Serdica

Mathematical Journal

Сердика

Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://serdica.math.bas.bg
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **50** (2024), 345–364

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

UNIQUENESS OF A POWER OF AN ENTIRE FUNCTION WITH ITS DIFFERENTIAL-DIFFERENCE POLYNOMIALS SHARING A VALUE CM*

Soumon Roy, Sudip Saha

Communicated by M. Savov

Abstract. In this paper, we investigate the scenario when power of a transcendental entire function shares a value CM with its differential-difference polynomial. Our result extend and generalize a recent result of Adud and Chakraborty [1].

1. Introduction. Throughout this paper, we assume that the reader has a basic understanding of the fundamental concepts and key theorems of Nevanlinna theory [12]. A meromorphic function $\mathfrak{f}$ can be either analytic or have, at most a countable number of poles in the complex plane. If $\mathfrak{f}$ has no poles, it is equivalent to an entire function. The order and hyper order of a

https://doi.org/10.55630/serdica.2024.50.345-364

2020 *Mathematics Subject Classification:* 30D30, 30D20, 30D35.
Key words: value sharing, weighted sharing, shift, difference polynomial.
*The research work of Mr. Soumon Roy is supported by the Department of Higher Education, Science and Technology & Biotechnology, Govt. of West Bengal under the sanction of order no. 1303(sanc.) STBT-11012(26)/17/2021-ST SEC dated 14/03/2022.

meromorphic function $\mathfrak{f}$ are represented by $\rho(\mathfrak{f})$ and $\rho_1(\mathfrak{f})$ respectively. The hyper order is defined as follow:

$$\rho_1(\mathfrak{f}) := \limsup_{r\to\infty} \frac{\log\log T(r,\mathfrak{f})}{\log r}.$$

In the context of non-constant meromorphic functions $\mathfrak{f}$ and $\mathfrak{g}$ defined on the complex plane $\mathbb{C}$, with $b \in \mathbb{C}\cup\{\infty\}$. When we say that $\mathfrak{f}$ and $\mathfrak{g}$ share the value b CM (counting multiplicities), we mean that the equations $\mathfrak{f}(z) = b$ and $\mathfrak{g}(z) = b$ have the same zeros at points z in $\mathbb{C}$ with corresponding multiplicities considered. This implies that the zero sets of $\mathfrak{f} - b$ and $\mathfrak{g} - b$ coincide in a precise manner, maintaining the order of each zero. Conversely, if $\mathfrak{f}$ and $\mathfrak{g}$ share the value b IM (ignoring multiplicities), it means that while the zeros of $\mathfrak{f} - b$ and $\mathfrak{g} - b$ coincide as sets, the multiplicities of these zeros are not taken into account. A "*shift* of $\mathfrak{f}(z)$" refers to the function $\mathfrak{f}(z + d)$, where d is non zero constant. A polynomial that involves $\mathfrak{f}(z)$, it's derivatives, or shift is termed a "differential-difference polynomial". A complex homogeneous differential-difference polynomials of $\mathfrak{f}(z)$ can be written as

$$\omega(\mathfrak{f}) = \sum_{v=1}^{t} d_v\big(\mathfrak{f}^{(v)}(z+\tau_v)\big)^n + \sum_{k=1}^{s} l_k(\mathfrak{f}(z+\sigma_k))^n,$$

where $n \in \mathbb{N}$, $t,\ s \in \mathbb{N}$ and $\tau_v,\ \sigma_k$ are the complex constants and $d_v,\ l_k$ are non-zero constants.

In the uniqueness theory of meromorphic functions, Rolf Nevanlinna established two key results: the Five Value Theorem and the Four Value Theorem. The five value theorem asserts that if two non-constant meromorphic functions $\mathfrak{f}$ and $\mathfrak{g}$ share five distinct values on the extended complex plane IM, then $\mathfrak{f} \equiv \mathfrak{g}$. Likewise *if two meromorphic functions* $\mathfrak{f}$ *and* $\mathfrak{g}$ *share four distinct values on the extended complex plane CM, then* $\mathfrak{f} \equiv T \circ \mathfrak{g}$*, where* T *is a Möbius transformation.*

Later, in [8], Gundersen improved 4 CM present in the four value theorem to 2 CM + 2 IM. Moreover, Gundersen [7], showed that 4 CM cannot be improved to 4 IM, while 1 CM + 3 IM remains an open problem till today.

For the uniqueness of the entire functions, if we consider a special situation where $\mathfrak{g}$ is the first derivative of $\mathfrak{f}$, one usually needs sharing of only two values CM for their uniqueness. In 1977, Rubel and Yang [18] first showed that *if a non-constant entire function* $\mathfrak{f}$ *and its derivative* $\mathfrak{f}'$ *share two distinct values* a, b *CM, then* $\mathfrak{f} \equiv \mathfrak{f}'$.

In 1979, Mues and Steinmetz [17] observed that in Rubel and Yang's result, the CM sharing can be further relaxed to IM sharing. They proved that *if a non-constant entire function* $\mathfrak{f}$ *and its derivative* $\mathfrak{f}'$ *share two distinct values* a*,* b *IM, then* $\mathfrak{f} \equiv \mathfrak{f}'$. It is well known that in Rubel and Yang's result, the two value sharing can not be further relaxed. We recall the following example. Let

$$\mathfrak{h}(z) = e^{e^z} \int_0^z e^{-e^t}(1 - e^t)dt.$$

Here, one can check that $\mathfrak{h}$ and $\mathfrak{h}'$ share 1 CM, but $(\mathfrak{h}' - 1) = e^z(\mathfrak{h} - 1)$. In this connection, we recall a famous conjecture proposed by R. Brück [2].

Conjecture. *Let* $\mathfrak{h}$ *be an entire function and*

$$\rho_1(\mathfrak{h}) := \limsup_{r\to\infty} \frac{\log\log T(r, \mathfrak{h})}{\log r}$$

be the hyper-order of $\mathfrak{h}$ *such that* $\rho_1(\mathfrak{h}) < \infty$ *and is not a positive integer. Let* $d \in \mathbb{C}$*. If* $\mathfrak{h}$ *and* $\mathfrak{h}'$ *share the value* d *CM, then*

$$\frac{\mathfrak{h}' - d}{\mathfrak{h} - d} = k,$$

where k *is a non-zero constant.*

We have now cover some fundamental aspects of value sharing principles in Nevanlinna theory [12].

