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ON THE TYPE OF GENERALIZED HYPERCOMPLEX STRUCTURES*

Anna Fino, Gueo Grantcharov

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Abstract. The generalized hypercomplex structures defined within the framework of generalized geometry include hypercomplex and holomorphic symplectic structures as particular cases. They have a S^2-family of generalized complex structures, and in this paper we study the types of these structures and the corresponding twistor space. We show that there are generalized hypercomplex structures on the $4n$-dimensional tori, which do not contain a structure of maximal (complex) type. Moreover, we show that the Kodaira-Thurston surface which has a holomorphic symplectic structure, admits also a generalized hypercomplex structure in which all generalized complex structures are of type 1.

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1. Introduction. The notion of generalized complex geometry was introduced by N. Hitchin in the context of generalized Calabi-Yau manifolds with fluxes [38]. Later, it was developed by M. Gualtieri [37]. The main idea is to replace the tangent bundle TM of a smooth manifold M with the generalized tangent bundle, i.e. the direct sum of the tangent and the cotangent bundle $TM \oplus T^*M$, and the Lie bracket with the Courant bracket

$$[X+\xi, Y+\eta] = [X,Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi),$$

$$X, Y \in \Gamma(TM), \quad \xi, \eta \in \Gamma(T^*M).$$

The Courant bracket admits more symmetries than just diffeomorphisms. The extra symmetries are called B-field transformations and are generated by closed 2-forms on M.

Such symmetries lead to a ganeralization of the Courant bracket depending on a closed 3-form H called twisted Courant bracket defined by

$$[X+\xi, Y+\eta]_H = [X,Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) + i_Xi_YH,$$

$$X, Y \in \Gamma(TM), \quad \xi, \eta \in \Gamma(T^*M).$$

and B-field transformations change H to $H + dB$.

A generalized almost complex structure $\mathcal{I}$ is an endomorphism of $TM \oplus T^*M$ with $\mathcal{I}^2 = -\mathrm{Id}$, compatible with the natural split signature inner product arising from the pairing of vectors and co-vectors. $\mathcal{I}$ is called generalized complex when it is integrable with respect to the (twisted) Courant bracket. Additionally, a choice of a positive definite metric on TM leads to a symmetric isomorphism $\mathcal{G}$ of $TM \oplus T^*M$. When it commutes with $\mathcal{I}$ there is a second generalized almost complex structure $\tilde{\mathcal{I}} = \mathcal{G}\mathcal{I}$. If $\tilde{\mathcal{I}}$ is also integrable, the pair $\mathcal{I}, \tilde{\mathcal{I}}$ is called a (twisted) generalized Kähler. The genereralized Kähler structures were introduced in [37] and are equivalent to bihermitian structures introduced first by physicists as (2,2)-supersymmetric sigma models [32]. Since their appearance, the generalized complex and generalized Kähler structures are in the mainstream of research in differential geometry and geometric structures. For some of the recent advances see e.g. [2, 4, 7, 8, 9, 10, 11, 22, 20, 21, 19, 30, 33, 35, 31, 40, 43, 44, 48, 50, 51, 52]. An important question in generalized Kähler geometry is the construction of compact examples. Apart from the deformation of Kähler structures, there are many explicit examples of non-trivial (twisted) generalized Kähler structures,

e.g., [1, 5, 6, 15, 16, 14, 18, 25, 27, 28, 29, 34]. In particular, non-Kähler examples are given by compact solvmanifolds, in contrast with the case of compact nilmanifolds which cannot admit any invariant generalized Kähler structure unless they are tori [17].

The generalized complex structures were also generalized to various other geometric structures on $TM \oplus T^*M$, one of which is generalized hypercomplex. A generalized almost hypercomplex structure is defined by a triple $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ of anti-commuting generalized almost complex structures such that $\mathcal{K} = \mathcal{I}\mathcal{J}$. In this case the S^2-family $\mathcal{I}_{a,b,c} = a\mathcal{I} + b\mathcal{J} + c\mathcal{K}$, $a^2 + b^2 + c^2 = 1$, is a family of generalized almost complex structures. When the family consists of generalized complex structures, we obtain a generalized hypercomplex structure. By [39], it is enough to have two anti-commuting integrable generalized almost complex structure in order to ensure that every structure in the family is integrable. In this case the space $Z = M \times S^2$ carries a tautologically defined generalized complex structure and is called the twistor space of M (see Section 4).

The first appearance of generalized hypercomplex structures was in fact as part of a generalized hyperkähler structure in [12]. Such structures arise from generalized hypercomplex structures in the same way as the generalized Kähler arise from generalized complex - by adding a compatible positive definite metric on TM which leads to a second generalized almost hypercomplex structure, and when the structures are integrable we have generalized hyperkähler structure. In physics the generalized hyperkähler manifolds correspond to (4,4)-supersymmetric sigma models. For the most recent physics application see [46, 47, 53]

Since the only known non-Kähler compact examples of generalized hyperkähler manifolds are locally products of compact Lie groups and hyperkähler manifolds, it is natural to study a less restrictive conditions. A prospective candidate is the generalized hypercomplex structure as studied first by M. Stienon [49], since both hypercomplex and holomorphic symplectic structures are particular cases of it. For more results see also [13, 26, 39, 42].

The purpose of this note is to initiate a study of the types of the generalized complex structures in the corresponding hypercomplex S^2-family and the generalized complex structure on its twistor space. After the preliminaries we start in Section 3 with two generalized complex structures on a vector space and determine when they are part of a generalized hypercomplex family. Then we discuss the possible types of the structures in the hypercomplex family when the two are chosen among complex and symplectic structures up to a B-transform. We extend the condition under which two generalized complex structures define a hypercomplex family from a vector space to a manifold, thus generalizing the

result of [39]. In the proof we use the twistor space and the ideas of D. Kaledin [41]. This is the first part of Section 4, in the second part we apply it to characterise the types of local generalized hypercomplex structures when the generalized complex structure on the twistor space is locally either of complex type or of the minimal possible type – one.

The main results of the paper are two examples in Section 5. One is of generalized hypercomplex structure on $4n$-dimensional tori in which the generic structure in the family is of symplectic type and there is no structure of complex type. The second one is a generalized hypercomplex structure on Kodaira-Thurston surface where every structure in the family is of type 1. Note that Kodaira-Thurston surface is one of the first examples of non-Kähler holomorphic symplectic manifold, so it admits generalized hypercomplex structures of different type.

