implies $\lim_{k\to\infty} \|u_{k+1} - u\| = 0$ and therefore, we get $u_k \to u$ as $k \to \infty$. □

Remark. The inequality (22) provides bounds for solutions of equations (1)–(2) for $t \in J$, based on known functions, that majorizes the iterations.

4. Continuous dependence via new three-step iteration. In this section, we will discuss the continuous dependencies of the solutions of the problem (1) on initial data, functions, and parameters.

4.1. Dependence on initial data. Suppose $u(t)$ and $\overline{u}(t)$ are solutions of (1) with initial data

$$u^{(j)}(0) = d_j, \quad j = 0, 1, 2, \ldots, n-1, \tag{24}$$

and

$$\overline{u}^{(j)}(0) = \overline{d}_j, \quad j = 0, 1, 2, \ldots, n-1, \tag{25}$$

respectively, where $d_j, \overline{d}_j$ are elements of the space $C_M(J, L)$.

Now, we define the operator for the equation (1) with initial conditions (25) as, for $t \in J$

$$\begin{aligned}(\overline{T}\overline{u})(t) = &\sum_{j=0}^{n-1} \frac{\overline{d}_j}{j!}\, t^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}\Big(\overline{u}^{[0]}(s), \overline{u}^{[1]}(s), \overline{u}^{[2]}(s), \ldots, \overline{u}^{[m]}(s)\Big) ds.\end{aligned} \tag{26}$$

We discuss the continuous dependence of solutions of equation (1) on initial data.

Theorem 4. *Suppose the function $\mathcal{F}$ in equation* (1) *satisfies the condition* (10). *Consider the sequences $\{u_k\}_{k=0}^{\infty}$ and $\{\overline{u}_k\}_{k=0}^{\infty}$ generated by new three-step iteration method associated with operators T in* (12) *and $\overline{T}$ in* (26), *respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. If the sequence $\{\overline{u}_k\}_{k=0}^{\infty}$ converges to $\overline{u}$, then we have*

$$\|u - \overline{u}\| \le \frac{5Q}{(1-\Delta)}, \tag{27}$$

where

$$Q = \sum_{j=0}^{n-1} \frac{\|d_j - \overline{d}_j\|}{j!}\, b^j.$$

Proof. Let $\{\overline{u}_k\}_{k=0}^{\infty}$ converges to $\overline{u}$. From iteration (6) and equations (12); (26), assumptions and Lemma 2, we obtain

$$\begin{aligned}&\|w_k(t) - \overline{w}_k(t)\| \\ &= \|(Tu_k)(t) - (\overline{T}\overline{u}_k)(t)\| \\ &= \Big\| \sum_{j=0}^{n-1} \frac{d_j}{j!}\, t^j + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}\Big(u_k^{[0]}(s), u_k^{[1]}(s), u_k^{[2]}(s), \ldots, u_k^{[m]}(s)\Big) ds\end{aligned}$$

$$-\sum_{j=0}^{n-1}\frac{\overline{d}_j}{j!}\,t^j-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mathcal{F}\Big(\overline{u}_k^{[0]}(s),\overline{u}_k^{[1]}(s),\overline{u}_k^{[2]}(s),\ldots,\overline{u}_k^{[m]}(s)\Big)ds\Big\|$$

$$\le\sum_{j=0}^{n-1}\frac{\|d_j-\overline{d}_j\|}{j!}\,b^j+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Big\|\mathcal{F}\Big(u_k^{[0]}(s),u_k^{[1]}(s),u_k^{[2]}(s),\ldots,u_k^{[m]}(s)\Big)$$

$$-\mathcal{F}\Big(\overline{u}_k^{[0]}(t),\overline{u}_k^{[1]}(s),\overline{u}_k^{[2]}(s),\ldots,\overline{u}_k^{[m]}(s)\Big)\Big\|ds$$

$$\le Q+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sum_{i=1}^{m}c_i\|u_k^{[i]}-\overline{u}_k^{[i]}\|ds$$

$$\le Q+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sum_{i=1}^{m}c_i\sum_{j=0}^{i-1}M^j\|u_k-\overline{u}_k\|ds. \tag{28}$$

Now, we estimate,

$$\begin{aligned}\|v_k(t)-\overline{v}_k(t)\|&=\|(1-\xi_k)\big(w_k(t)-\overline{w}_k(t)\big)+\xi_k\big((Tw_k)(t)-(\overline{T}\overline{w}_k)(t)\big)\|\\&\le\Big[(1-\xi_k)\|w_k(t)-\overline{w}_k(t)\|+\xi_k\|(Tw_k)(t)-(\overline{T}\overline{w}_k)(t)\|\Big]\\&\le(1-\xi_k)\|w_k(t)-\overline{w}_k(t)\|\\&\quad+\xi_k\Big[Q+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sum_{i=1}^{m}c_i\sum_{j=0}^{i-1}M^j\|w_k-\overline{w}_k\|ds\Big].\end{aligned}\tag{29}$$

Also, we estimate

$$\begin{aligned}\|u_{k+1}(t)-\overline{u}_{k+1}(t)\|&=\|Tv_k(t)-\overline{T}\overline{v}_k(t)\|\\&\le Q+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sum_{i=1}^{m}c_i\sum_{j=0}^{i-1}M^j\|v_k-\overline{v}_k\|ds.\end{aligned}\tag{30}$$

Now, by taking supremum in the inequalities (29), (30) and (31), we obtain

$$\begin{aligned}\|w_k-\overline{w}_k\|&\le Q+\frac{b^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{m}c_i\sum_{j=0}^{i-1}M^j\|u_k-\overline{u}_k\|\\&\le Q+\Delta\|u_k-\overline{u}_k\|\end{aligned}\tag{31}$$

Also,

$$\|v_k-\overline{v}_k\|\le(1-\xi_k)\|w_k-\overline{w}_k\|+\xi_k\Big[Q+\Delta\|w_k-\overline{w}_k\|\Big]$$

$$= \xi_k Q + \Big[1 - \xi_k\Big(1 - \Delta\Big)\Big]\|w_k - \overline{w}_k\|. \tag{32}$$

Similarly,

$$\|u_{k+1} - \overline{u}_{k+1}\| \leq Q + \Delta\|v_k - \overline{v}_k\|. \tag{33}$$