Definition 1.1. *Let* f *be an entire function. If there exists a positive number* ρ *such that* $|f(z)| < e^{r^\rho}$ *for* $|z| = r > r_0$*. If this inequality is true for a certain* ρ *then it is true for* $\rho' > \rho$*, thus there exists an infinite number of* $\rho > 0$ *satisfying the inequality. The lower bound of these* ρ *is called the "order of* f*".*

The order of meromorphic function $\mathfrak{f}$ *is denoted by* $\rho(\mathfrak{f})$ *and defined as follow:*

$$\rho(\mathfrak{f}) := \limsup_{r\to\infty} \frac{\log T(r, \mathfrak{f})}{\log r}.$$

Definition 1.2. *We denote by* $N(r, d; \mathfrak{f}| = 1)$ *the counting function of simple* d *points of* $\mathfrak{f}$*.*

Definition 1.3. *We denote by* $N(r, \infty; \mathfrak{f}| = 1)$ *the counting function of simple poles of* $\mathfrak{f}$*.*

Definition 1.4. *If* n *be a positive integer, we denote by* $\overline{N}(r, d; \mathfrak{f}| \geq n)$ *the counting function of these* d *points of* $\mathfrak{f}$ *whose multiplicities are greater than or equal to* n*, where each* d *point is counted only one time.*

Definition 1.5. *If n be a positive integer, we denote by $\overline{N}(r,\infty;\mathfrak{f}|\geq n)$ the counting function of poles of $\mathfrak{f}$ whose multiplicities are greater than or equal to n, where each pole is counted only one time.*

Definition 1.6. *Let $\mathfrak{f}$ and $\mathfrak{g}$ share the value d IM. We denote by $\overline{N_*}(r,d;\mathfrak{f},\mathfrak{g})$ the counting function of these d points of $\mathfrak{f}$ whose multiplicities are not equal to the multiplicities of the corresponding d points of $\mathfrak{g}$, where each d points are counted only one time.*

Obviously $\overline{N_*}(r,d;\mathfrak{f},\mathfrak{g})\equiv\overline{N_*}(r,d;\mathfrak{g},\mathfrak{f})$.

Definition 1.7. *Let $\mathfrak{f}$ and $\mathfrak{g}$ share the value d IM. We denote by $\overline{N_*}(r,\mathfrak{f},\mathfrak{g})$ the counting function of poles of $\mathfrak{f}$ whose multiplicities are not equal to the multiplicities of the corresponding poles of $\mathfrak{g}$, where each poles are counted only one time.*

Obviously $\overline{N_*}(r,\mathfrak{f},\mathfrak{g})\equiv\overline{N_*}(r,\mathfrak{g},\mathfrak{f})$.

Definition 1.8. *$N_2(r,d;\mathfrak{f})$ is defined by*

$$N_2(r,d;\mathfrak{f})=\overline{N}(r,d;\mathfrak{f})+\overline{N}(r,d;\mathfrak{f}|\geq 2).$$

We now introduce the concept of weighted value sharing pertaining to meromorphic function.

Definition 1.9. *Let n be a non negative integer or infinity. We denote $E_n(d,\mathfrak{f})$ the set of all d points of $\mathfrak{f}$ where a d point of multiplicity p is counted p times if $p\leq n$ and is counted $p+1$ times if $p>n$, for $d\in\mathbb{C}\cup\{\infty\}$.*

If $E_n(d,\mathfrak{f})=E_n(d,\mathfrak{g})$, then we say that $\mathfrak{f}$, $\mathfrak{g}$ share the value d with weight n. In other words if $\mathfrak{f}$, $\mathfrak{g}$ share the value d with weight n, then z_0 is a zero of $\mathfrak{f}-d$ with multiplicity p $(\leq n)$ if and only if z_0 is a zero of $\mathfrak{g}-d$ with multiplicity p $(\leq n)$. Again z_0 is a zero of $\mathfrak{f}-d$ with multiplicity p $(>n)$ if and only if z_0 is a zero of $\mathfrak{g}-d$ with multiplicity q $(>n)$ where p need not be equal to q.

So it can be written as if $\mathfrak{f}$, $\mathfrak{g}$ share (d,n) this means that $\mathfrak{f}$, $\mathfrak{g}$ share the value d with weight n.

Obviously if $\mathfrak{f}$, $\mathfrak{g}$ share (d,n) then $\mathfrak{f}$, $\mathfrak{g}$ share (d,m) for any integer m, $0\leq m<n$. Also we denote $\mathfrak{f}$, $\mathfrak{g}$ share the value d IM or CM if and only if $\mathfrak{f}$, $\mathfrak{g}$ share $(d,0)$ or (d,∞), respectively.

Definition 1.10 ([21]). *For a positive integer m and $d\in\mathbb{C}\cup\{\infty\}$, we have*

$$\delta_m(d;\mathfrak{f})=1-\limsup_{r\to\infty}\frac{N_m(r,d;\mathfrak{f})}{T(r,\mathfrak{f})}$$

and

$$\theta(d;\mathfrak{f}) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,d;\mathfrak{f})}{T(r,\mathfrak{f})}.$$

Clearly $0 \leq \delta(d,\mathfrak{f}) \leq \delta_m(d;\mathfrak{f}) \leq \delta_{m-1}(d;\mathfrak{f}) \leq \cdots \leq \delta_2(d;\mathfrak{f}) \leq \delta_1(d;\mathfrak{f}) = \theta(d;\mathfrak{f})$.

Brück himself verified the conjecture for $d = 0$ [2] and later Gundersen proved that the conjecture is true for finite order entire functions [11]. Recently, many researchers put their attention to consider the complex difference equations and the uniqueness of transcendental entire functions sharing values with their shifts. Using difference analogues of logarithmic derivative lemma, Heittokangas et al. established the following theorems:

Theorem A ([13]). *Let* $\mathfrak{h}$ *be a non constant meromorphic function such that its order of growth*

$$\rho(\mathfrak{h}) := \limsup_{r\to\infty} \frac{\log T(r,\mathfrak{h})}{\log r} < 2,$$

and let η *be a nonzero complex number and* $d \in \mathbb{C}$*. If* $\mathfrak{h}(z)$ *and* $\mathfrak{h}(z+\eta)$ *share the values* d *CM and* ∞ *CM, then*

$$\frac{\mathfrak{h}(z+\eta) - d}{\mathfrak{h}(z) - d} = k,$$

where k *is a non-zero constant.*

In the same paper [13], Heittokangas et al. provided the example $\mathfrak{h}(z) = e^{z^2} + 1$, which shows that $\rho(\mathfrak{h}) < 2$ cannot be relaxed to $\rho(\mathfrak{h}) \leq 2$.