2. Preliminaries on linear generalized complex structures. A generalized complex structure on a vector space V is a linear complex structure $\mathcal{I}$ on $V \oplus V^*$ orthogonal with respect to the pairing

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X)), \quad X, Y \in V,\, \xi, \eta \in V^*. \tag{1}$$

Since $\mathcal{I}^2 = -\mathrm{Id}$, the complexification of $V \oplus V^*$ splits as a direct sum of $\pm i$-eingenspaces L and $\overline{L}$. Moreover, as $\mathcal{I}$ is orthogonal, we obtain that for $v, w \in L$

$$\langle v, w\rangle = \langle \mathcal{I}v, \mathcal{I}w\rangle = \langle iv, iw\rangle = -\langle v, w\rangle.$$

Therefore L is a maximal isotropic subspace with respect to the pairing. Conversely, prescribing such an L as the i-eigenspace determines a unique generalized complex structure on V. As a consequence a generalized complex structure on a vector space V is equivalent to a maximal isotropic subspace L such that $L \subset (V \oplus V^*) \otimes \mathbb{C}$ such that $L \cap \overline{L} = \{0\}$.

Example 2.1. Examples of generalized complex structure are given by complex and symplectic structures. A complex structure J on a real vector space V induces a generalized complex structure on V which can be written in matrix form using the splitting $V \oplus V^*$ as

$$\mathcal{I}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}.$$

The i-eigenspace of $\mathcal{I}_J$ is $L = V^{0,1} + V^{*1,0}$.

A symplectic form ω on V also induces a generalized complex structure on V by letting

$$\mathcal{I}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

In this case the i-eigenspace of $\mathcal{I}_\omega$ is given by

$$L = \{X - i\,\omega(X) \mid X \in V\}.$$

A real 2-form B acts naturally on $V \oplus V^*$ by the B-field transform

$$e \mapsto e - B(\pi(e)),$$

where π is the projection on V. If $\mathcal{I}$ is a generalized complex structure on V whose i-eingenspace is L, the image of L under the action of a B-field is given by

$$L_B = \{e - i\,B(e) \mid e \in E\}$$

and it is still isotropic. The associated generalized complex structure $\mathcal{I}_B$ is called the B-field transform of I and is given in matrix form by

$$\mathcal{I}_B = Ad(\exp(B))\mathcal{I} = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \mathcal{I} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}.$$

Any two isotropic splittings of $V \oplus V^*$ are related by B-field transforms. More in general, every element $\beta \in \Lambda^2 V$ also acts in a similar way

$$X + \xi \mapsto X + \xi + \iota_\xi \beta$$

and the β-transform of a generalized complex structure is still a generalized complex structure.

A further characterization of a generalized complex structure on V can be given viewing forms as spinors. The Clifford algebra of $V \oplus V^*$ is defined using the natural form $\langle \cdot, \cdot \rangle$, i.e. for $v \in V \oplus V^* \subset \mathrm{Cl}(V \oplus V^*)$ we have $v^2 = \langle v, v \rangle$. There is a natural action of the Clifford algebra on $\Lambda^\bullet V^*$ by

$$(X + \xi) \cdot \alpha = \iota_X \alpha + \xi \wedge \alpha.$$

On $\Lambda^\bullet V \oplus V^* \subset \mathrm{Cl}(V \oplus V^*)$ one can define the following bilinear form

$$(\xi_1, \xi_2) \mapsto (\sigma(\xi_1) \wedge \xi_2)_{\mathrm{top}},$$

where σ is the anti-automorphism of $\mathrm{Cl}(V \oplus V^*)$ given on decomposable elements by

$$\sigma(v_1 \cdot v_2 \cdot \cdots \cdot v_k) = v_k \cdot \cdots \cdot v_2 \cdot v_1$$

and top denotes the top degree of the form.

Given a form $\rho \in \Lambda^\bullet V^* \otimes \mathbb{C}$ of possibly mixed degree one can consider the annihilator

$$L_\rho = \{v \in (V \oplus V^*) \otimes \mathbb{C} \mid v \cdot \rho = 0\}.$$

Since $\overline{L_\rho} = L_{\overline{\rho}}$ and $0 = v^2 \cdot \rho = \langle v, v\rangle \rho$, for every $v \in L_\rho$, the subspace L_ρ is isotropic. If L_ρ is maximal, i.e. $\dim_{\mathbb{C}} L_\rho = \dim_{\mathbb{R}} V$, the element $\rho \in \Lambda^\bullet V^*$ is called a pure form.

Given a maximal isotropic subspace $L \subset V \oplus V^*$ one can always find a pure form annihilating it and conversely if two forms annihilate the same isotropic subspace, then they are multiple of each other. Therefore there is a one to one correspondence between maximal isotropic subspaces and lines of pure forms. Algebraically, if ρ is a pure form, then it must be of the form $e^{B+i\omega}\Omega$, where B and ω are real 2-forms and Ω is a decomposable complex form.

A pure form $e^{B+i\omega}\Omega$ determines a generalized complex structure if and only if $(\rho, \overline{\rho}) \neq 0$. This holds only if V is even dimensional ($\dim V = 2n$), in which case $(\rho, \overline{\rho}) = \Omega \wedge \overline{\Omega} \wedge \omega^{n-k}$, where k is the degree of Ω. Therefore there exist no generalized complex structure on a odd dimensional space and a generalized complex structure is equivalent to a line $K \subset \Lambda^\bullet V^* \otimes \mathbb{C}$ generated by a form $e^{B+i\omega}\Omega$, such that Ω is a decomposable complex form of degree k, B and ω are real 2-forms and $\Omega \wedge \overline{\Omega} \wedge \omega^{n-k} \neq 0$. The degree k of the form Ω is called the type of the generalized complex structure and the line K annihilating L is said the canonical line.

Remark 2.2. Note that the canonical line in $\Lambda^\bullet V^* \otimes \mathbb{C}$ that gives the generalized complex structure for a complex structure is $\Lambda^{n,0}V^*$ and the line for a symplectic form ω is generated by $e^{i\omega}$.

If ρ is a generator of the canonical line of a generalized complex structure $\mathcal{I}$, $e^B \wedge \rho$ is a generator of a B-field transform of $\mathcal{I}$ and $\iota_{e^\beta}\rho$ is the β-transform of $\mathcal{I}$. However, the β-transform doesn't preserve the Courant bracket.

3. Linear generalized hypercomplex structures. We introduce now the notion of generalized hypercomplex structure on a vector space V.

Definition 3.1. *A generalized hypercomplex structure on V is a triple of anti-commuting complex structures $\mathcal{I}, J, K$ on $V \oplus V^*$ with $\mathcal{K} = \mathcal{I}J$, which are compatible with the symmetrized pairing* (1)

In fact, a pair of anti-commuting $\mathcal{I}, J$ defines uniquely $\mathcal{K}$ and leads to generalized hypercomplex structure. Then the set

$$\mathcal{I}_{a,b,c} = \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}$$

forms an S^2-family of complex structures compatible with $\langle \cdot, \cdot \rangle$. We consider hypercomplex structures to be the same if they define the same family.