Therefore, from (31), (32) and (33) along with hypotheses $\Delta < 1$, and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$, the resulting inequality becomes

$$\begin{aligned}
\|u_{k+1} - \overline{u}_{k+1}\| &\leq Q + \Delta\|v_k - \overline{v}_k\| \\
&\leq Q + \xi_k Q + \Big[1 - \xi_k\Big(1 - \Delta\Big)\Big]\|w_k - \overline{w}_k\| \\
&\leq Q + \xi_k Q + \Big[1 - \xi_k\Big(1 - \Delta\Big)\Big]\big[Q + \Delta\|u_k - \overline{u}_k\|\big] \\
&\leq 2Q + \Delta\xi_k Q + \Delta\Big[1 - \xi_k\Big(1 - \Delta\Big)\Big]\|u_k - \overline{u}_k\| \\
&\leq 4\xi_k Q + \xi_k Q + \Big[1 - \xi_k\Big(1 - \Delta\Big)\Big]\|u_k - \overline{u}_k\| \\
&\leq \Big[1 - \xi_k\Big(1 - \Delta\Big)\Big]\|u_k - \overline{u}_k\| + \xi_k(1 - \Delta)\frac{5Q}{(1 - \Delta)} \qquad (34)
\end{aligned}$$

We denote

$$\begin{aligned}
\beta_k &= \|u_k - \overline{u}_k\| \geq 0, \\
\mu_k &= \xi_k(1 - \Delta) \in (0, 1), \\
\gamma_k &= \frac{5Q}{(1 - \Delta)} \geq 0.
\end{aligned}$$

Since, $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Thus, the (34) satisfies all the conditions of Lemma 3 and hence, we get

$$\begin{aligned}
&0 \leq \limsup_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \gamma_k \\
&\Rightarrow 0 \leq \limsup_{k\to\infty} \|u_k - \overline{u}_k\| \leq \limsup_{k\to\infty} \frac{5Q}{(1 - \Delta)} \\
&\Rightarrow 0 \leq \limsup_{k\to\infty} \|u_k - \overline{u}_k\| \leq \frac{5Q}{(1 - \Delta)}. \qquad (35)
\end{aligned}$$

Using the assumptions $\lim_{k\to\infty} u_k = u$, $\lim_{k\to\infty} \overline{u}_k = \overline{u}$, therefore the inequality (35) gives

$$\|u - \overline{u}\| \leq \frac{5Q}{(1 - \Delta)}. \tag{36}$$

The inequality (36) shows the dependency of solutions to the problem (1) on the given initial data. □

4.2. Closeness of solutions. Consider the problem (1)–(2) and the corresponding problem

$$({}^cD^{\alpha}_{0+})\overline{u}(t) = \overline{\mathcal{F}}\Big(\overline{u}^{[0]}(t), \overline{u}^{[1]}(t), \overline{u}^{[2]}(t), \ldots, \overline{u}^{[m]}(t)\Big), \tag{37}$$

for $t \in J = [0, b]$, $n-1 < \alpha \leq n$ $(n \in \mathbb{N})$, $m \in \mathbb{N} \cup \{0\}$, with the given initial conditions

$$\overline{u}^{(j)}(0) = \overline{d}_j, \; j = 0, 1, 2, \ldots, n-1, \tag{38}$$

where $\overline{\mathcal{F}}$ is defined as $\mathcal{F}$ and $\overline{d}_j$ $(j = 0, 1, 2, \ldots, n-1)$ are given elements in $C_M(J, L)$.

Now, we define the operator for the IVP (37)–(38) as for $t \in J$

$$\begin{aligned}(\overline{T}\overline{u})(t) = \sum_{j=0}^{n-1} \frac{\overline{d}_j}{j!}\, t^j \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{\mathcal{F}}\Big(\overline{u}^{[0]}(s), \overline{u}^{[1]}(s), \overline{u}^{[2]}(s), \ldots, \overline{u}^{[m]}(s)\Big) ds.\end{aligned} \tag{39}$$

In the next theorem, we discuss the closeness of solutions of the problems (1)–(2) and (37)–(38).

Theorem 5. *Consider the sequences* $\{u_k\}_{k=0}^{\infty}$ *and* $\{\overline{u}_k\}_{k=0}^{\infty}$ *generated by new three-step iteration method associated with operators* T *in* (12) *and* $\overline{T}$ *in* (39)*, respectively with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in* $[0, 1]$ *satisfying* $\frac{1}{2} \leq \xi_k$ *for all* $k \in \mathbb{N} \cup \{0\}$*. Assume that*

(*i*) *all conditions of Theorem* 3 *hold, and* $u(t)$ *and* $\overline{u}(t)$ *are solutions of* (1)–(2) *and* (37)–(38) *respectively.*

(*ii*) *there exist non negative constant* ϵ *such that* $\forall\ t \in J$*;*

$$\begin{aligned}\Big\| \mathcal{F}\Big(u^{[0]}(t), u^{[1]}(t), u^{[2]}(t), \ldots, u^{[m]}(t)\Big) \\ - \overline{\mathcal{F}}\Big(u^{[0]}(t), u^{[1]}(t), u^{[2]}(t), \ldots, u^{[m]}(t)\Big)\Big\| \leq \epsilon.\end{aligned} \tag{40}$$

If the sequence $\{\overline{u}_k\}_{k=0}^{\infty}$ *converges to* $\overline{u}$*, then we have*

$$\|u-\overline{u}\| \leq \frac{5\Big[Q+\frac{\epsilon b^{\alpha}}{\Gamma(\alpha+1)}\Big]}{(1-\Delta)}. \tag{41}$$