Let $\mathfrak{h}(z)$ be a nonconstant meromorphic function and η be a nonzero complex constant. Then $\mathfrak{h}(z+\eta)$ is called the shift of $\mathfrak{h}(z)$. Also, $\Delta\mathfrak{h}(z) = \mathfrak{h}(z+\eta) - \mathfrak{h}(z)$ is called the difference operator of $\mathfrak{h}(z)$. Moreover,

$\Delta_\eta^0\mathfrak{h}(z) := \mathfrak{h}(z), \Delta_\eta^1\mathfrak{h}(z) := \Delta\mathfrak{h}(z)$, and $\Delta_\eta^k\mathfrak{h}(z) := \Delta_\eta^{k-1}(\Delta_\eta^1(\mathfrak{h}(z)))$, for $k \in \mathbb{N}$, $k \geq 2$.

In [4], Chen proved a difference analogue of the Brück conjecture as follows:

Theorem B ([4]). *Let* $\mathfrak{h}(z)$ *be a transcendental entire function of finite order. Also, assume that* $\mathfrak{h}$ *has a finite Borel exceptional value* $\alpha \in \mathbb{C}$*. Let* η *be a nonzero complex constant such that* $\mathfrak{h}(z+\eta) \not\equiv \mathfrak{h}(z)$*. If* $\Delta\mathfrak{h}(z)$ *and* $\mathfrak{h}(z)$ *share a finite value* $(d \not\equiv \alpha)$ *CM, then*

$$\frac{\Delta\mathfrak{h}(z) - d}{\mathfrak{h}(z) - d} = \frac{d}{d-\alpha}.$$

In [15], Huang and Zhang studied a parallel result corresponding to Theorem A as follows:

Theorem C ([15]). *Let $\mathfrak{h}(z)$ be a transcendental entire function of order of growth,*

$$\rho(\mathfrak{h}) := \limsup_{r\to\infty} \frac{\log T(r,\mathfrak{h})}{\log r} < 2,$$

and let η be a nonzero complex number and $k \in \mathbb{N}$. Assume that $\Delta_\eta^k \mathfrak{h}(z) \not\equiv 0$. If $\mathfrak{h}(z)$ and $\Delta_\eta^k \mathfrak{h}(z)$ share 0 CM, then

$$\Delta_c^k \mathfrak{h}(z) = p\mathfrak{h}(z),$$

for some non-zero constant p.

A homogeneous complex differential-difference polynomial of $\mathfrak{h}(z)$ is a polynomial expression that involves $\mathfrak{h}(z)$, its derivatives, and shift operators.

A complex homogeneous differential-difference polynomials of $\mathfrak{h}(z)$ can be written as

$$\omega(\mathfrak{h}) = \sum_{v=1}^{t} d_v \big(\mathfrak{h}^{(v)}(z+\tau_v)\big)^n + \sum_{k=1}^{s} l_k(\mathfrak{h}(z+\sigma_k))^n,$$

where $n \in \mathbb{N}$, $t,\ s \in \mathbb{N}$ and $\tau_v,\ \sigma_k$ are the complex constants and $d_v,\ l_k$ are non-zero constants.

Recently Adud and Chakraborty [1] established the following result:

Theorem D ([1]). *Let $\mathfrak{h}$ be a transcendental entire function and the order of $\mathfrak{h}$,*

$$\rho(\mathfrak{h}) := \limsup_{r\to\infty} \frac{\log T(r,\mathfrak{h})}{\log r} < 2$$

and $\omega(\mathfrak{h})$ be a complex homogeneous differential-difference polynomials defined by

$$\omega(\mathfrak{h}) = \sum_{v=1}^{t} d_v \big(\mathfrak{h}^{(v)}(z+\tau_v)\big)^n + \sum_{k=1}^{s} l_k(\mathfrak{h}(z+\sigma_k))^n,$$

where $n \in \mathbb{N}$, $t,\ s \in \mathbb{N}$ and $\tau_v,\ \sigma_k$ are the complex constants and $d_v,\ l_k$ are non-zero constants.

If $\mathfrak{h}^n$ and $\omega(\mathfrak{h})$ share 0 CM, then $\omega(\mathfrak{h}) = c\,\mathfrak{h}^n$ for some non zero constant c.

In this paper, we will investigate the value sharing where $\mathfrak{f}^n$ and $\omega(\mathfrak{f})$ both share the value a CM. Additionally, we will examine the scenario in which two differential-difference polynomials also share the value a CM.

2. Main results.

Theorem 2.1. *Let $\mathfrak{f}(z)$ be a transcendental entire function of finite order and $\omega(\mathfrak{f})$ be defined by $\omega(\mathfrak{f})=\sum_{v=1}^{t} d_v\big(\mathfrak{f}^{(v)}(z+\tau_v)\big)^n+\sum_{k=1}^{s} l_k(\mathfrak{f}(z+\sigma_k))^n$, where $n\geq 2$, $t,\ s\in\mathbb{N}$, $\tau_v,\ \sigma_k$ are the complex constants and $d_v,\ l_k$ are non-zero constants, be a homogeneous differential-difference polynomials with non zero constants. Let $\omega(\mathfrak{f})$ and $\mathfrak{f}^n$ share a CM, where $(a\neq 0,\infty)$. If $\delta_2(0;\omega(\mathfrak{f}))+\delta_2(0;\mathfrak{f}^n)>1$, then $\omega(\mathfrak{f})\equiv\mathfrak{f}^n$ or $\mathfrak{f}^n\omega(\mathfrak{f})\equiv 1$.*

Next we will delve deeper into the value sharing problem between two special type of differential-difference polynomials and derive a comprehensive general result.

