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EXTREMAL FUNCTIONS AND APPROXIMATE INVERSION FORMULAS FOR THE WEINSTEIN TYPE SEGAL-BARGMANN SPACE

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Abstract. In 1961, Bargmann introduced the classical Fock space $\mathscr{F}(\mathbb{C}^d)$ and in 1984, Cholewinski introduced the generalized Fock space $\mathscr{F}_\alpha^*(\mathbb{C}^d)$, were $\alpha = (\alpha_1, \dots, \alpha_d)$. These two spaces are the aim of many works, and have many applications in mathematics, in physics, and in quantum mechanics. In this work, we introduce and study the Fock space $\mathscr{F}_{\alpha_d,*}(\mathbb{C}^d)$ associated to the Weinstein operator Δ_W with $\alpha_d > -1/2$. This space satisfies the inclusions $\mathscr{F}_\alpha^*(\mathbb{C}^d) \subset \mathscr{F}_{\alpha_d,*}(\mathbb{C}^d) \subset \mathscr{F}(\mathbb{C}^d)$. We prove that the space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is a reproducing kernel Hilbert space (RKHS). Next, we examine the extremal functions associated to the difference operator $\mathscr{D}$ and to the primitive operator $\mathscr{P}$, respectively. Furthermore, we establish approximate inversion formulas for the operators $\mathscr{D}$ and $\mathscr{P}$, on $\mathscr{F}_{\alpha_d,*}(\mathbb{C}^d)$.

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1. Introduction. In [1], Bargmann has studied the Fock space $\mathscr{F}(\mathbb{C}^d)$, is a Hilbert space consisting of entire functions on $\mathbb{C}^d$, square integrable with respect to the measure

$$\mathrm{d}m(z) := \frac{1}{\pi^d} e^{-|z|^2} \mathrm{d}x\mathrm{d}y, \quad z = x + iy,$$

where $|z|^2 = \sum_{i=1}^{d} |z_i|^2$ and $\mathrm{d}x\mathrm{d}y = \prod_{i=1}^{d} \mathrm{d}x_i \mathrm{d}y_i$. This space is equipped with the inner product

$$\langle f, g\rangle_{\mathscr{F}(\mathbb{C}^d)} := \int_{\mathbb{C}^d} f(z)\overline{g(z)}\mathrm{d}m(z),$$

and has the reproducing kernel $\mathscr{K}(w, z) = e^{w.\overline{z}}$. Next, the space $\mathscr{F}(\mathbb{C}^d)$ is the aim of many works [2, 6].

In [7], Cholewinski has constructed a generalized Fock space $\mathscr{F}^*_\alpha(\mathbb{C})$ consisting of even entire functions on $\mathbb{C}$, square integrable with respect to the measure

$$\mathrm{d}m^e_\alpha(z) := \frac{1}{\pi 2^\alpha \Gamma(\alpha + 1)} |z|^{2\alpha+2} K_\alpha(|z|^2)\mathrm{d}z,$$

where K_α, $\alpha > -1/2$, is the modified Bessel function of the second kind and index α, also called the Macdonald function [8].

In [17], Sifi and Soltani has constructed a generalized Fock space $\mathscr{F}_\alpha(\mathbb{C})$ consisting of entire functions f on $\mathbb{C}$, such that

$$\int_{\mathbb{C}} |f_e(z)|^2 \mathrm{d}m^e_\alpha(z) + 2(\alpha + 1) \int_{\mathbb{C}} |f_o(z)|^2 |z|^{-2} \mathrm{d}m^e_{\alpha+1}(z) < \infty,$$

where $f_e(z) = \frac{1}{2}(f(z) + f(-z))$ and $f_o(z) = \frac{1}{2}(f(z) - f(-z))$. Next, the space $\mathscr{F}_\alpha(\mathbb{C})$ is the aim of many works [21, 24].

In this work, building on the ideas of Bargmann and Soltani, we define the Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ associated to the Weinstein kernel, to be the space of entire functions on $\mathbb{C}^d$, even with respect to the last variable, square integrable with respect to the measure,

$$\mathrm{d}m_\alpha(z) := \frac{1}{\pi^d 2^\alpha \Gamma(\alpha + 1)} e^{-|z|^2} e^{|z_d|^2} |z_d|^{2\alpha+2} K_\alpha(|z_d|^2)\mathrm{d}z, \quad z = (z_1, \dots, z_d) \in \mathbb{C}^d.$$

The space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is equipped with the inner product

$$\langle f, g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} := \int_{\mathbb{C}^d} f(z)\overline{g(z)}\mathrm{d}m_\alpha(z), \quad z = x + iy.$$

The generalized Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, has also a reproducing kernel

$$\mathscr{K}_\alpha(w, z) = \Psi_\alpha(w, \overline{z}),$$

where $\Psi_\alpha(\cdot,\cdot)$ is the Weinstein kernel defined in Section 2. Then if $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, we have

$$f(z) = \langle f, \mathscr{K}_\alpha(\cdot, z)\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}, \quad z \in \mathbb{C}^d.$$

Let $T : \mathscr{F}_{\alpha,*}(\mathbb{C}^d) \longrightarrow \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ be a bounded linear operator. For any $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ and any $\lambda > 0$, the Tikhonov regularization problem ([14], Theorem 2.5, Section 2),

$$\inf_{f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \left\{\lambda\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} + \|Tf - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\right\}$$

has a unique extremal function denoted $F^*_{\lambda,T}(h)$ and is given by

$$F^*_{\lambda,T}(h) = (\lambda I + T^*T)^{-1}T^*h,$$

where I is the unit operator and T^* is the adjoint of T.

Building on the ideas of Saitoh [14, 15, 16] and Soltani [18, 19, 20, 22, 23], we study the extremal functions associated with the difference operator $\mathscr{D}$ and the primitive operator $\mathscr{P}$ in the multi-variable case, respectively. Finally, we deduce approximate inversion formulas for the difference operator $\mathscr{D}$ and the primitive $\mathscr{P}$, respectively.

The results concerning the extremal functions $F^*_{\lambda,\mathscr{D}}(h)$ and $F^*_{\lambda,\mathscr{P}}(h)$ are special cases of [16]. These results are also an extension of similar results in the classical Fock space $\mathscr{F}(\mathbb{C}^d)$ (see [22]).