P r o o f. From iteration (6) and equations (12); (39), assumptions, and Lemma 2, we obtain

$$\begin{aligned}
&\|w_k(t)-\overline{w}_k(t)\| \\
&= \|(Tu_k)(t)-(\overline{T}\overline{u}_k)(t)\| \\
&= \Big\| \sum_{j=0}^{n-1} \frac{d_j}{j!}\, t^j + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}\Big(u_k^{[0]}(s), u_k^{[1]}(s), u_k^{[2]}(s), \dots, u_k^{[m]}(s)\Big) ds \\
&\quad - \sum_{j=0}^{n-1} \frac{\overline{d}_j}{j!}\, t^j - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{\mathcal{F}}\Big(\overline{u}_k^{[0]}(s), \overline{u}_k^{[1]}(s), \overline{u}_k^{[2]}(s), \dots, \overline{u}_k^{[m]}(s)\Big) ds \Big\| \\
&\leq \sum_{j=0}^{n-1} \frac{\|d_j-\overline{d}_j\|}{j!}\, b^j \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big\| \mathcal{F}\Big(u_k^{[0]}(s), u_k^{[1]}(s), u_k^{[2]}(s), \dots, u_k^{[m]}(s)\Big) \\
&\quad - \overline{\mathcal{F}}\Big(\overline{u}_k^{[0]}(s), \overline{u}_k^{[1]}(s), \overline{u}_k^{[2]}(s), \dots, \overline{u}_k^{[m]}(s)\Big) \Big\| ds \\
&\leq Q + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big\| \mathcal{F}\Big(u_k^{[0]}(s), u_k^{[1]}(s), u_k^{[2]}(s), \dots, u_k^{[m]}(s)\Big) \\
&\quad - \overline{\mathcal{F}}\Big(u_k^{[0]}(s), u_k^{[1]}(s), u_k^{[2]}(s), \dots, u_k^{[m]}(s)\Big) \Big\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big\| \overline{\mathcal{F}}\Big(u_k^{[0]}(s), u_k^{[1]}(s), u_k^{[2]}(s), \dots, u_k^{[m]}(s)\Big) \\
&\quad - \overline{\mathcal{F}}\Big(\overline{u}^{[0]}(s), \overline{u}^{[1]}(s), \overline{u}^{[2]}(s), \dots, \overline{u}^{[m]}(s)\Big) \Big\| ds \\
&\leq Q + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \epsilon ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{m} c_i \|u_k^{[i]} - \overline{u}_k^{[i]}\| ds
\end{aligned}$$

(42)

$$\leq Q + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \epsilon ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{m} c_i \sum_{j=0}^{i-1} M^j \|u_k - \overline{u}_k\| ds.$$

Considering the derivations from equations (23) and (25), the inequality becomes;

$$\|w_k - \overline{w}_k\| \leq Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)} + \Delta\|u_k - \overline{u}_k\|, \tag{43}$$

Likewise, it is observed that

$$\begin{aligned}\|v_k - \overline{v}_k\| \leq \Delta\xi_k\Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big] + \Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big] \\ + \Delta\Big[1 - \xi_k(1-\Delta)\Big]\|u_k - \overline{u}_k\|.\end{aligned} \tag{44}$$

Using the ideas from (33) and (34), along with hypotheses $\Delta < 1$ and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$, the resultant inequality becomes

$$\begin{aligned}\|u_{k+1} - \overline{u}_{k+1}\| &= \|Tv_k - \overline{T}\overline{v}_k\| \\ &\leq Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)} + \Delta\|v_k - \overline{v}_k\| \\ &\leq \Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big] + \|v_k - \overline{v}_k\| \\ &\leq \Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big] + \Delta\xi_k\Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big] + \Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big] \\ &\quad + \Delta\Big[1 - \xi_k(1-\Delta)\Big]\|u_k - \overline{u}_k\| \\ &\leq 2\Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big] + \xi_k\Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big] + \Big[1 - \xi_k(1-\Delta)\Big]\|u_k - \overline{u}_k\| \\ &\leq 4\xi_k\Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big] + \xi_k\Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big] + \Big[1 - \xi_k(1-\Delta)\Big]\|u_k - \overline{u}_k\| \\ &\leq \Big[1 - \xi_k(1-\Delta)\Big]\|u_k - \overline{u}_k\| + \xi_k(1-\Delta)\frac{5\Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big]}{(1-\Delta)}. \end{aligned} \tag{45}$$

We denote

$$\begin{aligned}\beta_k &= \|u_k - \overline{u}_k\| \geq 0, \\ \mu_k &= \xi_k(1-\Delta) \in (0,1), \\ \gamma_k &= \frac{5\Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big]}{(1-\Delta)} \geq 0.\end{aligned}$$

The assumption $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily observed that (45) meets all the requirements of Lemma 3 and consequently we have

$$0 \leq \limsup_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \gamma_k$$

$$\Rightarrow 0 \leq \limsup_{k\to\infty} \|u_k - \overline{u}_k\| \leq \limsup_{k\to\infty} \frac{5\Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big]}{(1-\Delta)}$$

$$\Rightarrow 0 \leq \limsup_{k\to\infty} \|u_k - \overline{u}_k\| \leq \frac{5\Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big]}{(1-\Delta)}. \tag{46}$$

Using the assumptions $\lim_{k\to\infty} u_k = u$, $\lim_{k\to\infty} \overline{u}_k = \overline{u}$, we get from (46) that

$$\|u - \overline{u}\| \leq \frac{5\Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big]}{(1-\Delta)}. \tag{47}$$

It indicates,the dependence of the solution of IVP (1)–(2) on both the function associated with the given equation and initial data. □

Remark. The inequality (47) relates the solutions of the problems (1)–(2) and (37)–(38) in the sense that if $\mathcal{F}$ and $\overline{\mathcal{F}}$ are close as $\epsilon \to 0$, then the solutions of the problems (1)–(2) and (37)–(38) close to one another (i.e., $\|u - \overline{u}\| \to 0$), and continuously depends on the functions involved and the initial data.

4.3. Dependence on parameters. We next consider the following problems

$$({}^cD^\alpha_{0+})u(t) = \mathcal{F}\Big(u^{[0]}(t), u^{[1]}(t), u^{[2]}(t), \ldots, u^{[m]}(t), \mu_1\Big), \tag{48}$$

for $t \in J = [0, b]$, $n - 1 < \alpha \leq n$ $(n \in \mathbb{N})$, $m \in \mathbb{N} \cup \{0\}$, with the given initial conditions

$$u^{(j)}(0) = d_j, \quad j = 0, 1, 2, \ldots, n-1, \tag{49}$$

and

$$({}^cD^\alpha_{0+})\overline{u}(t) = \mathcal{F}\Big(\overline{u}^{[0]}(t), \overline{u}^{[1]}(t), \overline{u}^{[2]}(t), \ldots, \overline{u}^{[m]}(t), \mu_2\Big), \tag{50}$$

for $t \in J = [0, b]$, $n-1 < \alpha \leq n$ $(n \in \mathbb{N})$, $m \in \mathbb{N} \cup \{0\}$, with the given initial conditions

$$\overline{u}^{(j)}(a) = \overline{d}_j, \; j = 0, 1, 2, \ldots, n-1, \tag{51}$$

where $\mathcal{F} : [C_M(J, L)]^{m+1} \times \mathbb{R} \to C_M(J, L)$ is continuous function, d_j, $\overline{d}_j$ $(j = 0, 1, 2, \ldots, n-1)$ are given elements in $C_M(J, L)$ and μ_1, μ_2 are real parameters.