Theorem 2.2. *Let $\mathfrak{f}(z)$ be a transcendental entire function of finite order and $\omega(\mathfrak{f})$ be defined by $\omega(\mathfrak{f})=\sum_{v=1}^{t} d_v\big(\mathfrak{f}^{(v)}(z+\tau_v)\big)^n+\sum_{k=1}^{s} l_k(\mathfrak{f}(z+\sigma_k))^n$ and $\Omega(\mathfrak{f})=\sum_{j=1}^{p} m_j\big(\mathfrak{f}^{(j)}(z+\mu_j)\big)^n+\sum_{l=1}^{q} n_l(\mathfrak{f}(z+\gamma_l))^n$, where $n\ (\geq 2)$, $t,\ s,\ p,\ q\in\mathbb{N}$; τ_v, σ_k, μ_j, γ_l are constants and d_v, l_k, m_j, n_l are non-zero constants. Let $\omega(\mathfrak{f})$ and $\Omega(\mathfrak{f})$ share a CM, where $(a\neq 0,\infty)$. If $\delta_2(0;\omega(\mathfrak{f}))+\delta_2(0;\Omega(\mathfrak{f}))>1$, then $\omega(\mathfrak{f})\equiv\Omega(\mathfrak{f})$ or $\omega(\mathfrak{f})\Omega(\mathfrak{f})\equiv 1$.*

3. Some useful lemmas. The difference logarithmic derivative lemma was proved by Chiang and Feng [3, Corollary 2.5] as well as Halburd and Korhonen [10, Theorem 2.1], [11, Theorem 5.6]. This lemma plays a crucial role in considering the difference analogues of Nevanlinna theory. Here, we recall the version of Halburd and Korhonen [11, Theorem 5.6].

Lemma 3.1 ([11]). *Let $\mathfrak{h}(z)$ be a transcendental meromorphic function of finite order. Then*

$$m\left(r,\frac{\mathfrak{h}(z+d)}{\mathfrak{h}(z)}\right)=S(r,\mathfrak{h}).$$

Lemma 3.2 ([3, Corollary 2.5]). *Let $\mathfrak{h}(z)$ be a meromorphic function of finite order $\rho(\mathfrak{h})$. For each $\epsilon>0$, then*

$$T(r,\mathfrak{h}(z+d))=T(r,\mathfrak{h})+O(r^{\rho(\mathfrak{h})-1+\epsilon})+O(\log r).$$

Thus if $\mathfrak{h}$ is a transcendental meromorphic function of finite order $\rho(\mathfrak{h})$, then $T(r,\mathfrak{h}(z+d))=T(r,\mathfrak{h})+S(r,\mathfrak{h})$.

Lemma 3.3 ([7]). *Let $\mathfrak{h}$ be a transcendental meromorphic function of finite order. Then*

$$m\left(r, \frac{\mathfrak{h}^{(k)}(z+d)}{\mathfrak{h}(z+e)}\right) = S(r, \mathfrak{h}),$$

for all z satisfies $|z| = r \notin E$, E is a set with finite logarithmic measure, where d and e are complex constants and k is a non-negative integer.

4. Proof of main theorems.

4.1. Proof of Theorem 2.1.

Proof. Since $\omega(\mathfrak{f})$ and $\mathfrak{f}^n(z)$ share a CM, then $\mathcal{F} = \dfrac{\omega(\mathfrak{f})}{a}$ and $\mathcal{G} = \dfrac{\mathfrak{f}^n(z)}{a}$ share 1 CM.

Let

$$\mathcal{P} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}-1}\right).$$

Then it can be written as

$$\mathcal{P} = \frac{U'}{U} - 2\frac{V'}{V},$$

where $U = \dfrac{\mathcal{F}'}{\mathcal{G}'}$ and $V = \dfrac{\mathcal{F}-1}{\mathcal{G}-1}$.

First we assume that $\mathcal{P} \not\equiv 0$.

Since $\mathcal{F}$ and $\mathcal{G}$ share 1 CM, then using 1st fundamental theorem we have,

$$\begin{aligned} N\left(r, 1; \mathcal{F}| = 1\right) &\leq N\left(r, 0; \mathcal{P}\right) \\ &\leq T\left(r, 0; \mathcal{P}\right) \\ &\leq T\left(r, \mathcal{P}\right) + O(1) \\ &= m(r, \mathcal{P}) + N(r, \mathcal{P}) + O(1) \\ &= N(r, \mathcal{P}) + S(r, \mathcal{F}) + S(r, \mathcal{G}). \end{aligned}$$

Therefore,

$$N(r, 1; \mathcal{F}| = 1) \leq N(r, \mathcal{P}) + S(r, \mathfrak{f}). \tag{4.1}$$

Here we can easily verify that the possible poles of $\mathcal{P}$ can come from,

(1) multiple zeros of $\mathcal{F}$ and $\mathcal{G}$.

(2) zeros of $\mathcal{F}'$ which do not come zeros from $\mathcal{F}(\mathcal{F}-1)$.

(3) zeros of $\mathcal{G}'$ which do not come zeros from $\mathcal{G}(\mathcal{G}-1)$.

Therefore,

$$N(r, \mathcal{P}) \leq \overline{N}(r, 0; \mathcal{F}| \geq 2) + \overline{N}(r, 0; \mathcal{G}| \geq 2) + \overline{N_0}(r, 0; \mathcal{F}') + \overline{N_0}(r, 0; \mathcal{G}'), \tag{4.2}$$

where $\overline{N_0}(r,0;\mathcal{F}')$ is the reduced counting function of those zeros of $\mathcal{F}'$ which are not come from the zeros of $\mathcal{F}(\mathcal{F}-1)$ and $\overline{N_0}(r,0;\mathcal{G}')$ is the reduced counting function of those zeros of $\mathcal{G}'$ which are not come from the zeros of $\mathcal{G}(\mathcal{G}-1)$.