The content of the paper is as follows. In Section 2, we introduce the Weinstein type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. In Section 3, we examine the extremal functions associated with the difference operator $\mathscr{D}$ and the primitive operator $\mathscr{P}$. In Sections 4 and 5, we establish approximate inversion formulas for the difference operator $\mathscr{D}$ and the primitive operator $\mathscr{P}$, respectively.

Throughout this paper, we shall use the standard multi-index notations.

- For any $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}^d$, $\nu! = \prod_{i=1}^{d} \nu_i!$. Here $\mathbb{N} = \{0, 1, 2, \dots\}$.
- For all $z = (z_1, \dots, z_d)$, $w = (w_1, \dots, w_d) \in \mathbb{C}^d$, $w.z = \sum_{i=1}^{d} w_i z_i$,

  $$|z| = \left(\sum_{i=1}^{d} |z_i|^2\right)^{1/2}.$$
- For all $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}^d$, $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, $z^\nu = z_d^{\nu_d} \prod_{j=1}^{d} z_j^{\nu_j}$.

• For any $\nu \in \mathbb{N}^d$, the partial ordering $\geq$ on $\mathbb{N}^d$, which is defined by

$$\nu \geq \mathbf{1} \iff \nu_i \geq 1, \forall\, i = 1, \dots, d, \text{ with } \mathbf{1} = (1, \dots, 1).$$

2. Weinstein type Fock space. In this section, we define the Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ associated with the Weinstein kernel in d-dimensions. Next, we prove that the previous space is a reproducing kernel Hilbert space (RKHS).

Let Δ_W be the Weinstein operator [3, 4, 11] defined in the upper half space $\mathbb{R}^d_+ := \mathbb{R}^{d-1} \times (0, \infty)$, by

$$\Delta_W := \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} + \frac{2\alpha + 1}{x_d} \frac{\partial}{\partial x_d} = \Delta_{d-1} + B_\alpha, \quad d \geq 2,\ \alpha > -1/2,$$

where Δ_{d-1} is the Laplacian operator on $\mathbb{R}^{d-1}$ and B_α is the Bessel operator with respect to the variable x_d defined on $(0, \infty)$ by

$$B_\alpha := \frac{\partial^2}{\partial x_d^2} + \frac{2\alpha + 1}{x_d} \frac{\partial}{\partial x_d}.$$

This operator has important applications in both pure and applied mathematics especially in fluid mechanics [27] and give rise to a generalization of multi-variable analytic structures like the Fourier transform, and the standard convolution [3, 4, 5, 12]. For any $\xi \in \mathbb{R}^d_+$, the system

$$B_\alpha u(x) = \xi_d^2 u(x), \quad \frac{\partial^2 u}{\partial x_j^2}(x) = \xi_j^2 u(x), \quad j = 1, \dots, d-1,$$

$$u(0) = 1, \quad \frac{\partial u}{\partial x_d}(0) = 0, \quad \frac{\partial u}{\partial x_j}(0) = \xi_j, \quad j = 1, \dots, d-1.$$

admits a unique solution $\Psi_\alpha(x, \xi)$, given by

$$\Psi_\alpha(x, \xi) = e^{\xi' . x'} \mathcal{I}_\alpha(\xi_d x_d),$$

where $\mathcal{I}_\alpha$ is the modified spherical Bessel function [26] given by

$$\mathcal{I}_\alpha(t) = \Gamma(\alpha + 1) \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k + \alpha + 1)} \left(\frac{t}{2}\right)^{2k}. \tag{2.1}$$

The Weinstein kernel Ψ_α can be extended analytically on $\mathbb{C}^d \times \mathbb{C}^d$, and it can be expanded in a power series in the form

$$\Psi_\alpha(w, z) = \sum_{\nu \in \mathbb{N}^d} \frac{w^\nu z^\nu}{c_\nu(\alpha)}, \quad c_\nu(\alpha) = \frac{2^{2\nu_d} \nu!}{\Gamma(\alpha + 1)} \Gamma(\nu_d + \alpha + 1). \tag{2.2}$$

In the statement, and later in this work we use the following notations.
• m_α, being the measure given by

$$\mathrm{d}m_\alpha(z) := \frac{1}{\pi^d 2^\alpha \Gamma(\alpha+1)} e^{-|z|^2} e^{|z_d|^2} |z_d|^{2\alpha+2} K_\alpha(|z_d|^2)\mathrm{d}z, \tag{2.3}$$

where

$$K_\alpha(z) := \frac{\pi}{2}\frac{I_{-\alpha}(z) - I_\alpha(z)}{\sin(\alpha\pi)}, \quad \alpha \in \mathbb{C}\setminus\mathbb{Z}, \quad |\arg(z)| < \pi,$$

$$K_n(z) := \lim_{\alpha\to n} K_\alpha(z), \quad n \in \mathbb{Z},$$

and

$$I_\alpha(z) := \frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\alpha \mathcal{I}_\alpha(z).$$

Here $\mathcal{I}_\alpha$ is the modified spherical Bessel function given by (2.1).
• $L^2_\alpha(\mathbb{C}^d)$, is the Hilbert space of measurable functions f on $\mathbb{C}^d$, such that

$$\|f\|_{L^2_\alpha(\mathbb{C}^d)} := \left[\int_{\mathbb{C}^d} |f(z)|^2 \mathrm{d}m_\alpha(z)\right]^{1/2} < \infty.$$

• $\mathscr{H}_*(\mathbb{C}^d)$, is the space of entire functions on $\mathbb{C}^d$, which is even with respect to the last variable.

We define the pre Hilbertian space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ as $\mathscr{F}_{\alpha,*}(\mathbb{C}^d) := \mathscr{H}_*(\mathbb{C}^d) \cap L^2_\alpha(\mathbb{C}^d)$. The space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is equipped with the inner product

$$\langle f, g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} := \int_{\mathbb{C}^d} f(z)\overline{g(z)}\mathrm{d}m_\alpha(z),$$

and the norm $\|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} := \sqrt{\langle f, f\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}}$, where m_α being the measure given by (2.3).