Let $u(t)$, $\overline{u}(t) \in C_M(J, L)$. We then define the operators for the equations (48) and (50), respectively as for $t \in J$,

$$\begin{aligned}(Tu)(t) &= \sum_{j=0}^{n-1} \frac{d_j}{j!}\, t^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}\Big(u^{[0]}(s), u^{[1]}(s), u^{[2]}(s), \ldots, u^{[m]}(s), \mu_1\Big) ds\end{aligned} \tag{52}$$

and

$$\begin{aligned}(\overline{T}\overline{u})(t) &= \sum_{j=0}^{n-1} \frac{\overline{d}_j}{j!}\, t^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}\Big(\overline{u}^{[0]}(s), \overline{u}^{[1]}(s), \overline{u}^{[2]}(s), \ldots, \overline{u}^{[m]}(s), \mu_2\Big) ds.\end{aligned} \tag{53}$$

The following theorem establishes the continuous dependence of solutions on parameters.

Theorem 6. *Consider the sequences* $\{u_k\}_{k=0}^{\infty}$ *and* $\{\overline{u}_k\}_{k=0}^{\infty}$ *generated by new three-step iteration method associated with operators* T *in* (52) *and* $\overline{T}$ *in* (53) *respectively with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in* $[0, 1]$ *satisfying* $\frac{1}{2} \leq \xi_k$ *for all* $k \in \mathbb{N} \cup \{0\}$. *Assume that*

(i) $u(t)$ *and* $\overline{u}(t)$ *are solutions of* (48)–(49) *and* (50)–(51) *respectively.*

(ii) *there exist constants* $\overline{c}_i$, r, $i = 0, 1, 2, \ldots, m$ *such that the function* $\mathcal{F}$ *satisfy the conditions:*

$$\begin{aligned}\Big\| \mathcal{F}\Big(u^{[0]}(t), u^{[1]}(t), u^{[2]}(t), \ldots, u^{[m]}(t), \mu_1\Big) \\ - \mathcal{F}\Big(\overline{u}^{[0]}(t), \overline{u}^{[1]}(t), \overline{u}^{[2]}(t), \ldots, \overline{u}^{[m]}(t), \mu_1\Big)\Big\| \leq \sum_{i=1}^{m} \overline{c}_i \|u^{[i]} - \overline{u}^{[i]}\|.\end{aligned}$$

and

$$\Big\|\mathcal{F}\Big(u^{[0]}(t),u^{[1]}(t),u^{[2]}(t),\ldots,u^{[m]}(t),\mu_1\Big) - \mathcal{F}\Big(u^{[0]}(t),u^{[1]}(t),u^{[2]}(t),\ldots,u^{[m]}(t),\mu_2\Big)\Big\| \le r|\mu_1-\mu_2|.$$

If the sequence $\{\overline{u}_k\}_{k=0}^{\infty}$ *converges to* $\overline{u}$*, then we have*

$$\|u-\overline{u}\| \le \frac{5\Big[Q+|\mu_1-\mu_2|\frac{rb^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\overline{\Delta}\Big)}, \tag{54}$$

where $\overline{\Delta} = \dfrac{b^\alpha}{\Gamma(\alpha+1)}\displaystyle\sum_{i=1}^{m}\overline{c}_i\sum_{j=0}^{i-1}M^j < 1.$

P r o o f. From iteration (6) and equations (52), (53), assumptions and Lemma 2, we obtain

$$\begin{aligned}
&\|w_k(t)-\overline{w}_k(t)\| \\
&= \|(Tu_k)(t)-(\overline{T}\overline{u}_k)(t)\| \\
&= \Big\|\sum_{j=0}^{n-1}\frac{d_j}{j!}\,t^j + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{F}\Big(u_k^{[0]}(s),u_k^{[1]}(s),u_k^{[2]}(s),\ldots,u_k^{[m]}(s),\mu_1\Big)ds \\
&\quad -\sum_{j=0}^{n-1}\frac{\overline{d}_j}{j!}\,t^j - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{F}\Big(\overline{u}_k^{[0]}(s),\overline{u}_k^{[1]}(s),\overline{u}_k^{[2]}(s),\ldots,\overline{u}_k^{[m]}(s),\mu_2\Big)ds\Big\| \\
&\le \sum_{j=0}^{n-1}\frac{\|d_j-\overline{d}_j\|}{j!}b^j + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\|\mathcal{F}\Big(u_k^{[0]}(s),u_k^{[1]}(s),u_k^{[2]}(s),\ldots,u_k^{[m]}(s),\mu_1\Big) \\
&\qquad - \mathcal{F}\Big(\overline{u}_k^{[0]}(s),\overline{u}_k^{[1]}(s),\overline{u}_k^{[2]}(s),\ldots,\overline{u}_k^{[m]}(s),\mu_2\Big)\Big\|ds \\
&\le Q + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\|\mathcal{F}\Big(u_k^{[0]}(s),u_k^{[1]}(s),u_k^{[2]}(s),\ldots,u_k^{[m]}(s),\mu_1\Big) \\
&\qquad - \mathcal{F}\Big(\overline{u}_k^{[0]}(s),\overline{u}_k^{[1]}(s),\overline{u}_k^{[2]}(s),\ldots,\overline{u}_k^{[m]}(s),\mu_1\Big)\Big\|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\|\mathcal{F}\Big(\overline{u}_k^{[0]}(s),\overline{u}_k^{[1]}(s),\overline{u}_k^{[2]}(s),\ldots,\overline{u}_k^{[m]}(s),\mu_1\Big) \\
&\qquad - \mathcal{F}\Big(\overline{u}_k^{[0]}(s),\overline{u}_k^{[1]}(s),\overline{u}_k^{[2]}(s),\ldots,\overline{u}_k^{[m]}(s),\mu_2\Big)\Big\|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq Q + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} r|\mu_1 - \mu_2| ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{m} \overline{c}_i \big\| u_k^{[i]} - \overline{u}_k^{[i]} \big\| ds \\
&\leq Q + |\mu_1 - \mu_2| \frac{rb^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{m} \overline{c}_i \sum_{j=0}^{i-1} M^j \|u_k - \overline{u}_k\| ds
\end{aligned} \tag{55}
$$