Note that the existence of a generalized hypercomplex structure on the vector space V implies that the dimension of V has to be a multiple of 4. This is for instance a consequence of the results in Section 4 of [45].

The two basic examples of generalized hypercomplex structure are the usual hypercomplex structure on V and holomorphic symplectic structure, corresponding to a complex structure and two symplectic structures ω_1, ω_2 such that $\omega_1 + i\omega_2$ is a non-degenerate $(2,0)$-form [49]. The existence of B-transform symmetry makes the general case a little more complicated. Note that a usual hyperkahler structures gives rise to 2 different generalized hypercomplex structures according to our definition. Our first observation is

Lemma 3.2. *Let $\mathcal{I}_1 \neq \pm\mathcal{I}_2$ be two generalized complex structures on V. Then they determine a generalized hypercomplex structure iff there is $p, |p| < 1$, such that*

$$\mathcal{I}_1\mathcal{I}_2 + \mathcal{I}_2\mathcal{I}_1 = 2p\, Id. \tag{2}$$

P r o o f. The lemma is well known for the usual hypercomplex structures (see e.g. [24]). The family is constructed by noticing that

$$\mathcal{K} = \frac{1}{2\sqrt{1-p^2}}(\mathcal{I}_1\mathcal{I}_2 - \mathcal{I}_2\mathcal{I}_1)$$

is a generalized complex structure which anti-commutes with $\mathcal{I}_1, \mathcal{I}_2$. Then $\mathcal{I}_1$ and $\mathcal{K}$ define a generalized hypercomplex family which is the same as the one defined by $\mathcal{I}_2$ and $\mathcal{K}$. The compatibility with the pairing $\langle \cdot, \cdot \rangle$ follows from the compatibility of $\mathcal{I}_1$ and $\mathcal{I}_2$, and the equation (2). □

Now consider the action of B-transforms. If

$$\mathcal{I} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

is a symplectic generalized complex structure, then

$$Ad(\exp(B))(\mathcal{I}) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix}.$$

Similarly if $\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$ is a generalized complex structure of complex type then

$$Ad(\exp(B))(\mathcal{I}) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}$$

$$= \begin{pmatrix} J & 0 \\ BJ + J^*B & -J^* \end{pmatrix}.$$

From here we note that B-transformation of a structure of complex type like $\mathcal{I}$ depends only on the $(2,0)+(0,2)$-part of B, since for $(1,1)$-form $BJ+J^*B = 0$. Then the following Lemma is natural:

Lemma 3.3. *Let $\mathcal{I}_1, \mathcal{I}_2$ be two generalized complex structures on V satisfying (2) which arise from complex structures on V up to a B-transforms. Then the corresponding generalized hypercomplex family is given by*

$$\{\mathcal{I}_{a,b,c} = Ad(\exp(B + B_{a,b,c}))I_{a,b,c} \mid (a,b,c) \in S^2\}$$

such that:

(i) $\{I_{a,b,c} \,|\, (a,b,c) \in S^2\}$ are generalized complex structures which arise from complex structures defining a usual hypercomplex structure on V;

(ii) B is some fixed 2-form independent of (a,b,c);

(iii) $B_{a,b,c}$ are self-dual 2-forms depending on (a,b,c) – i.e forms of type $(2,0)+(0,2)$ with respect to every $I_{a',b',c'}$ for $(a',b',c') \in S^2$.

P r o o f. First note that a B-transformation acts non-trivially on generalized complex structures of complex type only when B is of type $(2,0)+(0,2)$, so

$$Ad(\exp(B))\mathcal{I} = Ad(\exp(B^{(2,0)+(0,2)}))\mathcal{I},$$

where $B^{(2,0)+(0,2)}$ is the $(2,0)+(0,2)$-component of B. Since

$$Ad(\exp(B))(\mathcal{I}_1\mathcal{I}_2) = Ad(\exp(B))(\mathcal{I}_1)Ad(\exp(B))(\mathcal{I}_2),$$

we have to check condition (2) for generalized complex structures of the type

$$\mathcal{I}_1 = \begin{pmatrix} J_1 & 0 \\ 0 & -J_1^* \end{pmatrix}$$

and

$$\mathcal{I}_2 = \begin{pmatrix} J_2 & 0 \\ BJ_2 + J_2^*B & -J_2^* \end{pmatrix}.$$

First note that

$$BJ_2 + J_2^*B = 2B^{(2,0)+(0,2)}_{J_2}.$$

A direct calculation shows that (2) is satisfied if and only if $J_1J_2 + J_2J_1 = 2pId|_V$ and $J_1^*B^{(2,0)+(0,2)}_{J_2} + B^{(2,0)+(0,2)}_{J_2}J_1 = 0$. The second identity is $(B^{(2,0)+(0,2)}_{J_2})^{(1,1)}_{J_1} = 0$. From above, we can consider $B = B^{(2,0)+(0,2)}_{J_2}$, so then $B \in \Lambda^{(2,0)+(0,2)}_{J_2} \cap \Lambda^{(2,0)+(0,2)}_{J_1}$. Then one can easily check that B is of type $(2,0)+(0,2)$ with respect to any structure in the family determined by J_1, J_2 and conversely – every structure in a hypercomplex family can be changed by a B-transform in such way and still the family defines generalized hypercomplex structure. This proves the lemma. □

We call such generalized hypercomplex structures *B-twisted hypercomplex*, since the B-transformations depend on the point $(a,b,c) \in S^2$.

Next we consider the case when two generalized complex structures of symplectic type satisfy (2). To compare with the case of complex structures we consider first the types of the structures in the S^2-family of generalized complex structures, defined by a holomorphic symplectic structure. Let

$$\mathcal{I} = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

for a complex structure J and a symplectic structure ω such that $J^*\omega = \omega J$. Then

$$\mathcal{K} = \mathcal{I}J = \begin{pmatrix} 0 & -J\omega^{-1} \\ -J^*\omega & 0 \end{pmatrix}.$$

From $J^*\omega = \omega J$ we get $J^* = \omega J \omega^{-1}$ and $J = \omega^{-1} J^* \omega$. We check directly that

$$\begin{aligned}
(b\omega + c\omega J)(-b\omega^{-1} + c\omega^{-1}J^*) &= -(b^2+c^2)\mathrm{Id},\\
(b\omega J - c\omega)(-b\omega^{-1} + c\omega^{-1}J^*) &= -(b^2+c^2)J^*,\\
(-b\omega^{-1} + c\omega^{-1}J^*)(b\omega J - c\omega) &= -(b^2+c^2)J.
\end{aligned}$$

Using this we see that

$$\mathcal{I}_{a,b,c} = a\mathcal{I} + b\mathcal{J} + c\mathcal{K} = \begin{pmatrix} aJ & -b\omega^{-1} + c\omega^{-1}J^* \\ b\omega + c\omega J & -aJ^* \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 0 \\ \dfrac{a}{b^2+c^2}(b\omega J - c\omega) & 1 \end{pmatrix} \begin{pmatrix} 0 & -b\omega^{-1} + c\omega^{-1}J^* \\ \dfrac{1}{b^2+c^2}(b\omega + c\omega J) & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -\dfrac{a}{b^2+c^2}(b\omega J - c\omega) & 1 \end{pmatrix}$$

and so every structure in the family except $\pm\mathcal{I}$ is of symplectic type.