Theorem 2.1. *If $f, g \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^\nu$ and $g(z) = \sum_{\nu\in\mathbb{N}^d} b_\nu z^\nu$, then*

$$\langle f, g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} a_\nu \overline{b_\nu} c_\nu(\alpha),$$

where $c_\nu(\alpha)$ are the constants given by (2.2)*.*

Proof. Let $\sigma \in (0,\infty)$, and let denote the set $E_\sigma := \{z \in \mathbb{C}^d : |z_j| \leq$

$\sigma,\ j=1,\dots,d\}$. We compute

$$M(\sigma)=\int_{E_\sigma}|f(z)|^2\mathrm{d}m_\alpha(z).$$

Then,

$$\langle f,f\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}=\lim_{\sigma\to\infty}M(\sigma). \tag{2.4}$$

Using the dominated convergence theorem, we obtain

$$M(\sigma)=\sum_{\nu,\mu\in\mathbb{N}^d}a_\nu\overline{a_\mu}\int_{E_\sigma}z^\nu\overline{z}^\mu\mathrm{d}m_\alpha(z),$$

with $\nu=(\nu_1,\dots,\nu_d)$ and $\mu=(\mu_1,\dots,\mu_d)$. Or

$$\int_{E_\sigma}z^\nu\overline{z}^\mu\mathrm{d}m_\alpha(z)=\prod_{j=1}^{d-1}\left[\frac{1}{\pi}\int_{|z_j|\le\sigma}z_j^{\nu_j}\overline{z_j}^{\mu_j}e^{-|z_j|^2}\mathrm{d}z_j\right]$$
$$\times\left[\frac{1}{\pi2^\alpha\Gamma(\alpha+1)}\int_{|z_d|\le\sigma}z_d^{2\nu_d}\overline{z_d}^{2\mu_d}|z_d|^{2\alpha+2}K_\alpha(|z_d|^2)\mathrm{d}z_d\right].$$

Using polar coordinates $z_j=r_je^{i\theta_j}$, we find

$$\int_{E_\sigma}z^\nu\overline{z}^\mu\mathrm{d}m_\alpha(z)=\prod_{j=1}^{d}\theta_{\nu_j,\mu_j}(\sigma)\delta_{\nu_j,\mu_j},$$

where δ_{ν_j,μ_j} is the Kronecker symbol,

$$\theta_{\nu_j,\nu_j}(\sigma)=\int_0^{\sigma^2}u^{\nu_j}e^{-u}\mathrm{d}u,\quad 1\le j\le d-1,$$

and

$$\theta_{\nu_d,\nu_d}(\sigma)=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\int_0^\sigma r^{4\nu_d+2\alpha+3}K_\alpha(r^2)\mathrm{d}r.$$

Thus

$$M(\sigma)=\sum_{\nu\in\mathbb{N}^d}|a_\nu|^2\prod_{j=1}^{d}\theta_{\nu_j,\nu_j}(\sigma). \tag{2.5}$$

But

$$\lim_{\sigma\to\infty}\theta_{\nu_j,\nu_j}(\sigma)=\nu_j!,\quad 1\le j\le d-1,$$

and from ([8], p. 50) we have

$$\lim_{\sigma\to\infty}\theta_{\nu_d,\nu_d}(\sigma)=\frac{2^{2\nu_d}\nu_d!}{\Gamma(\alpha+1)}\Gamma(\nu_d+\alpha+1).$$

Using (2.4) and (2.5), we obtain from the monotone convergence theorem that

$$\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 c_\nu(\alpha). \tag{2.6}$$

By polarization, we obtain the result for the inner product of two functions. □

Remark 2.2. Bargmann [1] introduced the classical Fock space $\mathscr{F}(\mathbb{C}^d)$ and Cholewinski [7] introduced the generalized Fock space $\mathscr{F}^*_\beta(\mathbb{C}^d) = \mathscr{F}^*_{\beta_1}(\mathbb{C}) \otimes \cdots \otimes \mathscr{F}^*_{\beta_{d-1}}(\mathbb{C}) \otimes \mathscr{F}^*_\alpha(\mathbb{C})$, where $\beta = (\beta_1, \dots, \beta_{d-1}, \alpha)$. The Weinstein type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ introduced in this work satisfies the inclusions

$$\mathscr{F}^*_\beta(\mathbb{C}^d) \subset \mathscr{F}_{\alpha,*}(\mathbb{C}^d) \subset \mathscr{F}(\mathbb{C}^d). \tag{2.7}$$

Proof. Let $\beta = (\beta_1, \dots, \beta_{d-1}, \alpha)$, and let $f \in \mathscr{F}^*_\beta(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^\nu$. From [1], we have

$$\|f\|^2_{\mathscr{F}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 \nu!.$$

Using the inequality $\nu! \leq c_\nu(\alpha)$, we obtain

$$\|f\|^2_{\mathscr{F}(\mathbb{C}^d)} \leq \sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 c_\nu(\alpha) = \|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}. \tag{2.8}$$

On the other hand, from [7], we have

$$\|f\|^2_{\mathscr{F}^*_\beta(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 b_\nu(\beta),$$

where

$$b_\nu(\beta) = \frac{2^{2\nu_d}\nu_d!\Gamma(\nu_d+\alpha+1)}{\Gamma(\alpha+1)} \prod_{j=1}^{d-1} \frac{2^{2\nu_j}\nu_j!\Gamma(\nu_j+\beta_j+1)}{\Gamma(\beta_j+1)}.$$

Using the inequality $c_\nu(\alpha) \leq b_\nu(\beta)$, we obtain

$$\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 b_\nu(\beta) = \|f\|^2_{\mathscr{F}^*_\beta(\mathbb{C}^d)}. \tag{2.9}$$

Then, from (2.8) and (2.9) we deduce (2.7). □

Lemma 2.3. *Let* $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, *then*

$$|f(z)| \leq [\Psi_\alpha(z,\overline{z})]^{1/2} \|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}, \quad z \in \mathbb{C}^d.$$

Proof. Let $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^\nu$. From the Cauchy-Schwarz inequality,

$$|f(z)| \leq \left[\sum_{\nu\in\mathbb{N}^d} \frac{|z^\nu|^2}{c_\nu(\alpha)}\right]^{1/2} \left[\sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 c_\nu(\alpha)\right]^{1/2} = [\Psi_\alpha(z,\overline{z})]^{1/2} \, \|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

which gives the result. □

Theorem 2.4. *The space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is a Hilbert space and the set*

$$\left\{\frac{z^\nu}{\sqrt{c_\nu(\alpha)}}\right\}_{\nu\in\mathbb{N}^d}$$

forms a Hilbertian basis for the space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. Then $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert space $L^2_\alpha(\mathbb{C}^d)$, and $\{f_n\}_{n\in\mathbb{N}}$ converges in $L^2_\alpha(\mathbb{C}^d)$ to a function $f \in L^2_\alpha(\mathbb{C}^d)$. On the other hand, from Lemma 2.3, we have

$$|f_{n+p}(z) - f_n(z)| \leq [\Psi_\alpha(z,\overline{z})]^{1/2} \, \|f_{n+p} - f_n\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}, \quad n,p \in \mathbb{N}.$$