Recalling the derivations obtained in equations (31) and (32), the above inequality becomes

$$
\|w_k - \overline{w}_k\| \leq Q + |\mu_1 - \mu_2| \frac{rb^\alpha}{\Gamma(\alpha+1)} + \overline{\Delta} \|u_k - \overline{u}_k\|, \tag{56}
$$

and similarly, it is seen that

$$
\begin{aligned}
\|v_k - \overline{v}_k\| \leq \overline{\Delta} \xi_k \Big[Q + |\mu_1 - \mu_2| \frac{rb^\alpha}{\Gamma(\alpha+1)} \Big] + \Big[Q + |\mu_1 - \mu_2| \frac{rb^\alpha}{\Gamma(\alpha+1)} \Big] \\
+ \overline{\Delta} \Big[1 - \xi_k \Big(1 - \overline{\Delta} \Big) \Big] \|u_k - \overline{u}_k\|.
\end{aligned} \tag{57}
$$

Therefore, using the idea from (33) and (34) along with hypotheses $\overline{\Delta} < 1$, and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$, the resulting inequality becomes

$$
\begin{aligned}
\|u_{k+1} - \overline{u}_{k+1}\| &= \|Tv_k - \overline{T}\overline{v}_k\| \\
&\leq \Big[Q + |\mu_1 - \mu_2| \frac{rb^\alpha}{\Gamma(\alpha+1)} \Big] + \overline{\Delta} \|v_k - \overline{v}_k\| \\
&\leq \Big[Q + |\mu_1 - \mu_2| \frac{rb^\alpha}{\Gamma(\alpha+1)} \Big] + \|v_k - \overline{v}_k\| \\
&\leq \Big[1 - \xi_k \Big(1 - \overline{\Delta} \Big) \Big] \|u_k - \overline{u}_k\| + \xi_k \Big(1 - \overline{\Delta} \Big) \frac{5 \Big[Q + |\mu_1 - \mu_2| \frac{rb^\alpha}{\Gamma(\alpha+1)} \Big]}{\Big(1 - \overline{\Delta} \Big)}.
\end{aligned} \tag{58}
$$

We denote

$$
\begin{aligned}
\beta_k &= \|u_k - \overline{u}_k\| \geq 0, \\
\mu_k &= \xi_k \Big(1 - \overline{\Delta} \Big) \in (0,1), \\
\gamma_k &= \frac{5 \Big[Q + |\mu_1 - \mu_2| \frac{rb^\alpha}{\Gamma(\alpha+1)} \Big]}{\Big(1 - \overline{\Delta} \Big)} \geq 0.
\end{aligned}
$$

The assumption $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (58) satisfies all the conditions of Lemma 3 and hence we have

$$0 \leq \limsup_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \gamma_k$$

$$\Rightarrow 0 \leq \limsup_{k\to\infty} \|u_k - \overline{u}_k\| \leq \limsup_{k\to\infty} \frac{5\Big[Q + |\mu_1 - \mu_2| \frac{rb^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1 - \overline{\Delta}\Big)}$$

(59) $$\Rightarrow 0 \leq \limsup_{k\to\infty} \|u_k - \overline{u}_k\| \leq \frac{5\Big[Q + |\mu_1 - \mu_2| \frac{rb^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1 - \overline{\Delta}\Big)}.$$

Using the assumptions, $\lim_{k\to\infty} u_k = u$, $\lim_{k\to\infty} \overline{u}_k = \overline{u}$, we get from (59) that

(60) $$\|u - \overline{u}\| \leq \frac{5\Big[Q + |\mu_1 - \mu_2| \frac{rb^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1 - \overline{\Delta}\Big)},$$

which shows the dependence of solutions of the problem (1)–(2) on parameters μ_1 and μ_2. □

Remark. The result deals with the property of a solution called "dependence of solutions on parameters". Here the parameters are scalars and also note that the initial conditions do not involve parameters. The dependence on parameters is an important aspect in various physical problems.

5. Example.

Consider the following problem:

(61) $$\left({}^cD_{0+}^{\frac{1}{2}}\right)u(t) = \frac{1}{2} + \frac{t}{3} + \frac{t}{57}\sin(t)u^{[1]}(t) + \frac{t^2}{99}\cos(t)u^{[2]}(t),$$
$$t \in [0,1],\ 0 < \alpha = \frac{1}{2} \leq 1,$$

with the given initial condition

(62) $$u(0) = 0.$$

Comparing this equation with the equation (1), we have $d_0 = 0$ and

$$\mathcal{F}\Big(u^{[0]}(t), u^{[1]}(t), u^{[2]}(t)\Big) = \frac{1}{2} + \frac{t}{3} + \frac{t}{57}\sin(t)u^{[1]}(t) + \frac{t^2}{99}\cos(t)u^{[2]}(t).$$

Now, we define the operator for the IVP (61)–(62) as follows:

(63)
$$(Tu)(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}}\Big[\frac{1}{2}+\frac{t}{3}+\frac{t}{57}\sin(t)u^{[1]}(t)+\frac{t^2}{99}\cos(t)u^{[2]}(t)\Big]ds.$$