Now we check when $\mathcal{I}_1, \mathcal{I}_2$ of symplectic type define a holomorphic symplectic structure up to a B-transform.

Lemma 3.4. *Let $\mathcal{I}_1, \mathcal{I}_2$ be two generalized complex structures of symplectic type on $V \oplus V^*$ satisfying* (2)*, such that*

$$\mathcal{I}_1 = \begin{pmatrix} 0 & -\omega_1^{-1} \\ \omega_1 & 0 \end{pmatrix}$$

and

$$\mathcal{I}_2 = \begin{pmatrix} \omega_2^{-1}B & -\omega_2^{-1} \\ \omega_2 + B\omega_2^{-1}B & -B\omega_2^{-1} \end{pmatrix}.$$

Let $A = B\omega_2^{-1}$ and $D = \omega_1\omega_2^{-1} + pId$. Then

$$A^2 + D^2 = (p^2-1)Id \qquad AD = DA. \tag{3}$$

Conversely, for given symplectic forms ω_1, ω_2, a closed form B, and a real number p such that the endomorphisms A and D defined as above satisfy (3)*, the corresponding $\mathcal{I}_1$ and $\mathcal{I}_2$ define a generalized hypercomplex structure.*

P r o o f. The proof is a direct calculation:

$$\mathcal{I}_1\mathcal{I}_2 = \begin{pmatrix} -\omega_1^{-1}\omega_2 - \omega_1^{-1}B\omega_2^{-1}B & \omega_1^{-1}B\omega_2^{-1} \\ \omega_1\omega_2^{-1}B & -\omega_1\omega_2^{-1} \end{pmatrix}$$

and

$$\mathcal{I}_2\mathcal{I}_1 = \begin{pmatrix} -\omega_2^{-1}\omega_1 & -\omega_2^{-1}B\omega_1^{-1} \\ -B\omega_2^{-1}\omega_1 & -\omega_2\omega_1^{-1} - B\omega_2^{-1}B\omega_1^{-1} \end{pmatrix}.$$

Then (2) is equivalent to the system

$$\text{(4)} \qquad \begin{cases} \omega_2^{-1}\omega_1 + \omega_1^{-1}\omega_2 + \omega_1^{-1}B\omega_2^{-1}B = -2p\,\mathrm{Id}, \\ \omega_1\omega_2^{-1} + \omega_2\omega_1^{-1} + B\omega_2^{-1}B\omega_1^{-1} = -2p\,\mathrm{Id}, \\ \omega_1^{-1}B\omega_2^{-1} - \omega_2^{-1}B\omega_1^{-1} = 0, \\ \omega_1\omega_2^{-1}B - B\omega_2^{-1}\omega_1 = 0. \end{cases}$$

Since

$$\omega_1(\omega_2^{-1}\omega_1 + \omega_1^{-1}\omega_2 + \omega_1^{-1}B\omega_2^{-1}B)\omega_1^{-1} = \omega_1\omega_2^{-1} + \omega_2\omega_1^{-1} + B\omega_2^{-1}B\omega_1^{-1}$$

and

$$-\omega_1(\omega_1^{-1}B\omega_2^{-1} - \omega_2^{-1}B\omega_1^{-1})\omega_1 = \omega_1\omega_2^{-1}B - B\omega_2^{-1}\omega_1,$$

the third and the fourth (resp. the first and second) equations in (4) are equivalent. So it is sufficient to show that the second and third equation in (4) are equivalent to the equations (3).

By the definition of D and the second equation in (4) we have

$$\text{(5)} \qquad D = \omega_1\omega_2^{-1} + p\,\mathrm{Id} = -\omega_2\omega_1^{-1} - B\omega_2^{-1}B\omega_1^{-1} - p\,\mathrm{Id}.$$

Thus, by (5) and the definition of A we get

$$AD = B\omega_2^{-1}(-\omega_2\omega_1^{-1} - B\omega_2^{-1}B\omega_1^{-1} - pId) = -B\omega_1^{-1} - B\omega_2^{-1}B\omega_2^{-1}B\omega_1^{-1} - pB\omega_2^{-1}.$$

Similarly we have

$$DA = -\omega_2\omega_1^{-1}B\omega_2^{-1} - B\omega_1^{-1}B\omega_2^{-1}B\omega_2^{-1} - pB\omega_2^{-1}.$$

Multiplying the third equation in (4) by ω_2 to the left gives

$$\omega_2\omega_1^{-1}B\omega_2^{-1} = B\omega_1^{-1}$$

and multiplying by B to the left gives $B\omega_1^{-1}B\omega_2^{-1} = B\omega_2^{-1}B\omega_1^{-1}$, so

$$B\omega_1^{-1}B\omega_2^{-1}B\omega_2^{-1} = B\omega_2^{-1}B\omega_1^{-1}B\omega_2^{-1} = B\omega_2^{-1}B\omega_2^{-1}B\omega_1^{-1}.$$

This proves $AD = DA$. For the second relation between A and D we obtain

$$A^2+D^2 = (B\omega_2^{-1})^2+(\omega_1\omega_2^{-1}+p\,\mathrm{Id})^2 = B\omega_2^{-1}B\omega_2^{-1}+\omega_1\omega_2^{-1}\omega_1\omega_2^{-1}+2p\omega_1\omega_2^{-1}+p^2\,\mathrm{Id}.$$

By the second equation in (4) $2p\,\mathrm{Id} = -\omega_1\omega_2^{-1} - \omega_2\omega_1^{-1} - B\omega_2^{-1}B\omega_1^{-1}$ and multiplying by $\omega_1\omega_2^{-1}$

$$2p\,\omega_1\omega_2^{-1} = -\omega_1\omega_2^{-1}\omega_1\omega_2^{-1} - \mathrm{Id} - B\omega_2^{-1}B\omega_2^{-1}$$

so

$$B\omega_2^{-1}B\omega_2^{-1} + \omega_1\omega_2^{-1}\omega_1\omega_2^{-1} + 2p\,\omega_1\omega_2^{-1} = -\mathrm{Id}.$$

Therefore $A^2 + D^2 = (p^2 - 1)\mathrm{Id}$. The converse follows directly. □

From here we see that two generalized complex structures of symplectic type satisfying (2) will define a holomorphic symplectic structure when they are not twisted by a B-transform. However, in general this is not true. We call the generalized hypercomplex structures which are B-transformations with B not depending on $(a, b, c) \in S^2$ of the structures in Lemma 3.4 *B-transformed hypersymplectic*.