This inequality shows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to f. Since the function $z \to [\Psi_\alpha(z,\overline{z})]^{1/2}$ is continuous on $\mathbb{C}^d$, then $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on all compact sets of $\mathbb{C}^d$. Consequently, by Weierstrass uniform convergence theorem [10], we deduce that $f \in \mathscr{H}_*(\mathbb{C}^d)$, then $f \in L^2_\alpha(\mathbb{C}^d) \cap \mathscr{H}_*(\mathbb{C}^d)$ and hence $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. □

From Lemma 2.3, the map $f \to f(z)$, $z \in \mathbb{C}^d$, is a continuous linear functional on the Hilbert space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. Thus from the Riesz theorem [13], the space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ has a reproducing kernel.

Theorem 2.5. *The function $\mathscr{K}_\alpha$, given for $w, z \in \mathbb{C}^d$ by*

$$\mathscr{K}_\alpha(w,z) = \Psi_\alpha(w,\overline{z}),$$

is a reproducing kernel for the Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, that is $\mathscr{K}_\alpha(.,z) \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, and for all $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, we have $\langle f, \mathscr{K}_\alpha(.,z)\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = f(z)$.

Proof. It is easy to see that $\mathscr{K}_\alpha(\cdot,z) \in \mathscr{H}_*(\mathbb{C}^d)$. And from (2.2) and (2.6), we deduce that

$$\|\mathscr{K}_\alpha(.,z)\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} \left|\frac{(\overline{z})^\nu}{c_\nu(\alpha)}\right|^2 c_\nu(\alpha) = \sum_{\nu\in\mathbb{N}^d} \frac{|(\overline{z})^\nu|^2}{c_\nu(\alpha)} = \Psi_\alpha(z,\overline{z}) < \infty.$$

This $\mathscr{K}_\alpha(.,z) \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. On the other hand, if $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(w) = \sum_{\nu\in\mathbb{N}^d} a_\nu w^\nu$, it follows from Theorem 2.1 that

$$\langle f, \mathscr{K}_\alpha(.,z)\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} a_\nu \frac{z^\nu}{c_\nu(\alpha)} c_\nu(\alpha) = f(z),$$

which completes the proof. □

3. Extremal functions. In this section, building on the ideas of Saitoh [14, 15, 16] we solve the Tikhonov regularization problem associated to the Weinstein type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, for the difference and primitive operators. The results that are written here are special cases of [16].

Let $T : \mathscr{F}_{\alpha,*}(\mathbb{C}^d) \longrightarrow \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ be a bounded linear operator. For any $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ and any $\lambda > 0$, the Tikhonov regularization problem [14, Theorem 2.5, Section 2]

$$\inf_{f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \left\{\lambda\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} + \|Tf - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\right\}$$

has a unique extremal function denoted $F^*_{\lambda,T}(h)$ and is given by

$$F^*_{\lambda,T}(h) = (\lambda I + T^*T)^{-1}T^*h, \tag{3.1}$$

where I is the unit operator and T^* is the adjoint of T.

Example 3.1 (The difference operator)**.** Let $\mathscr{D}$ be the difference operator defined for $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^\nu$, by

$$\mathscr{D}f(z) := \sum_{\nu\in\mathbb{N}^d} a_{\nu+\mathbf{1}} z^\nu. \tag{3.2}$$

In particular cases, for $f \in \mathscr{F}(\mathbb{C})$ (the classical Fock space), the difference operator [9, 22] is given

$$\mathscr{D}f(z) := \begin{cases} \frac{1}{z}(f(z) - f(0)), & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

And, for $f \in \mathscr{F}^*_\alpha(\mathbb{C})$ (the Cholewinski-type Fock space), the difference operator [25] is given

$$\mathscr{D}f(z) := \begin{cases} \frac{1}{z^2}(f(z) - f(0)), & z \neq 0, \\ \frac{1}{2}f''(0), & z = 0. \end{cases}$$

In this example, we determine the extremal function $F^*_{\lambda,\mathscr{D}}(h)$ associated with the difference operator $\mathscr{D}$ on the space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$.

Theorem 3.2. *Let $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^\nu$, we have*

(*i*) $\|\mathscr{D}f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{2\sqrt{\alpha+1}}\|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$,

(*ii*) $\mathscr{D}^*f(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{c_{\nu-\mathbf{1}}(\alpha)}{c_\nu(\alpha)} a_{\nu-\mathbf{1}} z^\nu$,

(*iii*) $\mathscr{D}^*\mathscr{D}f(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{c_{\nu-\mathbf{1}}(\alpha)}{c_\nu(\alpha)} a_\nu z^\nu$,

(*iv*) *For any $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ and for any $\lambda > 0$, the Tikhonov regularization problem*

$$\inf_{f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \left\{\lambda\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} + \|\mathscr{D}f - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\right\},$$

has a unique minimizer given by $F^*_{\lambda,\mathscr{D}}(h)(z) = \langle h, \varphi_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$, where

$$\varphi_z(w) = \sum_{\nu\in\mathbb{N}^d} \frac{(\overline{z})^{\nu+\mathbf{1}}}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)} w^\nu.$$

Proof. Let $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^\nu$.