5.1. Existence and uniqueness. We have

$$\begin{aligned}
&\Big|\mathcal{F}\Big(u^{[0]}(t),u^{[1]}(t),u^{[2]}(t)\Big)-\mathcal{F}\Big(\overline{u}^{[0]}(t),\overline{u}^{[1]}(t),\overline{u}^{[2]}(t)\Big)\Big|\\
&\le \Big|\Big[\frac{1}{2}+\frac{t}{3}+\frac{t}{57}\sin(t)u^{[1]}(t)+\frac{t^2}{99}\cos(t)u^{[2]}(t)\Big]\\
&\qquad -\Big[\frac{1}{2}+\frac{t}{3}+\frac{t}{57}\sin(t)\overline{u}^{[1]}(t)+\frac{t^2}{99}\cos(t)\overline{u}^{[2]}(t)\Big]\Big|\\
&\le \Big|\frac{t}{57}\Big||\sin(t)|\Big|u^{[1]}-\overline{u}^{[1]}\Big|+\Big|\frac{t^2}{99}\Big||\cos(t)|\Big|u^{[2]}-\overline{u}^{[2]}\Big|\\
&\le \frac{1}{57}\Big|u^{[1]}-\overline{u}^{[1]}\Big|+\frac{1}{99}\Big|u^{[2]}-\overline{u}^{[2]}\Big|. \qquad (64)
\end{aligned}$$

Thus, we have $c_1 = \frac{1}{57}$ and $c_2 = \frac{1}{99}$.
Furthermore, for $L = 1$, and $M = 5$;

$$\begin{aligned}
\Delta = \frac{b^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{m} c_i\sum_{j=0}^{i-1} M^j &= \frac{1^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{2} c_i\sum_{j=0}^{i-1} 5^j\\
&= \frac{1}{\Gamma(\alpha+1)}\Big[\frac{1}{57}(1)+\frac{1}{99}(1+5)\Big]\\
&= \frac{1}{\Gamma(\alpha+1)}\Big[\frac{1}{57}+\frac{2}{33}\Big]\\
&= \frac{49}{627\times\Gamma(\alpha+1)}
\end{aligned}$$

For $\alpha = \frac{1}{2}$

$$\begin{aligned}
\Delta &= \frac{49}{627\times 0.8862269}\\
&\approx 0.0881827419 < 1. \qquad (65)
\end{aligned}$$

The sequence $\{u_k\}$ associated with the iterative process (6) for the operator T in (63) converges to a unique solution $u \in C_M(J,L)$, as every requirement of Theorem 3 are satisfied.

We now discuss the rate of convergence for the Picard, Mann, Ishikawa, S-iteration, and new three-step iteration processes. Referring to [3, 4, 18, 20, 30] for definitions of a_k, b_k, c_k, d_k, and e_k under S-iteration, Picard iteration, Mann iteration, Ishikawa iteration, and a new three-step iteration.

(a) $a_k = \theta^k \Big[1 - (1-\theta)\alpha\beta\Big]^k \|u_1 - x^*\|$,

(b) $b_k = \theta^k \|u_1 - x^*\|$,

(c) $c_k = \Big[1 - (1-\theta)\beta\Big]^k \|u_1 - x^*\|$,

(d) $d_k = \Big[1 - (1-\theta)^2\beta\Big]^k \|u_1 - x^*\|$,

(e) $e_k = \theta^{2k} \Big[1 - (1-\theta)\alpha\beta\Big]^k \|u_1 - x^*\|$,

where $\theta \in [0,1)$ is contracting factor. For given $u_1 \in \mathbb{R}$, the convergence of sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$, $\{d_k\}$ and $\{e_k\}$ depends only on the factors $\Delta_1 = \theta^k \Big[1 - (1-\theta)\alpha\beta\Big]^k$, $\Delta_2 = \theta^k$, $\Delta_3 = \Big[1 - (1-\theta)\beta\Big]^k$, $\Delta_4 = \Big[1 - (1-\theta)^2\beta\Big]^k$ and

Iteration (k)	S-iteration (Δ_1)	P-iteration (Δ_2)	M-iteration (Δ_3)	I-iteration (Δ_4)	3 steps-iteration (Δ_5)
1	0.0680811054	0.0881827419	0.5440913710	0.5842946439	0.0060035785
2	0.0046350369	0.0077761960	0.2960354199	0.3414002309	0.0000360430
3	0.0003155584	0.0006857263	0.1610703175	0.1994783264	0.0000002164
4	0.0000214836	0.0000604692	0.0876369699	0.1165541177	0.0000000013
5	0.0000014626	0.0000053323	0.0476825191	0.0681019467	**0.0000000000**
6	0.0000000996	0.0000004702	0.0259436472	0.0397916027	0
7	0.0000000068	0.0000000415	0.0141157146	0.0232500203	0
8	0.0000000005	0.0000000037	0.0076802385	0.0135848623	0
9	**0.0000000000**	0.0000000003	0.0041787515	0.0079375623	0
10	0	**0.0000000000**	0.0022736226	0.0046378751	0
⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮
38	0	0	0.0000000001	0.0000000014	0
39	0	0	**0.0000000000**	0.0000000008	0
40	0	0	0	0.0000000005	0
41	0	0	0	0.0000000003	0
42	0	0	0	0.0000000002	0
43	0	0	0	0.0000000001	0
44	0	0	0	0.0000000001	0
45	0	0	0	**0.0000000000**	0
46	0	0	0	0	0

$\Delta_5 = \theta^{2k}\Big[1-(1-\theta)\alpha\beta\Big]^k$ respectively. The above table compares the values of the factors. Δ_1, Δ_2, Δ_3, Δ_4 and Δ_5 under respective iteration processes, with $\theta = \Delta = 0.0881827419$ and $\alpha = \beta = \dfrac{1}{2}$:

The above table and definitions (4, 5) clearly show that $\lim\limits_{k\to\infty} \dfrac{e_k}{a_k} = 0$, $\lim\limits_{k\to\infty} \dfrac{e_k}{b_k} = 0$, $\lim\limits_{k\to\infty} \dfrac{e_k}{c_k} = 0$ and $\lim\limits_{k\to\infty} \dfrac{e_k}{d_k} = 0$. We found that the new three-step iteration approach is faster than the S-iteration, Picard, Mann, and Ishikawa iterations.

Remark. From the data obtained, we observe that the S-iteration process converges to a fixed point at 9^{th} step, the Picard iteration process at 10^{th} step, the Mann iteration process at 39^{th} step, the Ishikawa iteration process at 45^{th} step, and the New Three-Step iteration process at 5^{th} step.