Now we want to see when two generalized complex structures as in (2) define a holomorphic symplectic structure:

Lemma 3.5. *Suppose the generalized complex structures $\mathcal{I}_1, \mathcal{I}_2$ are as in Lemma 3.4 satisfying* (3)*. Then the generalized hypercomplex structure they define contains a structure of maximal type (i.e. a B-transformed complex structure) if and only if there are real numbers a, b, c such that*

$$cB = a\omega_2 + b\omega_1. \tag{6}$$

In case $c \neq 0$ the equations (4) *is equivalent to the single equation:*

$$(\omega_2^{-1}\omega_1 + \gamma Id)^2 = \theta Id, \tag{7}$$

for some real numbers γ and $\theta < 0$.

P r o o f. If we take $\mathcal{K} = \dfrac{1}{2\sqrt{1-p^2}}[\mathcal{I}_1, \mathcal{I}_2]$, then the hypercomplex family defined by $\mathcal{I}_1$ and $\mathcal{I}_2$ is in the span of $\mathcal{I}_1, \mathcal{I}_2, \mathcal{K}$, so by the assumption of the

Lemma there are a, b, c such that the type of $\mathcal{I}_{a,b,c} = a\mathcal{I}_1 + b\mathcal{I}_2 + c\mathcal{K}$ is maximal. A direct computation of the Poisson vector of $\mathcal{I}_{a,b,c}$ gives

$$-a\omega_1^{-1} - b\omega_2^{-1} + \frac{c}{\sqrt{1-p^2}}\omega_1^{-1}B\omega_2^{-1} = 0.$$

Multiplying by ω_1 to the left and ω_2 to the right we obtain (6) with c replacing $\frac{c}{\sqrt{1-p^2}}$. From here we see that the equation $AD = DA$ in the system (3) is satisfied and the first equation in (3) is

$$(B\omega_2^{-1})^2 + (\omega_1\omega_2^{-1} + pId)^2 = (p^2-1)\mathrm{Id}$$

If $c \neq 0$, the equation (6) becomes

$$B = \alpha\omega_1 + \beta\omega_2$$

and substituting it in the first equation of (3) we obtain

$$(\alpha^2+1)(\omega_1\omega_2^{-1})^2 + 2(\alpha\beta + p)\omega_1\omega_2^{-1} = -(1+\beta^2)\mathrm{Id}$$

which by completing the square gives (7) with

$$\gamma = \frac{\alpha\beta + p}{\alpha^2+1}, \quad \theta = \left(\frac{\alpha\beta+p}{\alpha^2+1}\right)^2 - \frac{1+\beta^2}{1+\alpha^2}.$$

The fact that $\theta < 0$ follows from $-1 < p < 1$. □

Now we consider the case that one of the structures in the family has maximal type, i.e. is equivalent up to a B-transformed complex structure. By fixing the complex structure, we see that a B-transformed holomorphic symplectic structure is described by the following:

Lemma 3.6. *Let $\mathcal{I}_1, \mathcal{I}_2$ be two generalized complex structures on $V \oplus V^*$ satisfying* (2)*, such that*

$$\mathcal{I}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$$

and

$$\mathcal{I}_2 = \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix}$$

Then the generalized hypercomplex structure defined by $\mathcal{I}_1$ and $\mathcal{I}_2$ is a B-transformed holomorphic symplectic structure.

P r o o f. Again we have:

$$\mathcal{I}_1\mathcal{I}_2 = \begin{pmatrix} J\omega^{-1}B & -J\omega^{-1} \\ -J^*\omega - J^*B\omega^{-1}B & J^*B\omega^{-1} \end{pmatrix}$$

and

$$\mathcal{I}_2\mathcal{I}_1 = \begin{pmatrix} -\omega^{-1}BJ & -\omega^{-1}J^* \\ \omega J + B\omega^{-1}BJ & B\omega^{-1}J^* \end{pmatrix}.$$

So (2) is equivalent to:

$$\text{(8)} \qquad \begin{cases} \omega J = J^*\omega \\ \omega J - J^*\omega + B\omega^{-1}BJ - J^*B\omega^{-1}B = 0 \\ J\omega^{-1}B + \omega^{-1}BJ = 2pId \\ J^*B\omega^{-1} + B\omega^{-1}J^* = 2pId. \end{cases}$$

Similarly as before using the first equation we see that the third and fourth equations are equivalent to $J^*B + BJ = 2p\omega$.

Now consider the structure $\mathcal{K} = \dfrac{1}{\sqrt{1-p^2}}[\mathcal{I}_1, \mathcal{I}_2]$. By direct calculation using (8) we see that $\operatorname{rank}(\mathcal{K}) = \operatorname{rank}(\omega^{-1}J^*)$, which is maximal. Then $\mathcal{K}$ is a B-transform of some symplectic structure and the generalized hypercomplex structure defined by $\mathcal{I}_1$ and $\mathcal{I}_2$ is the same as the one defined by $\mathcal{I}_1$ and $\mathcal{K}$. So we can replace $\mathcal{I}_2$ by $\mathcal{K}$ in the Lemma and consider only the case $p = 0$. But then from above $J^*B + BJ = 0$ and $Ad(\exp(B))\mathcal{I}_1 = \mathcal{I}_1$. From here it follows that the generalized hypercomplex structure defined by $\mathcal{I}_1, \mathcal{I}_2$ arises from a holomorphic symplectic structure which is transformed by B. □

4. Generalized hypercomplex manifolds and their twistor spaces. Recall that the direct sum $TM \oplus T^*M$ of the tangent and the cotangent bundle of a manifold M carries a Courant bracket

$$[X+\xi, Y+\eta] = [X,Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi), \qquad X, Y \in \Gamma(TM), \quad \xi, \eta \in \Gamma(T^*M).$$

The Courant bracket admits more symmetries than just diffeomorphisms. The extra symmetries are called B-field transformations and are generated by closed 2-forms on M. They correspond to B-transformations of $T_pM \oplus T_p^*M$ at each point p in M. Note that the Courant bracket we define is skew-symmetric and

doesn't satisfy the Jacobi identity, but it is possible to define a bracket, which satisfies the Jacobi identity and is not skew-symmetric.