(*i*) We have

$$\|\mathscr{D}f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} |a_{\nu+\mathbf{1}}|^2 c_\nu(\alpha) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} |a_\nu|^2 c_{\nu-\mathbf{1}}(\alpha).$$

Using the fact that $c_\nu(\alpha) = 4(\nu_1\nu_2\cdots\nu_d)(\nu_d+\alpha)c_{\nu-\mathbf{1}}(\alpha)$, we deduce that

$$\|\mathscr{D}f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{4(\alpha+1)} \sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 c_\nu(\alpha) = \frac{1}{4(\alpha+1)}\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}.$$

(*ii*) If $g \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $g(z) = \sum_{\nu\in\mathbb{N}^d} b_\nu z^\nu$, then

$$\langle\mathscr{D}f, g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} a_{\nu+\mathbf{1}}\overline{b_\nu}c_\nu(\alpha) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} a_\nu\overline{b_{\nu-\mathbf{1}}}c_{\nu-\mathbf{1}}(\alpha) = \langle f, \mathscr{D}^*g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

where

$$\mathscr{D}^* g(z) = \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} \frac{c_{\nu-\mathbf{1}}(\alpha)}{c_\nu(\alpha)} b_{\nu-\mathbf{1}} z^\nu.$$

(iii) From (3.2) and (ii), we deduce that

$$\mathscr{D}^* \mathscr{D} f(z) = \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} \frac{c_{\nu-\mathbf{1}}(\alpha)}{c_\nu(\alpha)} a_\nu z^\nu.$$

(iv) We put $h(z) = \sum_{\nu \in \mathbb{N}^d} h_\nu z^\nu$ and $F^*_{\lambda,\mathscr{D}}(h)(z) = \sum_{\nu \in \mathbb{N}^d} d_\nu z^\nu$. From (3.1), we have $(\lambda I + \mathscr{D}^* \mathscr{D}) F^*_{\lambda,\mathscr{D}}(h)(z) = \mathscr{D}^* h(z)$, by (iii) we deduce that

$$d_\nu = 0, \quad \text{if } \exists\, \nu_i = 0,$$

$$d_\nu = \frac{c_{\nu-\mathbf{1}}(\alpha) h_{\nu-\mathbf{1}}}{\lambda c_\nu(\alpha) + c_{\nu-\mathbf{1}}(\alpha)}, \quad \nu \geq \mathbf{1}.$$

Thus,

$$F^*_{\lambda,\mathscr{D}}(h)(z) = \sum_{\nu \in \mathbb{N}^d} \frac{c_\nu(\alpha) h_\nu}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)} z^{\nu+\mathbf{1}} = \langle h, \varphi_z \rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}, \tag{3.3}$$

where

$$\varphi_z(w) = \sum_{\nu \in \mathbb{N}^d} \frac{(\overline{z})^{\nu+\mathbf{1}}}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)} w^\nu.$$

The theorem is proved. □

Example 3.3 (The primitive operator)**.** Let $\mathscr{P}$ be the primitive operator defined for $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu z^\nu$, by

$$\mathscr{P} f(z) := \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} \frac{a_{\nu-\mathbf{1}}}{\prod_{j=1}^{d-1} \nu_j} z^\nu = \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} \frac{a_{\nu-\mathbf{1}}}{(\nu - \mathbf{1})^*} z^\nu, \tag{3.4}$$

with $\nu * = \prod_{j=1}^{d-1} (\nu_j + 1)$.

In particular cases, for $f \in \mathscr{F}(\mathbb{C})$, the primitive operator [22] is given by

$$\mathscr{P} f(z) := \int_{[0,z]} f(w) \mathrm{d}w,$$

where $[0, z] = \{tz,\ \ t \in [0, 1]\}$. And, for $f \in \mathscr{F}^*_\alpha(\mathbb{C})$, the primitive operator is given

$$\mathscr{P}f(z) := 2\int_{[0,z]} wf(w)\mathrm{d}w.$$

In this example, we determine the extremal function $F^*_{\lambda,\mathscr{P}}(h)$ associated with the primitive operator $\mathscr{P}$ on the space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$.

Theorem 3.4. *Let $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^\nu$, we have*

(*i*) $\|\mathscr{P}f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq 2\sqrt{\alpha+2}\|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$,

(*ii*) $\mathscr{P}^*f(z) = \sum_{\nu\in\mathbb{N}^d} \frac{c_{\nu+\mathbf{1}}(\alpha)}{\nu^* c_\nu(\alpha)} a_{\nu+\mathbf{1}} z^\nu$,

(*iii*) $\mathscr{P}^*\mathscr{P}f(z) = \sum_{\nu\in\mathbb{N}^d} \frac{c_{\nu+\mathbf{1}}(\alpha)}{(\nu^*)^2 c_\nu(\alpha)} a_\nu z^\nu$,

(*iv*) *For any $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ and for any $\lambda > 0$, the Tikhonov regularization problem*

$$\inf_{f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \left\{\lambda\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} + \|\mathscr{P}f - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\right\},$$

has a unique minimizer given by $F^*_{\lambda,\mathscr{P}}(h)(z) = \langle h, \varphi_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$, where

$$\varphi_z(w) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{(\nu-\mathbf{1})^*(\overline{z})^{\nu-\mathbf{1}}}{\lambda((\nu-\mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha) + c_\nu(\alpha)} w^\nu.$$

Proof. Let $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^\nu$.

(*i*) We have

$$\|\mathscr{P}f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{|a_{\nu-\mathbf{1}}|^2}{((\nu-\mathbf{1})^*)^2} c_\nu(\alpha) = \sum_{\nu\in\mathbb{N}^d} \frac{|a_\nu|^2}{(\nu^*)^2} c_{\nu+\mathbf{1}}(\alpha).$$

Using the fact that $c_\nu(\alpha) = 4(\nu_1\nu_2\ldots\nu_d)(\nu_d+\alpha)c_{\nu-\mathbf{1}}(\alpha)$, we deduce that

$$\|\mathscr{P}f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq 4(\alpha+2)\sum_{\nu\in\mathbb{N}^d} |a_\nu|^2 c_\nu(\alpha) = 4(\alpha+2)\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}.$$

(*ii*) If $g \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $g(z) = \sum_{\nu\in\mathbb{N}^d} b_\nu z^\nu$, then

$$\langle\mathscr{P}f, g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{a_{\nu-\mathbf{1}}}{(\nu-\mathbf{1})^*}\overline{b_\nu} c_\nu(\alpha)$$

$$= \sum_{\nu\in\mathbb{N}^d} \frac{a_\nu}{\nu^*} \ \overline{b_{\nu+\mathbf{1}}} c_{\nu+\mathbf{1}}(\alpha) = \langle f, \mathscr{P}^* g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

where

$$\mathscr{P}^* g(z) = \sum_{\nu\in\mathbb{N}^d} \frac{c_{\nu+\mathbf{1}}(\alpha)}{\nu^* c_\nu(\alpha)} b_{\nu+\mathbf{1}} z^\nu.$$

(*iii*) From (3.4) and (*ii*), we deduce that

$$\mathscr{P}^*\mathscr{P} f(z) = \sum_{\nu\in\mathbb{N}^d} \frac{c_{\nu+\mathbf{1}}(\alpha)}{(\nu^*)^2 c_\nu(\alpha)} a_\nu z^\nu.$$