By the 2nd fundamental theorem we have,

$$(2-1)T(r,\mathcal{F}) \leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) - \overline{N_0}(r,0;\mathcal{F}') + S(r,\mathcal{F}).$$

Therefore,

$$T(r,\mathcal{F}) \leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) - \overline{N_0}(r,0;\mathcal{F}') + S(r,\mathfrak{f}). \tag{4.3}$$

Since $\mathcal{F}$ and $\mathcal{G}$ share 1 CM then,

$$\overline{N}(r,1;\mathcal{F}) = \overline{N}(r,1;\mathcal{F}|=1) + \overline{N}(r,1;\mathcal{G}|\geq 2). \tag{4.4}$$

From (4.1), (4.2), (4.3), (4.4) we have,

$$\begin{aligned} (4.5)\qquad & T(r,\mathcal{F}) \\ \leq\ & \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;\mathcal{G}|\geq 2) + \overline{N_0}(r,0;\mathcal{G}') \\ (4.6)\qquad & +\overline{N}(r,1;\mathcal{G}|\geq 2) + S(r,\mathfrak{f}). \end{aligned}$$

Since $\mathcal{F}$ and $\mathcal{G}$ share 1 CM then,

$$\overline{N_0}(r,0;\mathcal{G}') + \overline{N}(r,1;\mathcal{G}|\geq 2) + N(r,0;\mathcal{G}) - \overline{N}(r,0;\mathcal{G}) \leq N(r,0;\mathcal{G}'). \tag{4.7}$$

By using 1st fundamental theorem and logarithmic derivative and Lemma 3.1 [9] we have,

$$\begin{aligned} N(r,0;\mathcal{G}') &\leq N(r,0;\frac{\mathcal{G}'}{\mathcal{G}}) + N(r,0;\mathcal{G}) - \overline{N}(r,0;\mathcal{G}) \\ &\leq N(r,\frac{\mathcal{G}'}{\mathcal{G}}) + N(r,0;\mathcal{G}) - \overline{N}(r,0;\mathcal{G}) + S(r,\mathcal{G}) \\ &\leq \overline{N}(r,\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + N(r,0;\mathcal{G}) - \overline{N}(r,0;\mathcal{G}) + S(r,\mathcal{G}). \end{aligned}$$

So we have

$$N(r,0;\mathcal{G}') \leq N(r,0;\mathcal{G}) + S(r,\mathfrak{f}). \tag{4.8}$$

From (4.7) and (4.8) we have,

$$\overline{N_0}(r,0;\mathcal{G}') + \overline{N}(r,1;\mathcal{G}|\geq 2) \leq \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}). \tag{4.9}$$

From (4.5) and (4.9) we have,

$$T(r,\mathcal{F}) \leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}| \geq 2) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}).$$

Similarly

$$T(r,\mathcal{G}) \leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}| \geq 2) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}).$$

So,

$$T(r) \leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}| \geq 2) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}),$$

where $T(r) = \max\{T(r,\mathcal{F}), T(r,\mathcal{G})\}$.

If $T(r) = T(r,\mathcal{F})$ then

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}| \geq 2) \\ &\quad + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

which contradicts $\delta_2(0;\mathcal{F}) + \delta_2(0;\mathcal{G}) > 1$. So $\mathcal{P} \equiv 0$. Then

$$\mathcal{F} = \frac{A_1\mathcal{G} + B_1}{C_1\mathcal{G} + D_1},$$

where $A_1D_1 - B_1C_1 \neq 0$ and A_1, B_1, C_1, D_1 are complex numbers.

Case 1: Let $A_1C_1 = 0$. Here A_1 & C_1 are not both simultaneously zero.

Subcase 1.1: Let $A_1 = 0$ & $C_1 \neq 0$. Then $\frac{1}{\mathcal{F}} = \alpha\mathcal{G} + \beta$, where $\alpha = \frac{C_1}{B_1}$, $\beta = \frac{D_1}{B_1}$.

If 1 is picard exceptional value of $\mathcal{F}$ then $\mathcal{G}$ also, then by the 2nd fundamental theorem we have,

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}| \geq 2) \\ &\quad + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

which contradicts $\delta_2(0;\mathcal{F}) + \delta_2(0;\mathcal{G}) > 1$. So 1 can not be a picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$. Then $\alpha + \beta = 1$.

If $\alpha \neq 1$ then by the 2nd fundamental theorem we have,

$$(2-1)T(r,\mathcal{F}) \leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\frac{1}{1-\alpha};\mathcal{F}) + S(r,\mathfrak{f})$$

$$\leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}),$$

which contradicts the given condition in a similar way.

Thus if $\alpha = 1$ then $\mathcal{F}\mathcal{G} \equiv 1$ i.e.

$$\omega(\mathfrak{f})\mathfrak{f}^n \equiv 1.$$

Subcase 1.2: Let $A_1 \neq 0$ & $C_1 = 0$. So $D_1 \neq 0$. Then $\mathcal{F} = \gamma\mathcal{G} + \delta$, $\gamma = \dfrac{A_1}{D_1}$, $\delta = \dfrac{B_1}{D_1}$.

If 1 be picard exceptional value of $\mathcal{F}$, then $\mathcal{G}$ also.

Now by the 2nd fundamental theorem we have,

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

which contradicts the given condition in similar way.

So 1 can not be picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$. Then $\gamma + \delta = 1$.

If $\gamma \neq 1$ then by the 2nd fundamental theorem we have,

$$\begin{aligned} (2-1)T(r,\mathcal{F}) &\leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,1-\gamma;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

which contradicts the given condition in a similar way.

So if $\gamma = 1$ then $\mathcal{F} \equiv \mathcal{G}$ i.e.

$$\omega(\mathfrak{f}) \equiv \mathfrak{f}^n.$$

Case 2: Let $A_1C_1 \neq 0$. This implies $A_1 \neq 0$ & $C_1 \neq 0$.

Subcase 2.1: Let $B_1 = 0$ and $D_1 \neq 0$. So $\dfrac{1}{\mathcal{F}} = \alpha_1 + \dfrac{\beta_1}{\mathcal{G}}$, where $\alpha_1 = \dfrac{C_1}{A_1}$, $\beta_1 = \dfrac{D_1}{A_1}$.

If 1 is a picard exceptional value of $\mathcal{F}$ then $\mathcal{G}$ also.

Thus by the 2nd fundamental theorem we have,

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

which contradicts the given condition in a similar way.

So 1 can not be picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$. Then $\alpha_1 + \beta_1 = 1$.

If $\alpha_1 \neq 1$ then by the 2nd fundamental theorem we have,

$$\begin{aligned}(2-1)T(r,\mathcal{G}) &\leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,1-\frac{1}{\alpha_1};\mathcal{G}) + S(r,\mathcal{G}) \\ &\leq \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq \overline{N}(r,0;\mathcal{G}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}),\end{aligned}$$

which contradicts the given condition in a similar way.

If $\alpha_1 = 1$, then

$$\mathcal{F} \equiv 1,$$

which is not possible. So this case is not possible.