The graphical representations of the above table are shown in Fig. 1.

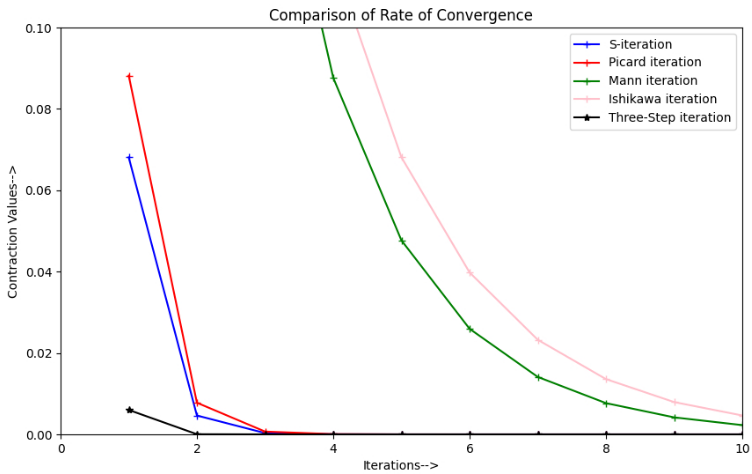

Fig. 1. Comparison of the Rate of Convergence

5.2. Error estimate. The error estimate for every $u_0 \in C_M(J, L)$ using the three-step iteration technique is as follows:

$$\begin{aligned}\|u_{k+1} - u\| &\le \frac{\Delta^{2k+2}}{e^{(1-\Delta)\sum_{i=0}^{k}\xi_i}}\|u_0 - u\| \\ &\le \frac{\left[\frac{49}{627\times\Gamma(\alpha+1)}\right]^{2k+2}}{e^{\left[1-\frac{49}{627\times\Gamma(\alpha+1)}\right]\sum_{i=0}^{k}\xi_i}}\|u_0 - u\|\end{aligned}$$

$$\leq \frac{\left[\frac{49}{627\times\Gamma(\alpha+1)}\right]^{2k+2}}{e^{\left[1-\frac{49}{627\times\Gamma(\alpha+1)}\right]\sum_{i=0}^{k}\frac{1}{1+i}}}\|u_0-u\|, \tag{66}$$

Recalling the Theorem 3, the error estimates for various iterative techniques can be obtained as below:
Picard iteration:

$$\begin{aligned}\|u_{k+1}-u\| &\leq \Delta^{k+1}\|u_0-u\| \\ &\leq \left(\frac{49}{627\times\Gamma(\alpha+1)}\right)^{k+1}\|u_0-u\|.\end{aligned}$$

Mann iteration:

$$\begin{aligned}\|u_{k+1}-u\| &\leq \frac{1}{e^{(1-\Delta)\sum_{i=0}^{k}\xi_i}}\|u_0-u\| \\ &\leq \frac{1}{e^{\left(1-\frac{49}{627\times\Gamma(\alpha+1)}\right)\sum_{i=0}^{k}\xi_i}}\|u_0-u\| \\ &\leq \frac{1}{e^{\left(1-\frac{49}{627\times\Gamma(\alpha+1)}\right)\sum_{i=0}^{k}\frac{1}{1+i}}}\|u_0-u\|\end{aligned}$$

Ishikawa iteration:

$$\begin{aligned}\|u_{k+1}-u\| &\leq \frac{1}{e^{(1-\Delta)\sum_{i=0}^{k}\xi_i(1-\eta_i)}}\|u_0-u\| \\ &\leq \frac{1}{e^{\left(1-\frac{49}{627\times\Gamma(\alpha+1)}\right)\sum_{i=0}^{k}\xi_i(1-\eta_i)}}\|u_0-u\| \\ &\leq \frac{1}{e^{\left(1-\frac{49}{627\times\Gamma(\alpha+1)}\right)\sum_{i=0}^{k}\frac{1}{1+i}\left(1-\frac{1}{2+i}\right)}}\|u_0-u\|\end{aligned}$$

Normal S-iteration:

$$\begin{aligned}\|u_{k+1}-u\| &\leq \frac{\Delta^{k+1}}{e^{(1-\Delta)\sum_{i=0}^{k}\xi_i}}\|u_0-u\| \\ &\leq \frac{\left(\frac{49}{627\times\Gamma(\alpha+1)}\right)^{k+1}}{e^{\left(1-\frac{49}{627\times\Gamma(\alpha+1)}\right)\sum_{i=0}^{k}\frac{1}{1+i}}}\|u_0-u\| \\ &\leq \frac{\left(\frac{49}{627\times\Gamma(\alpha+1)}\right)^{k+1}}{e^{\left(1-\frac{49}{627\times\Gamma(\alpha+1)}\right)\sum_{i=0}^{k}\frac{1}{1+i}}}\|u_0-u\|\end{aligned}$$

where we have chosen $\xi_i = \dfrac{1}{1+i}, \eta_i = \dfrac{1}{2+i} \in [0,1]$. The estimate in (66) constitutes an error bound (due to computation truncation at the k^{th} iteration).

5.3. Continuous dependence. For $d_0 = 0$, $\overline{d}_0 = \dfrac{1}{2}$, we have

$$\begin{aligned}
\|u - \overline{u}\| &\le \frac{5Q}{(1-\Delta)} \\
&\le \frac{5\sum_{j=0}^{n-1} \frac{\|d_j - \overline{d}_j\|}{j!}\; b^j}{(1-\Delta)} \\
&\le \frac{\frac{5}{2}}{\Big(1 - 0.0881827419\Big)} \\
&\simeq 2.7417774536. \qquad (67)
\end{aligned}$$

This shows the dependence of solutions of the equation (61) on given data.