(*iv*) We put $h(z) = \sum_{\nu\in\mathbb{N}^d} h_\nu z^\nu$ and $F^*_{\lambda,\mathscr{P}}(h)(z) = \sum_{\nu\in\mathbb{N}^d} d_\nu z^\nu$, from (3.1) we have $(\lambda I + \mathscr{P}^*\mathscr{P})F^*_{\lambda,\mathscr{P}}(h)(z) = \mathscr{P}^* h(z)$, by (*iii*) we deduce that

$$d_\nu = \frac{\nu^* c_{\nu+\mathbf{1}}(\alpha) h_{\nu+\mathbf{1}}}{\lambda(\nu^*)^2 c_\nu(\alpha) + c_{\nu+\mathbf{1}}(\alpha)}.$$

Thus,

$$F^*_{\lambda,\mathscr{P}}(h)(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{(\nu-\mathbf{1})^* c_\nu(\alpha) h_\nu}{\lambda((\nu-\mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha) + c_\nu(\alpha)} z^{\nu-\mathbf{1}} \tag{3.5}$$

$$= \langle h, \varphi_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

where

$$\varphi_z(w) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{(\nu-\mathbf{1})^* (\overline{z})^{\nu-\mathbf{1}}}{\lambda((\nu-\mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha) + c_\nu(\alpha)} w^\nu.$$

The theorem is proved. □

4. The difference operator. In this section, we examine the extremal function $F^*_{\lambda,\mathscr{D}}(h)(z)$, and we establish approximate inversion formulas for the difference operator $\mathscr{D}$. The results that are written here are an extension of similar results in the classical Fock space $\mathscr{F}(\mathbb{C}^d)$ (see [22]).

Lemma 4.1. *If* $\lambda > 0$ *and* $f, h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, *then*

(*i*) $|F^*_{\lambda,\mathscr{D}}(h)(z)| \leq \dfrac{1}{2\sqrt{\lambda}} \|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} [\Psi_\alpha(z,\overline{z})]^{1/2}$,

(*ii*) $|\mathscr{D}F^*_{\lambda,\mathscr{D}}(h)(z)| \leq \dfrac{1}{4\sqrt{\lambda(\alpha+1)}} \|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} [\Psi_\alpha(z,\overline{z})]^{1/2}$, (*iii*) $|F^*_{\lambda,\mathscr{D}}(\mathscr{D}f)(z)| \leq \frac{1}{4\sqrt{\lambda(\alpha+1)}}\|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}[\Psi_\alpha(z,\overline{z})]^{1/2}$,

(*iv*) $\|F^*_{\lambda,\mathscr{D}}(h)\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{4\lambda}\|h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$.

Proof. Let $\lambda > 0$ and $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $h(z) = \sum_{\nu\in\mathbb{N}^d} h_\nu z^\nu$.

(*i*) From (3.3), we have

$$|F^*_{\lambda,\mathscr{D}}(h)(z)| \leq \|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\|\varphi_z\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}.$$

And using the fact that $(x+y)^2 \geq 4xy$, we obtain

$$\|\varphi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d}\left|\frac{(\overline{z})^{\nu+\mathbf{1}}}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}\right|^2 c_\nu(\alpha) \leq \frac{1}{4\lambda}\sum_{\nu\in\mathbb{N}^d}\frac{|(\overline{z})^\nu|^2}{c_\nu(\alpha)} = \frac{1}{4\lambda}\Psi_\alpha(z,\overline{z}).$$

(*ii*) From (3.2) and (3.3), we have

$$\mathscr{D}F^*_{\lambda,\mathscr{D}}(h)(z) = \sum_{\nu\in\mathbb{N}^d}\frac{c_\nu(\alpha)h_\nu}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}z^\nu = \langle h,\phi_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}, \tag{4.1}$$

where

$$\phi_z(w) = \sum_{\nu\in\mathbb{N}^d}\frac{(\overline{z})^\nu}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}w^\nu,$$

and

$$\|\phi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d}\left|\frac{(\overline{z})^\nu}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}\right|^2 c_\nu(\alpha) \leq \frac{1}{4\lambda}\sum_{\nu\in\mathbb{N}^d}\frac{|(\overline{z})^\nu|^2}{c_{\nu+\mathbf{1}}(\alpha)},$$

and by using the fact that $c_\nu(\alpha) = 4(\nu_1\nu_2\cdots\nu_d)(\nu_d+\alpha)c_{\nu-\mathbf{1}}(\alpha)$, we deduce that

$$\|\phi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{16\lambda(\alpha+1)}\sum_{\nu\in\mathbb{N}^d}\frac{|(\overline{z})^\nu|^2}{c_\nu(\alpha)} = \frac{1}{16\lambda(\alpha+1)}\Psi_\alpha(z,\overline{z}).$$

(*iii*) If $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^\nu$, we have

$$F^*_{\lambda,\mathscr{D}}(\mathscr{D}f)(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}}\frac{c_{\nu-\mathbf{1}}(\alpha)a_\nu}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}z^\nu = \langle f,\Omega_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}, \tag{4.2}$$

and

$$|F^*_{\lambda,\mathscr{D}}(\mathscr{D}f)(z)| \leq \|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\|\Omega_z\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

where

$$\Omega_z(w) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{c_{\nu-\mathbf{1}}(\alpha)(\overline{z})^\nu}{c_\nu(\alpha)(\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha))} w^\nu,$$

and by using the fact that $c_\nu(\alpha) = 4(\nu_1\nu_2\cdots\nu_d)(\nu_d+\alpha)c_{\nu-\mathbf{1}}(\alpha)$, we conclude that

$$\begin{aligned}\|\Omega_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} &= \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \left|\frac{c_{\nu-\mathbf{1}}(\alpha)(\overline{z})^\nu}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}\right|^2 \frac{1}{c_\nu(\alpha)} \\ &\leq \frac{1}{4\lambda}\sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{c_{\nu-\mathbf{1}}(\alpha)}{(c_\nu(\alpha))^2}|(\overline{z})^\nu|^2 \\ &\leq \frac{1}{16\lambda(\alpha+1)}\Psi_\alpha(z,\overline{z}).\end{aligned}$$

(iv) From (3.3), we have

$$\|F^*_{\lambda,\mathscr{D}}(h)\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} c_\nu(\alpha)\left[\frac{c_{\nu-\mathbf{1}}(\alpha)|h_{\nu-\mathbf{1}}|}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}\right]^2.$$

Using the fact that $(x+y)^2 \geq 4xy$, we obtain

$$\|F^*_{\lambda,\mathscr{D}}(h)\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{4\lambda}\sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} c_{\nu-\mathbf{1}}(\alpha)|h_{\nu-\mathbf{1}}|^2 = \frac{1}{4\lambda}\|h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}.$$

This completes the proof of the lemma. □

Theorem 4.2. *If* $\lambda > 0$ *and* $f, h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, *then*

(i) $\lim_{\lambda\to0^+} \|\mathscr{D}F^*_{\lambda,\mathscr{D}}(h) - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = 0,$

(ii) $\lim_{\lambda\to0^+} \|F^*_{\lambda,\mathscr{D}}(\mathscr{D}f) - f_0\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = 0,$

where $f_0(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} a_\nu z^\nu$ *if* $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^\nu$.