Subcase 2.2: Let $B_1 \neq 0$ and $D_1 = 0$. So $\mathcal{F} = \gamma_1 + \frac{\delta_1}{\mathcal{G}}$, where $\gamma_1 = \frac{A_1}{C_1}$, $\delta_1 = \frac{B_1}{C_1}$.

If 1 is a picard exceptional value of $\mathcal{F}$, then $\mathcal{G}$ also.

Thus by 2nd fundamental theorem we have,

$$\begin{aligned}T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}),\end{aligned}$$

which contradicts the given condition in a similar way.

So 1 can not be picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$. Then $\gamma_1 + \delta_1 = 1$.

If $\gamma_1 \neq 1$ then by the 2nd fundamental theorem we have,

$$\begin{aligned}(2-1)T(r,\mathcal{G}) &\leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,1-\frac{1}{\gamma_1};\mathcal{G}) + S(r,\mathfrak{f}) \\ &\leq \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}),\end{aligned}$$

which contradicts the given condition in a similar way.

If $\gamma_1 = 1$ then

$$\mathcal{F} \equiv 1,$$

which is not possible. So this case is also not possible.

Subcase 2.3: Let $B_1 \neq 0$ and $D_1 \neq 0$. Then $\frac{B_1}{D_1} \neq 0$.

Let

$$\mathcal{F}-\frac{B_1}{D_1}=\frac{\mathcal{G}(A_1-\frac{B_1C_1}{D_1})}{C_1\mathcal{G}+D_1}.$$

Since $\mathcal{G}$ is an entire function then $\dfrac{B_1}{D_1}$ point of $\mathcal{F}$ only comes from zeros of $\mathcal{G}$.

Then by 2nd fundamental theorem we have,

$$\begin{aligned}(2-1)T(r,\mathcal{F}) &\leq \overline{N}(r,\infty;\mathcal{F})+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,\frac{B}{D};\mathcal{F})+S(r,\mathfrak{f})\\ &\leq \overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+S(r,\mathfrak{f})\\ &\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f}),\end{aligned}$$

which contradicts the given condition in a similar way.

This theorem is proved. □

4.2. Proof of Theorem 2.2.

Proof. Since $\omega(\mathfrak{f})$ and $\Omega(\mathfrak{f})$ share a CM, then $\mathcal{F}=\dfrac{\omega(\mathfrak{f})}{a}$ and $\mathcal{G}=\dfrac{\Omega(\mathfrak{f})}{a}$ share 1 CM.

Let

$$\mathcal{P}=\left(\frac{\mathcal{F}''}{\mathcal{F}'}-\frac{2\mathcal{F}'}{\mathcal{F}-1}\right)-\left(\frac{\mathcal{G}''}{\mathcal{G}'}-\frac{2\mathcal{G}'}{\mathcal{G}-1}\right).$$

So it can be written as

$$\mathcal{P}=\frac{U'}{U}-2\frac{V'}{V},$$

where $U=\dfrac{\mathcal{F}'}{\mathcal{G}'}$ and $V=\dfrac{\mathcal{F}-1}{\mathcal{G}-1}$.

Now we assume that $\mathcal{P}\not\equiv 0$.

Since $\mathcal{F}$ and $\mathcal{G}$ share 1 CM then using 1st fundamental theorem we have,

$$\begin{aligned}N\left(r,1;\mathcal{F}|=1\right) &\leq N\left(r,0;\mathcal{P}\right)\\ &\leq T\left(r,0;\mathcal{P}\right)\\ &\leq T\left(r,\mathcal{P}\right)+O(1)\\ &= m(r,\mathcal{P})+N(r,\mathcal{P})+O(1)\\ &= N(r,\mathcal{P})+S(r,\mathcal{F})+S(r,\mathcal{G}).\end{aligned}$$

Therefore,

$$N(r,1;\mathcal{F}|=1)\leq N(r,\mathcal{P})+S(r,\mathfrak{f}). \tag{4.10}$$

Here we can easily verify that the possible poles of $\mathcal{P}$ can come from,
(1) multiple zeros of $\mathcal{F}$ and $\mathcal{G}$.
(2) zeros of $\mathcal{F}'$ which do not come zeros from $\mathcal{F}(\mathcal{F}-1)$.
(3) zeros of $\mathcal{G}'$ which do not come zeros from $\mathcal{G}(\mathcal{G}-1)$.

Therefore,

$$N(r,\mathcal{P}) \leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;\mathcal{G}|\geq 2) + \overline{N_0}(r,0;\mathcal{F}') + \overline{N_0}(r,0;\mathcal{G}'), \tag{4.11}$$

where $\overline{N_0}(r,0;\mathcal{F}')$ is the reduced counting function of those zeros of $\mathcal{F}'$ which are not come from the zeros of $\mathcal{F}(\mathcal{F}-1)$ and $\overline{N_0}(r,0;\mathcal{G}')$ is the reduced counting function of those zeros of $\mathcal{G}'$ which are not come from the zeros of $\mathcal{G}(\mathcal{G}-1)$.

By the 2nd fundamental theorem we have,

$$(2-1)T(r,\mathcal{F}) \leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) - \overline{N_0}(r,0;\mathcal{F}') + S(r,\mathcal{F}).$$

Therefore,

$$T(r,\mathcal{F}) \leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) - \overline{N_0}(r,0;\mathcal{F}') + S(r,\mathfrak{f}). \tag{4.12}$$

Since $\mathcal{F}$ and $\mathcal{G}$ share 1 CM then,

$$\overline{N}(r,1;\mathcal{F}) = \overline{N}(r,1;\mathcal{F}|=1) + \overline{N}(r,1;\mathcal{G}|\geq 2). \tag{4.13}$$

From (4.10), (4.11), (4.12), (4.13) we have,

$$\begin{aligned} T(r,\mathcal{F}) \leq\; & \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;\mathcal{G}|\geq 2) \\ & + \overline{N_0}(r,0;\mathcal{G}') + \overline{N}(r,1;\mathcal{G}|\geq 2) + S(r,\mathfrak{f}). \end{aligned} \tag{4.14}$$

Since $\mathcal{F}$ and $\mathcal{G}$ share 1 CM then,

$$\begin{aligned} & T(r,\mathcal{F}) \qquad (4.15)\\ \leq\; & \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;\mathcal{G}|\geq 2) + \overline{N_0}(r,0;\mathcal{G}') \\ & + \overline{N}(r,1;\mathcal{G}|\geq 2) + S(r,\mathfrak{f}). \qquad (4.16) \end{aligned}$$