A *generalized almost complex structure* on M is an endomorphism $\mathcal{I}$ of $TM \oplus T^*M$ of square $-\mathrm{Id}$ and compatible with the canonical pairing. The $\mathcal{I}$ splits the complexification $(TM \oplus T^*M) \otimes \mathbb{C}$ into complex $\pm\sqrt{-1}$-eigenbundles $E^{(1,0)}$ and $E^{(0,1)}$ of type $(1,0)$ and $(0,1)$. A generalized almost complex structure is called integrable if one (and hence both) of these eigenbundles are closed under the complexified Courant bracket. It is equivalent also to vanishing of the generalized version of the Nijenhuis tensor, which is again a real tensor, defined via the Courant bracket. An integrable generalized almost complex structure is called *generalized complex structure*.

Now let $\mathcal{I}, J$ be anti-commuting generalized complex structures. By the result in [39], $\mathcal{K} = \mathcal{I}J = -\mathcal{J}I$ is also integrable and every structure in the family $\{\mathcal{J}_{\mathbf{a}} \,|\, \mathbf{a} \in S^2\}$ is also integrable.

Definition 4.1. *The* twistor space *of a generalized almost hypercomplex manifold M is the smooth manifold $Z = M \times S^2$ endowed with the generalized almost complex structure $\mathbb{I}$ defined at $T_{(p,\mathbf{a})}Z \oplus T^*_{(p,\mathbf{a})}Z$ as*

$$\mathbb{I}|_{(p,\mathbf{a})} = \mathcal{I}_{\mathbf{a}}|_{T_pM \oplus T^*_pM} \oplus \mathcal{J}|_{T_{\mathbf{a}}S^2 \oplus T^*_{\mathbf{a}}S^2},$$

where $\mathcal{J}$ is the generalized complex structure on S^2 determined by the canonical complex structure via $S^2 \cong \mathbb{C}P^1$.

The definition is "half" of the definition in [12] of a twistor space of generalized hyperkähler manifold. Note that we can use the Fubini-Studi symplectic form on S^2 for $\mathcal{J}$ in the definition and the following Theorem is still valid:

Theorem 4.2. *When M is generalized hypercomplex, the structure $\mathbb{I}$ on Z preserves the generalized tangent bundles of M and S^2 and is integrable, i.e. it defines a generalized complex structure.*

The proof is the same as for the usual twistor space for hypercomplex manifolds. Take sections X^h, Y^h of $T^cM \oplus (T^*)^cM$ and U, V of $T^cS^2 \oplus (T^*)^cS^2$ of the complexified generalized tangent bundles which are of type $(1,0)$ with respect to $\mathbb{I}$ and such that

$$X^h|_{(p,\mathbf{a})} = (X,\alpha)|p - i\mathbb{I}_{\mathbf{a}}(X,\alpha)|_p, Y^h|_{(p,\mathbf{a})} = (Y,\beta)|p - i\mathbb{I}_{\mathbf{a}}(Y,\beta)|_p$$

for some sections $(X,\alpha), (Y,\beta)$ of $TM \oplus T^*M$. Clearly such sections span the $(1,0)$-subspace of $T^cZ \oplus (T^*)^cZ$ at each point $(p,\mathbf{a})$. Then one can check directly

that $[X^h, Y^h]$ and $[U, V]$ are in the same $(1,0)$-subspace while $[X^h, U] = [X^h, V] = [Y^h, U] = [Y^h, V] = 0$.

It is useful for the twistor theory to identify S^2 with $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$ via the stereographic projection:

$$St : \lambda \in \mathbb{C} \cup \{\infty\} \to \left(\frac{1 - |\lambda|^2}{1 + |\lambda|^2}, \frac{i(\overline{\lambda} - \lambda)}{1 + |\lambda|^2}, \frac{\lambda + \overline{\lambda}}{1 + |\lambda|^2} \right) \in S^2.$$

If $\mathcal{I}_\lambda$ denotes the structure corresponding to λ via the inverse map $St^{-1}(a, b, c)$, then $\mathcal{I} = \mathcal{I}_0$ and $\mathcal{J} = \mathcal{I}_{\sqrt{-1}}$. Moreover there is a basis $Z_1, W_1, \dots, Z_n, W_n$ of $(1,0)$-vectors with respect to $\mathcal{I}$ such that $\mathcal{J}(Z_i) = -\overline{W_i}, \mathcal{J}(W_i) = \overline{Z_i}$. Then the structure $\mathcal{I}_\lambda$ has $(1,0)$-space given by

$$\mathrm{Span}\{Z_i + \lambda \overline{W_i}, W_i - \lambda \overline{Z_i} \,|\, i = 1, 2, \dots, n\}.$$

Note that $\mathcal{I}_\lambda$ exhaust all structures in the family except $-\mathcal{I}$, which corresponds to the point at infinity of the complex plane with parameter λ.

We want to compare the type of the generalized complex structure on Z with the types of the generalized hypercomplex structures consider in Section 3. To this end we need the following characterization of generalized hypercomplex structures, which generalizes the result in [39] mentioned above:

Theorem 4.3. *Two generalized complex structures $\mathcal{I}_1$ and $\mathcal{I}_2$ are part of a generalized hypercomplex family if and only if they satisfy*

$$\mathcal{I}_1 \mathcal{I}_2 + \mathcal{I}_2 \mathcal{I}_1 = 2p \, Id \tag{9}$$

for a real number p with $|p| < 1$.

P r o o f. The analog for the usual hypercomplex structures is proven by D. Kaledin [41] and here we use his approach. We first notice that by Lemma 2.3 we obtain a generalized almost complex family containing $\mathcal{I}_1$ and $\mathcal{I}_2$.

First consider the linear hypercomplex structure on $V \oplus V^*$ for $V = T_pM$ for some p, defined by anticommuting $\mathcal{I}, \mathcal{J}$ and (1,0)-vectors X, Y for $\mathcal{I}$ such that $\mathcal{J}(X) = \overline{Y}$, so $\mathcal{J}(Y) = -\overline{X}$. Since $\mathcal{J}$ maps the $(1,0)$-space for $\mathcal{I}$ isomorphically to the (0,1)-space, there is a basis of this (1,0)-space $X_1, Y_1, X_2, Y_2, \dots, X_n, Y_n$ with $\mathcal{J}(X_i) = \overline{Y_i}, \mathcal{J}(Y_i) = -\overline{X_i}$. Using the stereographic projection, one can consider $\mathcal{I}_{a,b,c} = \mathcal{I}_\lambda$ for $\lambda \in \mathbb{CP}^1$ being the image of $(a, b, c) \in S^2$. Then as mentioned above one can see that for $\lambda \neq \infty$ a (1,0)-basis for $\mathcal{I}_\lambda$ is given by

$$X_i^\lambda = X_i - \lambda \overline{Y_i}, \qquad Y_i^\lambda = Y_i + \lambda \overline{X_i}.$$

Here $\lambda = i$ corresponds to $\mathcal{J}$. Moreover $\lambda = \infty$ corresponds to $-\mathcal{I}$.