P r o o f. Let $\lambda > 0$ and $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $h(z) = \sum_{\nu\in\mathbb{N}^d} h_\nu z^\nu$.

(i) From (4.1), we have

$$\mathscr{D}F^*_{\lambda,\mathscr{D}}(h)(z) - h(z) = \sum_{\nu\in\mathbb{N}^d} \frac{-\lambda c_{\nu+\mathbf{1}}(\alpha)h_\nu}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)} z^\nu;$$

therefore,

$$\|\mathscr{D}F^*_{\lambda,\mathscr{D}}(h) - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} c_\nu(\alpha)\left[\frac{\lambda c_{\nu+\mathbf{1}}(\alpha)|h_\nu|}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}\right]^2,$$

again, by the dominated convergence theorem and the fact that

$$c_\nu(\alpha)\left[\frac{\lambda c_{\nu+\mathbf{1}}(\alpha)|h_\nu|}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}\right]^2 \le c_\nu(\alpha)|h_\nu|^2,$$

we deduce (i).

(ii) If $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^\nu$, then from (4.2) we have

$$F^*_{\lambda,\mathscr{D}}(\mathscr{D}f)(z) - f_0(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\ge\mathbf{1}} \frac{-\lambda c_\nu(\alpha)a_\nu}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)} z^\nu.$$

So, one has

$$\|F^*_{\lambda,\mathscr{D}}(\mathscr{D}f) - f_0\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d,\ \nu\ge\mathbf{1}} c_\nu(\alpha)\left[\frac{\lambda c_\nu(\alpha)|a_\nu|}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}\right]^2.$$

Now, we use the dominated convergence theorem and the fact that

$$c_\nu(\alpha)\left[\frac{\lambda c_\nu(\alpha)|a_\nu|}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}\right]^2 \le c_\nu(\alpha)|a_\nu|^2,$$

we deduce (ii). □

Since $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is a RKHS, the evaluation function is always bounded on $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. Then, from Theorem 4.2, we deduce the following pointwise approximate inversion formulas for the operator $\mathscr{D}$.

Corollary 4.3. *If $\lambda > 0$ and $f, h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, then*

(i) $\lim_{\lambda\to 0^+} \mathscr{D}F^*_{\lambda,\mathscr{D}}(h)(z) = h(z)$,

(ii) $\lim_{\lambda\to 0^+} F^*_{\lambda,\mathscr{D}}(\mathscr{D}f)(z) = f_0(z)$.

5. The primitive operator. In this section, we examine the extremal function $F^*_{\lambda,\mathscr{P}}(h)(z)$, and we establish approximate inversion formulas for the primitive operator $\mathscr{P}$. The results that are written here are an extension of similar results in the classical Fock space $\mathscr{F}(\mathbb{C}^d)$ (see [22]).

Lemma 5.1. *If $\lambda > 0$ and $f, h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, then*

(i) $|F^*_{\lambda,\mathscr{P}}(h)(z)| \leq \frac{1}{2\sqrt{\lambda}}\|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}[\Psi_\alpha(z,\overline{z})]^{1/2}$,

(ii) $|\mathscr{P}F^*_{\lambda,\mathscr{P}}(h)(z)| \leq \sqrt{\frac{\alpha+1}{\lambda}}\|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}[\Psi_\alpha(z,\overline{z})]^{1/2}$,

(iii) $|F^*_{\lambda,\mathscr{P}}(\mathscr{P}f)(z)| \leq \sqrt{\frac{\alpha+1}{\lambda}}\|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}[\Psi_\alpha(z,\overline{z})]^{1/2}$,

(iv) $\|F^*_{\lambda,\mathscr{P}}(h)\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{4\lambda}\|h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$.

P r o o f. Let $\lambda > 0$ and $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $h(z) = \sum_{\nu\in\mathbb{N}^d} h_\nu z^\nu$.

(i) From (3.5), we have

$$|F^*_{\lambda,\mathscr{P}}(h)(z)| \leq \|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\|\varphi_z\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}.$$

Since

$$\|\varphi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \left|\frac{(\nu-\mathbf{1})^*(\overline{z})^{\nu-\mathbf{1}}}{\lambda((\nu-\mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha)+c_\nu(\alpha)}\right|^2 c_\nu(\alpha),$$

and using the fact that $(x+y)^2 \geq 4xy$, we obtain

$$\|\varphi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{4\lambda}\sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}}\frac{|(\overline{z})^{\nu-\mathbf{1}}|^2}{c_{\nu-\mathbf{1}}(\alpha)} = \frac{1}{4\lambda}\Psi_\alpha(z,\overline{z}).$$

(ii) From (3.4) and (3.5), we have

$$\mathscr{P}F^*_{\lambda,\mathscr{P}}(h)(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}}\frac{c_\nu(\alpha)h_\nu}{\lambda((\nu-\mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha)+c_\nu(\alpha)}z^\nu \tag{5.1}$$

$$= \langle h,\phi_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

where

$$\phi_z(w) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}}\frac{(\overline{z})^\nu}{\lambda((\nu-\mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha)+c_\nu(\alpha)}w^\nu.$$

Then,

$$\|\phi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}}\left|\frac{(\overline{z})^\nu}{\lambda((\nu-\mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha)+c_\nu(\alpha)}\right|^2 c_\nu(\alpha),$$

and by using the fact that $c_\nu(\alpha) = 4(\nu_1\nu_2\cdots\nu_d)(\nu_d+\alpha)c_{\nu-\mathbf{1}}(\alpha)$, we deduce that

$$\|\phi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{4\lambda}\sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{|(\overline{z})^\nu|^2}{((\nu-\mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha)}$$

$$\leq \frac{\alpha+1}{\lambda}\sum_{\nu\in\mathbb{N}^d}\frac{|(\overline{z})^\nu|^2}{c_\nu(\alpha)} \leq \frac{\alpha+1}{\lambda}\Psi_\alpha(z,\overline{z}).$$