By using 1st fundamental theorem and logarithmic derivative and Lemma 3.1 [9] we have,

$$\begin{aligned} N(r,0;\mathcal{G}') &\leq N(r,0;\frac{\mathcal{G}'}{\mathcal{G}}) + N(r,0;\mathcal{G}) - \overline{N}(r,0;\mathcal{G}) \\ &\leq N(r,\frac{\mathcal{G}'}{\mathcal{G}}) + N(r,0;\mathcal{G}) - \overline{N}(r,0;\mathcal{G}) + S(r,\mathcal{G}) \end{aligned}$$

$$\leq \overline{N}(r,\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + N(r,0;\mathcal{G}) - \overline{N}(r,0;\mathcal{G}) + S(r,\mathcal{G}).$$

So we have

$$N(r,0;\mathcal{G}') \leq N(r,0;\mathcal{G}) + S(r,\mathfrak{f}). \tag{4.17}$$

From (4.15) and (4.17) we have,

$$\overline{N_0}(r,0;\mathcal{G}') + \overline{N}(r,1;\mathcal{G}| \geq 2) \leq \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}). \tag{4.18}$$

From (4.14) and (4.18) we have,

$$T(r,\mathcal{F}) \leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}| \geq 2) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}).$$

Similarly

$$T(r,\mathcal{G}) \leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}| \geq 2) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}).$$

So,

$$T(r) \leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}| \geq 2) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}),$$

where $T(r) = \max\{T(r,\mathcal{F}), T(r,\mathcal{G})\}$.

If $T(r) = T(r,\mathcal{F})$ then

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}| \geq 2) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

which contradicts $\delta_2(0;\mathcal{F}) + \delta_2(0;\mathcal{G}) > 1$. So $\mathcal{P} \equiv 0$. Then

$$\mathcal{F} = \frac{A_2\mathcal{G} + B_2}{C_2\mathcal{G} + D_2},$$

where $A_2D_2 - B_2C_2 \neq 0$ and A_2, B_2, C_2, D_2 are complex numbers.

Case 1: Let $A_2C_2 = 0$. Here A_2 & C_2 are not both simultaneously zero.

Subcase 1.1: Let $A_2 = 0$ & $C_2 \neq 0$. Then $\frac{1}{\mathcal{F}} = \alpha\mathcal{G} + \beta$, where $\alpha = \frac{C_2}{B_2}$, $\beta = \frac{D_2}{B_2}$.

If 1 is picard exceptional value of $\mathcal{F}$ then $\mathcal{G}$ also, then by the 2nd fundamental theorem we have,

$$T(r,\mathcal{F}) \leq \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f})$$

$$\begin{aligned}&\leq \overline{N}(r,0;\mathcal{F}|\geq 2)+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G}|\geq 2)+\overline{N}(r,0;\mathcal{G})+S(r,\mathfrak{f})\\&\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f}),\end{aligned}$$

which contradicts $\delta_2(0;\mathcal{F})+\delta_2(0;\mathcal{G})>1$. So 1 can not be a picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$. Then $\alpha+\beta=1$.

If $\alpha\neq 1$ then by the 2nd fundamental theorem we have,

$$\begin{aligned}(2-1)T(r,\mathcal{F}) &\leq \overline{N}(r,\infty;\mathcal{F})+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,\frac{1}{1-\alpha};\mathcal{F})+S(r,\mathfrak{f})\\&\leq \overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+S(r,\mathfrak{f}),\end{aligned}$$

which contradicts the given condition in a similar way.

Thus if $\alpha=1$ then $\mathcal{F}\mathcal{G}\equiv 1$ i.e.,

$$\omega(\mathfrak{f})\Omega(\mathfrak{f})\equiv 1.$$

Subcase 1.2: Let $A_2\neq 0$ & $C_2=0$. So $D_2\neq 0$. Then $\mathcal{F}=\gamma\mathcal{G}+\delta$, $\gamma=\frac{A_2}{D_2}$, $\delta=\frac{B_2}{D_2}$.

If 1 be picard exceptional value of $\mathcal{F}$, then $\mathcal{G}$ also.

Now by the 2nd fundamental theorem we have,

$$\begin{aligned}T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F})+S(r,\mathfrak{f})\\&\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f}),\end{aligned}$$

which contradicts the given condition in similar way.

So 1 can not be picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$. Then $\gamma+\delta=1$.

If $\gamma\neq 1$ then by the 2nd fundamental theorem we have,

$$\begin{aligned}(2-1)T(r,\mathcal{F}) &\leq \overline{N}(r,\infty;\mathcal{F})+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,1-\gamma;\mathcal{F})+S(r,\mathfrak{f})\\&\leq \overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+S(r,\mathfrak{f})\\&\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f}),\end{aligned}$$

which contradicts the given condition in a similar way.

So if $\gamma=1$ then $\mathcal{F}\equiv\mathcal{G}$ i.e.

$$\omega(\mathfrak{f})\equiv\Omega(\mathfrak{f}).$$

Case 2: Let $A_2C_2\neq 0$. This implies $A_2\neq 0$ & $C_2\neq 0$.

Subcase 2.1: Let $B_2 = 0$ and $D_2 \neq 0$. So $\frac{1}{\mathcal{F}} = \alpha_1 + \frac{\beta_1}{\mathcal{G}}$, where $\alpha_1 = \frac{C_2}{A_2}$, $\beta_1 = \frac{D_2}{A_2}$.

If 1 is a picard exceptional value of $\mathcal{F}$ then $\mathcal{G}$ also.

Thus by the 2nd fundamental theorem we have,

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

which contradicts the given condition in a similar way.

So 1 can not be picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$. Then $\alpha_1 + \beta_1 = 1$.

If $\alpha_1 \neq 1$ then by the 2nd fundamental theorem we have,

$$\begin{aligned} (2-1)T(r,\mathcal{G}) &\leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,1-\frac{1}{\alpha_1};\mathcal{G}) + S(r,\mathcal{G}) \\ &\leq \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq \overline{N}(r,0;\mathcal{G}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

which contradicts the given condition in a similar way.