Now consider a manifold with generalized complex structures $\mathcal{I}_1, \mathcal{I}_2$ satisfying $\mathcal{I}_1\mathcal{I}_2 + \mathcal{I}_2\mathcal{I}_1 = 2pId$ where $|p| < 1$ is constant function. Then

$$\mathcal{I}_1,\ (\mathcal{I}_2 + p\mathcal{I}_1)/\sqrt{1-p^2},\ [\mathcal{I}_1, \mathcal{I}_2]/2\sqrt{1-p^2}$$

is a triple of anti-commuting generalized almost complex structures in general. We'll show that they are actually integrable. To this end consider the Nijenhuis tensor of $\mathcal{I}_\lambda$ coupled with the pairing $\langle \cdot, \cdot \rangle$ on $TM \oplus T^*M$ for the structure $\mathcal{I}_\lambda$:

$$N_\lambda(U, V, W) = \langle [U, V] - \mathcal{I}_\lambda[\mathcal{I}_\lambda U, V] - \mathcal{I}_\lambda[U, \mathcal{I}_\lambda V] - [\mathcal{I}_\lambda U, \mathcal{I}_\lambda V], W \rangle,$$

where $[\cdot, \cdot]$ is the Courant bracket. The fact that this is a 3-form on $TM \oplus T^*M$ is proved in [39]. It is clear that $\mathcal{I}_\lambda$ is integrable iff $N_\lambda = 0$. Moreover $N_\lambda = 0$ iff $N_\lambda(U_\lambda, V_\lambda, W_\lambda) = 0$ for every (1,0)-vector fields $U_\lambda, V_\lambda, W_\lambda$ with respect to $\mathcal{I}_\lambda$. An important fact is that

$$N_\lambda(U_\lambda, V_\lambda, W_\lambda) = \langle [U_\lambda, V_\lambda], W_\lambda \rangle.$$

Since N_λ is a tensor, to prove the Theorem, it is enough to fix an arbitrary vector fields $U_\lambda, V_\lambda, W_\lambda$ among the basis X_i^λ, Y_i^λ above. Then

$$N_\lambda(U_\lambda, V_\lambda, W_\lambda) = \lambda^3 N_3 + \lambda^2 N_2 + \lambda N_1 + N_0$$

where N_i for each $i = 0, 1, 2, 3$ are values of a function on M of the type

$$p \to \langle [X, Y]|_p, Z|_p \rangle,$$

where X, Y, Z are among $X_j, Y_j, \overline{X_j}, \overline{Y_j}$, $j = 1, 2, \ldots, n$. Now as in D. Kaledin's argument, we can fix the point p and notice, that $N_\lambda(U_\lambda, V_\lambda, W_\lambda)$ becomes a polynomial of degree 3 in the variable λ, with 4 different roots corresponding to $\pm\mathcal{I}_1, \pm\mathcal{I}_2$, one of them at ∞. But this means that it vanishes identically. Since we choose p and $U_\lambda, V_\lambda, W_\lambda$ arbitrary, the theorem follows. □

By the result in [37] at generic points $(Z, \mathbb{I})$ is a product of complex and symplectic submanifolds and the dimension of the complex submanifold is also the type of $\mathbb{I}$. From the definition it follows that the type of $\mathbb{I}$ is at least one. The next observation follows from Lemma 3.3 and Lemma 3.4:

Theorem 4.4. *If the real dimension of M is $4n$ and the type of $\mathbb{I}$ on Z is $2n+1$ in some open set, then the family $\mathbb{I}|\mathbf{a}$ locally is a B-twisted hypercomplex structure as in Lemma* 3.3. *If the type of $\mathbb{I}$ is one at a point, then the family $\mathbb{I}|\mathbf{a}$, locally is of the type described in Lemma* 3.4. *If in addition for two of the structures in the S^2-family satisfy* (6) *and* (7), *then the structure is B-transformed holomorphic symplectic.*

P r o o f. We notice, that the conditions imply that the type is constant in an open set in both cases, since 1 is the minimal rank of $\mathbb{I}$ and being of minimal rank is an open condition. Such open set is of the form $U \times V$ with $U \subset M$ and $V \subset S^2$. Then in particular there are two generalized complex structures $\mathcal{I}_1$ and $\mathcal{I}_2$ which satisfy the conditions of Lemmas 3.3 or 3.4 for some p close enough to 1 since they will correspond to close λ values in V. □

The question about description of the generalized hypercomplex structure in case the type is between 2 and $2n$ is open at the moment.

5. Examples on tori and Kodaira-Thurston surface.

Example 5.1. Let $e^1, f^1, \dots, e^{2n}, f^{2n}$ be the standard basis of $(\mathbb{R}^{4n})^*$ and

$$\omega_2 = \sum_i (e^{2i-1} \wedge f^{2i-1} - e^{2i} \wedge f^{2i})$$

be the real part of the standard holomorphic symplectic form on $(\mathbb{R}^{4n})^*$ and

$$\omega_1 = \sum_i \lambda_i (e^{2i-1} \wedge f^{2i} + e^{2i} \wedge f^{2i-1})$$

and

$$B = \sum_i \mu_i (e^{2i-1} \wedge f^{2i} + e^{2i} \wedge f^{2i-1})$$

be "diagonal deformations" of the imaginary part such that $\lambda_i^2 + \mu_i^2 = 1$. Then the conditions of the Lemma 3.4 are satisfied with $p = 0$ and

$$A = \operatorname{diag}\left(\begin{pmatrix} 0 & 0 & -\mu_i & 0 \\ 0 & 0 & 0 & -\mu_i \\ \mu_i & 0 & 0 & 0 \\ 0 & \mu_i & 0 & 0 \end{pmatrix}\right), \quad D = \operatorname{diag}\left(\begin{pmatrix} 0 & 0 & -\lambda_i & 0 \\ 0 & 0 & 0 & -\lambda_i \\ \lambda_i & 0 & 0 & 0 \\ 0 & \lambda_i & 0 & 0 \end{pmatrix}\right).$$

From Lemma 3.4 we obtain two generalized complex structures $\mathcal{I}_1, \mathcal{I}_2$ which define a generalized hypercomplex structure. However, the conditions of Lemma 3.5 lead to the fact that this structure is a B-transformed holomorphic symplectic structure only if $\lambda_i = \lambda$ and $\mu_i = \mu$ for some λ and μ. On another side we see that the rank of the generalized complex structures in the family defined by this generalized hypercomplex structure could jump, since the rank of $a\omega_2 + b\omega_1 + cB$ could change for different a, b, c depending on particular λ_i, μ_i. However, no structure is of maximal type unless $\lambda_i = \lambda$ and $\mu_i = \mu$ for some λ and μ.