(*iii*) If $f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z)=\sum_{\nu\in\mathbb{N}^d} a_\nu z^\nu$, we have

$$F^*_{\lambda,\mathscr{P}}(\mathscr{P}f)(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{c_\nu(\alpha)a_{\nu-\mathbf{1}}}{\lambda((\nu-\mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha)+c_\nu(\alpha)} z^{\nu-\mathbf{1}} \tag{5.2}$$

$$= \langle f,\Omega_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

and

$$|F^*_{\lambda,\mathscr{P}}(\mathscr{P}f)(z)| \leq \|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\|\Omega_z\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

where

$$\Omega_z(w) = \sum_{\nu\in\mathbb{N}^d}\frac{c_{\nu+\mathbf{1}}(\alpha)(\overline{z})^\nu}{(\lambda(\nu^*)^2c_\nu(\alpha)+c_{\nu+\mathbf{1}}(\alpha))c_\nu(\alpha)}w^\nu.$$

Then,

$$\|\Omega_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d}\left|\frac{c_{\nu+\mathbf{1}}(\alpha)(\overline{z})^\nu}{\lambda(\nu^*)^2c_\nu(\alpha)+c_{\nu+\mathbf{1}}(\alpha)}\right|^2\frac{1}{c_\nu(\alpha)},$$

and by using the fact that $c_\nu(\alpha) = 4(\nu_1\nu_2\cdots\nu_d)(\nu_d+\alpha)c_{\nu-\mathbf{1}}(\alpha)$, we conclude that

$$\|\Omega_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{4\lambda}\sum_{\nu\in\mathbb{N}^d}\frac{c_{\nu+\mathbf{1}}(\alpha)|(\overline{z})^\nu|^2}{(\nu^*)^2c_\nu^2(\alpha)} \leq \frac{\alpha+1}{\lambda}\Psi_\alpha(z,\overline{z}).$$

(*iv*) From (3.5), we have

$$\|F^*_{\lambda,\mathscr{P}}(h)\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} c_{\nu-\mathbf{1}}(\alpha)\left[\frac{((\nu-\mathbf{1})^*)c_\nu(\alpha)|h_\nu|}{\lambda((\nu-\mathbf{1})^*)^2c_{\nu-\mathbf{1}}(\alpha)+c_\nu(\alpha)}\right]^2.$$

Using the fact that $(x+y)^2\geq 4xy$, we obtain

$$\|F^*_{\lambda,\mathscr{P}}(h)\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{4\lambda}\sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} c_\nu(\alpha)|h_\nu|^2 \leq \frac{1}{4\lambda}\|h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}.$$

This completes the proof of the lemma. □

Theorem 5.2. *If* $\lambda > 0$ *and* $f, h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, *then*

(*i*) $\lim\limits_{\lambda \to 0^+} \|\mathscr{P} F^*_{\lambda,\mathscr{P}}(h) - h_0\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = 0,$

(*ii*) $\lim\limits_{\lambda \to 0^+} \|F^*_{\lambda,\mathscr{P}}(\mathscr{P} f) - f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = 0.$

Proof. Let $\lambda > 0$ and $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $h(z) = \sum\limits_{\nu \in \mathbb{N}^d} h_\nu z^\nu$.

(*i*) From (5.1), we have

$$\mathscr{P} F^*_{\lambda,\mathscr{P}}(h)(z) - h_0(z) = \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} \frac{-\lambda((\nu - \mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha)}{\lambda((\nu - \mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha) + c_\nu(\alpha)} h_\nu z^\nu.$$

Therefore,

$$\|\mathscr{P} F^*_{\lambda,\mathscr{P}}(h) - h_0\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} c_\nu(\alpha) \left[\frac{\lambda((\nu - \mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha) |h_\nu|}{\lambda((\nu - \mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha) + c_\nu(\alpha)}\right]^2,$$

again, by the dominated convergence theorem and the fact that

$$c_\nu(\alpha) \left[\frac{\lambda((\nu - \mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha) |h_\nu|}{\lambda((\nu - \mathbf{1})^*)^2 c_{\nu-\mathbf{1}}(\alpha) + c_\nu(\alpha)}\right]^2 \leq c_\nu(\alpha) |h_\nu|^2,$$

we deduce (*i*).

(*ii*) If $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum\limits_{\nu \in \mathbb{N}^d} a_\nu z^\nu$, then from (5.2) we have

$$F^*_{\lambda,\mathscr{P}}(\mathscr{P} f)(z) - f(z) = \sum_{\nu \in \mathbb{N}^d} \frac{-\lambda(\nu^*)^2 c_\nu(\alpha)}{\lambda(\nu^*)^2 c_\nu(\alpha) + c_{\nu+\mathbf{1}}(\alpha)} a_\nu z^\nu.$$

So, one has

$$\|F^*_{\lambda,\mathscr{P}}(\mathscr{P} f) - f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu \in \mathbb{N}^d} c_\nu(\alpha) \left[\frac{\lambda(\nu^*)^2 c_\nu(\alpha) |a_\nu|}{\lambda(\nu^*)^2 c_\nu(\alpha) + c_{\nu+\mathbf{1}}(\alpha)}\right]^2.$$

Using the dominated convergence theorem and the fat that

$$c_\nu(\alpha) \left[\frac{\lambda(\nu^*)^2 c_\nu(\alpha) |a_\nu|}{\lambda(\nu^*)^2 c_\nu(\alpha) + c_{\nu+\mathbf{1}}(\alpha)}\right]^2 \leq c_\nu(\alpha) |a_\nu|^2,$$

we deduce (*ii*). □

Corollary 5.3. *If* $\lambda > 0$ *and* $f, h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, *then*

(*i*) $\lim\limits_{\lambda \to 0^+} \mathscr{P} F^*_{\lambda,\mathscr{P}}(h)(z) = h_0(z),$

(*ii*) $\lim_{\lambda\to 0^+} F^*_{\lambda,\mathscr{P}}(\mathscr{P}f)(z) = f(z)$.

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