5.4. Closeness of solutions. Next, we consider the perturbed equation:

$$(68) \qquad ({}^{c}D^{\frac{1}{2}}_{0^+})\overline{u}(t) = \frac{1}{2} + \frac{t}{3} + \frac{t^2}{7} + \frac{t}{57}\sin(t)\overline{u}^{[1]}(t) + \frac{t^2}{99}\cos(t)\overline{u}^{[2]}(t),$$
$$t \in [0,1],\ 0 < \alpha = \frac{1}{2} \le 1,$$

with the given initial conditions

$$(69) \qquad \overline{u}(0) = 0.$$

Similarly, comparing it with the equation (37), we have

$$\overline{\mathcal{F}}\Big(\overline{u}^{[0]}(t), \overline{u}^{[1]}(t), \overline{u}^{[2]}(t)\Big) = \frac{1}{2} + \frac{t}{3} + \frac{t^2}{7} + \frac{t}{57}\sin(t)\overline{u}^{[1]}(t) + \frac{t^2}{99}\cos(t)\overline{u}^{[2]}(t).$$

Define the mapping $\overline{T} : B \to B$ by

(70)
$$(\overline{T}\overline{u})(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}}\Big[\frac{1}{2} + \frac{s}{3} + \frac{s^2}{7} + \frac{s}{57}\sin(s)\overline{u}^{[1]}(s) + \frac{t^2}{99}\cos(s)\overline{u}^{[2]}(s)\Big]ds.$$

All the conditions of Theorem 3 are also met in the perturbed equation; consequently,the sequence $\{\overline{u}_k\}$ generated bythe new three-step iterative method (6)

for the operator $\overline{T}$ in (70) converges to a unique solution $\overline{u} \in C_M(J, L)$.
Now, we have the following estimate:

$$\begin{aligned}
&\left|\mathcal{F}\Big(u^{[0]}(t), u^{[1]}(t), u^{[2]}(t)\Big) - \overline{\mathcal{F}}\Big(u^{[0]}(t), u^{[1]}(t), u^{[2]}(t)\Big)\right| \\
&= \bigg|\Big[\frac{1}{2} + \frac{t}{3} + \frac{t}{57}\sin(t)u^{[1]}(t) + \frac{t^2}{99}\cos(t)u^{[2]}(t)\Big] \\
&\quad - \Big[\frac{1}{2} + \frac{t}{3} + \frac{t^2}{7} + \frac{t}{57}\sin(t)u^{[1]}(t) + \frac{t^2}{99}\cos(t)u^{[2]}(t)\Big]\bigg| \\
&= \left|\frac{t^2}{7}\right| \\
&\le \frac{1}{7} = \epsilon \qquad (t \le 1).
\end{aligned} \tag{71}$$

Consider the sequences $\{u_k\}_{k=0}^{\infty}$. with $u_k \to u$ as $k \to \infty$ and $\{\overline{u}_k\}_{k=0}^{\infty}$ with $\overline{u}_k \to \overline{u}$. The real sequence $\{\xi_k\}_{k=0}^{\infty}$ is constructed via a new three-step iteration approach associated with operators T in (63) and $\overline{T}$ in (70) as $k \to \infty$ in $[0, 1]$ satisfying $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. Then Theorem 5 gives that

$$\begin{aligned}
\|u - \overline{u}\| &\le \frac{5\Big[Q + \frac{\epsilon b^\alpha}{\Gamma(\alpha+1)}\Big]}{(1 - \Delta)} \\
&\le \frac{\frac{5}{2} + 5 \times \frac{1}{7}\frac{1}{\Gamma(\frac{3}{2})}}{(1 - 0.0881827419)} \\
&\le \frac{\frac{5}{2} + \frac{10}{7 \times \sqrt{\pi}}}{0.9118172581} \\
&\le \frac{3.3059851194}{0.9118172581} \\
&\simeq 3.6257101849.
\end{aligned} \tag{72}$$

This demonstrates the closeness and dependency of solutions on the initial data and the functions involved.

5.5. Dependence on parameters. At last, we will demonstrate how solutions depend on real parameters.

We consider the following differential equations involving real parameters μ_1 and μ_2:

$$({}^cD_{0+}^{\frac{1}{2}})u(t) = \frac{1}{2} + \frac{t}{3} + \frac{t}{57}\sin(t)u^{[1]}(t) + \frac{t^2}{99}\cos(t)u^{[2]}(t) + \mu_1, \tag{73}$$

$$({}^{c}D_{0^+}^{\frac{1}{2}})\overline{u}(t) = \frac{1}{2} + \frac{t}{3} + \frac{t}{57}\sin(t)\overline{u}^{[1]}(t) + \frac{t^2}{99}\cos(t)\overline{u}^{[2]}(t) + \mu_2, \tag{74}$$

$$t \in [0,1],\ 0 < \alpha = \frac{1}{2} \leq 1.$$

The above discussion shows that $\overline{c}_1 = \dfrac{1}{57}, \overline{c}_2 = \dfrac{1}{99}, r = 1$ and thus, $\Delta = \overline{\Delta}$. Using similar arguments and Theorem 6, we have

$$\begin{aligned}
\|u - \overline{u}\| &\leq \frac{5\left[Q + |\mu_1 - \mu_2|\frac{rb^\alpha}{\Gamma(\alpha+1)}\right]}{\left(1 - \overline{\Delta}\right)}, \\
&\leq \frac{5\left[\frac{1}{2} + |\mu_1 - \mu_2|\frac{1}{\Gamma(\frac{1}{2}+1)}\right]}{\left(1 - 0.0881827419\right)}, \\
&\leq \frac{\left[\frac{5}{2} + |\mu_1 - \mu_2|\frac{10}{\sqrt{\pi}}\right]}{0.9118172581}.
\end{aligned} \tag{75}$$

In particular, if we choose $\mu_1 = \dfrac{1}{4}$ and $\mu_2 = \dfrac{1}{6}$, then the above inequality (75) becomes

$$\begin{aligned}
\|u - \overline{u}\| &\leq \frac{\left[2.5 + 5.6418958355 \times |\frac{1}{4} - \frac{1}{6}|\right]}{0.9118172581} \\
&\leq \frac{2.9701579863}{0.9118172581} \\
&\simeq 3.25740488
\end{aligned} \tag{76}$$

6. Conclusions. At first, we used a new three-step iterative technique to show that the solution to the IVP (1)–(2) exists and is unique. We also investigated some qualitative characteristics of the solutions, such as their continuous dependency on initial data, closeness, and dependence on parameters and functions. At last, we supplied an appropriate example for confirming all of the results. Also provide the comparison table and graphical representation, suggesting that a new three-step iteration process has a faster rate of convergence than S-iteration, Picard, Mann, and Ishikawa iterations.

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