The above example of a linear generalized hypercomplex structure defines a generalized hypercomplex structure on $4n$-dimensional tori. To see this we just notice the basis $(e_1, f_1, \ldots, e_{2n}, f_{2n})$ could be selected among the standard vector fields and 1-forms $\dfrac{\partial}{\partial x_i}, dx_i, \dfrac{\partial}{\partial y_i}, dy_i$ for coordinates $(x_1, y_1, \ldots, x_{2n}, y_{2n})$ of the universal cover. Since all Courant brackets among them vanish, the generalized almost hypercomplex structure defined in Example 5.1 is in fact generalized hypercomplex. We can summarize these observations in:

Proposition 5.2. *The $4n$-dimensional tori admit a generalized hypercomplex structure in which the generic member of the family is of symplectic type, but the family contains no structure of complex type, hence is not B-transformed holomorphic symplectic.*

Next we provide an example of compact 4-manifold admitting a generalized hypercomplex structure which is neither a hypercomplex nor a holomorphic symplectic structure. The compact 4-manifold is the Kodaira-Thurston surface, i.e. the nilmanifold given by the compact quotient of $H_3 \times \mathbb{R}$ by a lattice, where H_3 is the real Heisenberg 3-dimensional Lie group.

In general the cotangent bundle T^*G of a Lie group G with Lie algebra $\mathfrak{g}$ has a canonical Lie group structure induced by the coadjoint action of the Lie group G on the Lie algebra $\mathfrak{g}$ and it has also a canonical bi-invariant neutral metric. Hermitian structures on T^*G such that left translations are holomorphic isometries are given by almost complex structures of $\mathfrak{g} \oplus \mathfrak{g}^*$ which are orthogonal with respect to the neutral inner product

$$h((X, \alpha), (Y, \beta)) = \frac{1}{2}(\beta(X) + \alpha(Y))$$

and satisfying $N_J = 0$, where N_J is Nihenhuis tensor defined with respect to the

Lie bracket:

$$[(X,\alpha),(Y,\beta)] = ([X,Y], -\beta\circ ad_X + \alpha\circ ad_Y). \tag{10}$$

By Proposition 3.1 in [3] there is a one-to-one correspondence between left-invariant hermitian structures (J,h) on T^*G, where h is the standard neutral metric on T^*G and left-invariant generalized complex structures on G.

Consider on $H_3\times\mathbb{R}$ the structure equations

$$\begin{aligned} de^j &= 0,\ j=0,1,2,\\ de^3 &= -e^1\wedge e^2. \end{aligned}$$

The cotangent bundle $T^*(H_3\times\mathbb{R})$ has, with respect to the Lie bracket (10), structure equations:

$$\begin{aligned} dE^j &= 0,\ j=1,2,3,5,8,\\ dE^4 &= -E^2\wedge E^3,\\ dE^6 &= -E^3\wedge E^8,\\ dE^7 &= E^2\wedge E^8, \end{aligned}$$

where (E^j) is the dual basis of (E_j) with

$$\begin{aligned} &E_1=(e_0,0),\ E_2=(e_1,0),\ E_3=(e_2,0),\ E_4=(e_3,0),\\ &E_5=(e^0,0),\ E_6=(e^1,0),\ E_7=(e^2,0),\ E_8=(e^3,0) \end{aligned}$$

and (e^i) is the dual basis of (e_i).

The Lie group $H_3\times\mathbb{R}$ has a pair of anti-commuting left-invariant generalized complex structures $\mathcal{I},\mathcal{J}$ of type 1, which are given respectively by the left-invariant complex structures on $T^*(H_3\times\mathbb{R})$ defined by the matrices

$$I=\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0,\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$J = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & b_1 & b_2 \\ -1 & -1 & -b_2 & b_1 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{b_2} & 1 & 0 & -b_1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -b_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{3}{b_2} & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -\frac{3}{b_2} & 0 \\ 0 & 0 & 0 & 1 & 0 & b_2 & -1 & 0 \\ -\frac{3}{b_2} & 0 & -1 & 0 & 0 & -b_1 & 0 & -1 \end{pmatrix},$$

where b_1 and $b_2 \neq 0$ are real numbers. It easy to check that I and J are both Hermitian with respect to the standard neutral metric h. Moreover, the generalized hypercomplex $\mathcal{I}_{a,b,c}$ is defined by the matrix

$$\begin{pmatrix} b{+}2c & -a{-}b{+}2c & 2cb_2 & -2cb_1 & 0 & 0 & bb_1 & bb_2 \\ a{-}b{+}2c & -b{-}2c & -bb_2 & bb_1 & 0 & 0 & 2cb_1 & 2cb_2 \\ \frac{-6c}{b_2} & \frac{3b}{b_2} & b{-}2c & 0 & -bb_1 & -2cb_1 & 0 & a{-}b{-}2c \\ 0 & 0 & 0 & b{-}2c & -bb_2 & -2cb_2 & -a{+}b{+}2c & 0 \\ 0 & 0 & 0 & \frac{3b}{b_2} & -b{-}2c & -a{+}b{-}2c & \frac{6c}{b_2} & 0 \\ 0 & 0 & 0 & \frac{6c}{b_2} & a{+}b{-}2c & b{+}2c & \frac{-3b}{b_2} & 0 \\ 0 & 0 & 0 & a{+}b{+}2c & -2cb_2 & bb_2 & -b{+}2c & 0 \\ \frac{-3b}{b_2} & \frac{-6c}{b_2} & -a{-}b{-}2c & 0 & 2cb_1 & -bb_1 & 0 & -b{+}2c \end{pmatrix}$$

and is of type 1.

So we obtain the following:

Proposition 5.3. *The Kodaira-Thurston surface admits both a holomorphic symplectic structure and a generalized hypercomplex structure in which the general complex structures in the generalized hypercomplex family are of type* 1.

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Anna Fino
Dipartimento di Matematica "G. Peano"
Università degli studi di Torino
Via Carlo Alberto 10
10123 Torino, Italy
e-mail: annamaria.fino@unito.it
and
Department of Mathematics and Statistics
Florida International University
Miami, FL 33199, USA
e-mail: afino@fiu.edu

Gueo Grantcharov
Department of Mathematics and Statistics
Florida International University
Miami, FL 33199, USA
e-mail: grantchg@fiu.edu

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