If $\alpha_1 = 1$, then

$$\mathcal{F} \equiv 1,$$

which is not possible. So this case is not possible.

Subcase 2.2: Let $B_2 \neq 0$ and $D_2 = 0$. So $\mathcal{F} = \gamma_1 + \frac{\delta_1}{\mathcal{G}}$, where $\gamma_1 = \frac{A_2}{C_2}$, $\delta_1 = \frac{B_2}{C_2}$.

If 1 is a picard exceptional value of $\mathcal{F}$, then $\mathcal{G}$ also.

Thus by 2nd fundamental theorem we have,

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

which contradicts the given condition in a similar way.

So 1 can not be picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$. Then $\gamma_1 + \delta_1 = 1$.

If $\gamma_1 \neq 1$ then by the 2nd fundamental theorem we have,

$$(2-1)T(r,\mathcal{G}) \leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,1-\frac{1}{\gamma_1};\mathcal{G}) + S(r,\mathfrak{f})$$

$$\leq \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f})$$
$$\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}),$$

which contradicts the given condition in a similar way.

If $\gamma_1 = 1$ then
$$\mathcal{F} \equiv 1,$$
which is not possible. So this case is also not possible.

Subcase 2.3: Let $B_2 \neq 0$ and $D_2 \neq 0$. Then $\frac{B_2}{D_2} \neq 0$.

Let
$$\mathcal{F} - \frac{B_2}{D_2} = \frac{\mathcal{G}(A_2 - \frac{B_2 C_2}{D_2})}{C_2\mathcal{G} + D_2}.$$
Since $\mathcal{G}$ is an entire function then $\frac{B_2}{D_2}$ point of $\mathcal{F}$ only comes from zeros of $\mathcal{G}$.

Then by 2nd fundamental theorem we have,

$$\begin{aligned}(2-1)T(r,\mathcal{F}) &\leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}\left(r, \frac{B_2}{D_2};\mathcal{F}\right) + S(r,\mathfrak{f}) \\ &\leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}),\end{aligned}$$

which contradicts the given condition in a similar way.

This theorem is proved. □

Acknowledgments. The author would like to thank Dr. Bikash Chakraborty for his valuable suggestions and comments, which significantly improved the presentation of this paper.

REFERENCES

[1] Md. Adud, B. Chakraborty. Some results related to differential-difference counterpart of the Brück conjecture. *Commun. Korean Math. Soc.* **39**, *1* (2024), 117–125.

[2] R. Brück. On entire functions which share one value CM with their first derivative. *Results Math.* **30**, *1–2* (1996), 21–24.

[3] Y.-M. Chiang, S.-J. Feng. On the Nevanlinna characteristic of $\mathfrak{f}(z+\eta)$ and difference equations in the complex plane. *Ramanujan J.* **16**, *1* (2008), 105–129.

[4] Z. Chen. On the difference counterpart of Brück's conjecture. *Acta Math. Sci. Ser. B (Engl. Ed.)* **34**, *3* (2014), 653–659.

[5] X. Dong and K. Liu. Some results on differential-difference analogues of Brück conjecture. *Math. Slovaca* **67**, *3* (2017), 691–700.

[6] M. Fang. Uniqueness of admissible meromorphic functions in the unit disc. *Sci. China Ser. A* **42**, *4* (1999), 367–381.

[7] G. G. Gundersen. Meromorphic functions that share three or four values. *J. London Math. Soc.* **20**, *3* (1979), 457–466.

[8] G. G. Gundersen. Meromorphic functions that share four values. *Trans. Amer. Math. Soc.* **277**, *2* (1983), 545–567.

[9] G. G. Gundersen, L.-Z. Yang. Entire functions that share one value with one or two of their derivatives. *J. Math. Anal. Appl.* **223**, *1* (1998), 88–95.

[10] R. G. Halburd, R. J. Korhonen. Difference analogue of the lemma on the logarithmic derivative with applications to difference equations. *J. Math. Anal. Appl.* **314**, *2* (2006), 477–487.

[11] R. G. Halburd, R. J. Korhonen. Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations. *J. Phys. A.* **40**, *6* (2007), R1–R38.

[12] W. K. Hayman. Meromorphic functions. Oxford Math. Monogr. Oxford, Clarendon Press, 1964.

[13] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, J. Zhang. Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity. *J. Math. Anal. Appl.* **355**, *1* (2009), 352–363.

[14] X.-H. Hua. Sharing values and a problem due to C. C. Yang. *Pacific J. Math.* **175**, *1* (1996), 71–81.

[15] Z.-B. Huang, R.-R. Zhang. Uniqueness of the differences of meromorphic functions. *Anal. Math.* **44**, *4* (2018), 461–473.

[16] I. Lahiri. Weighted value sharing and uniqueness of meromorphic functions. *Complex Variables Theory Appl.* **46**, *3* (2001), 241–253.

[17] E. Mues, N. Steinmetz. Meromorphe Funktionen die mit ihrer Ableitung Werte teilen. *Manuscripta Math.* **29**, *2–4* (1979), 195–206.

[18] L. A. Rubel, C. C. Yang. Values shared by an entire function and its derivative. In: Complex analysis (Eds J. D. Buckholtz, T. J. Suffridge) (Proc. Conf., Univ. Kentucky, Lexington, Ky., 1976), 101–103. Lecture Notes in Math., vol. **599**. Berlin, Springer, 1977, https://doi.org/10.1007/BFb0096830.

[19] C. C. Yang, X. H. Hua. Uniqueness and value sharing of meromorphic functions. *Ann. Acad. Sci. Fenn. Math.* **22**, *2* (1997), 395–406.

[20] H. X. Yi. Meromorphic functions that share one or two values. *Complex Variable Theory Appl.* **28**, *1* (1995), 1–11.

[21] Q. C. Zhang. Meromorphic function that shares one small function with its derivative. *JIPAM. J. Inequal. Pure Appl. Math.* **6**, *4* (2005), Article 116, 13 pp.

Soumon Roy
Nevanlinna Lab, Department of Mathematics
Ramakrishna Mission Vivekananda Centenary College
Rahara, West Bengal 700 118, India
e-mail: rsoumon@gmail.com

Sudip Saha
Department of Mathematics
Brainware University
Barasat, Kolkata, West Bengal 700 125, India
e-mail: sudipsaha814@gmail.com

Received October 23, 2023
Accepted November 14, 2024