Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **52** (2026), 61–84

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON THE LOCATION OF THE COMPLEX CONJUGATE ZEROS OF THE PARTIAL THETA FUNCTION

Vladimir Petrov Kostov

Communicated by S. Hakkaev

Abstract. We prove that for any $q \in (0,1)$, all complex conjugate pairs of zeros of the partial theta function $\theta(q,x) := \sum_{j=0}^{\infty} q^{j(j+1)/2}x^j$ with non-negative real part belong to the half-annulus $\{\mathrm{Re}(x) \geq 0,\ 1 < |x| < 5\}$, where the outer radius cannot be replaced by a number smaller than $e^{\pi/2} = 4.810477382\ldots$, and that for $q \in (0, 0.2^{1/4} = 0.6687403050\ldots]$, $\theta(q,.)$ has no zeros with non-negative real part. The complex conjugate pairs of zeros with negative real part belong to the left open half-disk of radius 49.8 centered at the origin.

1. Introduction. In this article we study the location of the complex conjugate pairs of zeros of the *partial theta function* defined as the sum of the series

$$\theta(q,x) := \sum_{j=0}^{\infty} q^{j(j+1)/2}x^j \tag{1}$$

https://doi.org/10.55630/serdica.2026.52.61-84

2020 *Mathematics Subject Classification:* 26A06.
Key words: partial theta function, Jacobi theta function, Jacobi triple product.

in which $(q, x) \in (0, 1) \times \mathbb{R}$. We consider x as a variable and q as a parameter. The name of the function matches its resemblance with the *Jacobi theta function* $\Theta(q, x) := \sum_{j=-\infty}^{\infty} q^{j^2} x^j$, because $\theta(q^2, x/q) = \sum_{j=0}^{\infty} q^{j^2} x^j$; the function is "partial", because summation to ∞ is performed from 0 and not from $-\infty$. It satisfies the functional equation

$$\theta(q, x) = 1 + qx\theta(q, qx) \ . \tag{2}$$

The interest in the study of the analytic properties of θ is justified by its applications in different domains. These include Ramanujan type q-series (see [31]), asymptotic analysis (see [2]), the theory of (mock) modular forms (see [4]) and statistical physics and combinatorics (see [28]). In [3] and [5] it is explained how the function θ can be applied to the research of problems about asymptotics and modularity of partial and false theta functions and their interaction with representation theory and conformal field theory.

The partial theta function has found its place in Ramanujan's lost notebook, see [1] and [31]; see also [24] about Appell-Lerch sums (mock theta functions). Andrews-Warnaar identities for the partial theta function are explored in [30], [32] and [29]. The relationship between θ and Artin-Tits monoids are the object of study in [6]. For the Padé approximants of θ see [23].

A recent interest in the properties of θ is motivated by its connection to section-hyperbolic polynomials. These are real polynomials with positive coefficients all whose roots are real negative and all whose finite sections (i.e. truncations) are also with all roots real negative. This research domain was initiated by Hardy, Petrovitch and Hutchinson (see [7], [26] and [8]) and continued in [25], [9] and [22].

The analytic properties of the partial theta function are interesting in their own. In [27] one can find an explicit combinatorial interpretation of the coefficients of the leading root of θ as a series in q. In papers [11]–[21] the author has explored some analytic properties of θ, including the situation when q and x are complex, with $(q, x) \in \mathbb{D}_1 \times \mathbb{C}$, and the case when they are real, with $q \in (-1, 0)$.

For any $q \in (0, 1)$ fixed, the partial theta function has infinitely-many negative and finitely-many complex conjugate pairs of zeros, see [14]. It turns out that for $q \in (0, 1)$, all complex conjugate pairs belong to the domain

$$\mathcal{E}_+ := \{x \in \mathbb{C} \ : \ \text{Re}\ x \in (-5792.7, 0), \ |\text{Im}\ x| < 132\} \ \cup \ \{|x| < 18\};$$

for $q \in (-1, 0)$, the same statement holds true about the domain

$$\mathcal{E}_- := \{x \in \mathbb{C} \ : \ |\mathrm{Re}\ x| < 364.2, \ |\mathrm{Im}\ x| < 132\},$$

see [13]. These results are not trivial at all given that as $q \to \pm 1^{\mp}$ convergence of the series (1) becomes worse.

Thus the complex zeros of θ remain "not far" from the origin for any $q \in (-1, 0) \cup (0, 1)$. One can ask also the question how close to the origin the zeros of θ can be found. In this direction the following results can be cited:

- ([20, Theorem 1]) *For any fixed $q \in (0, 1)$, the partial theta function has no zeros in the domain (see Fig. 1)*

$$\mathcal{D} := \{x \in \mathbb{C} \ : \ |x| \leq 3, \ \mathrm{Re}\ x \leq 0, \ |\mathrm{Im}\ x| \leq 3/\sqrt{2} = 2.121320344\ldots\}.$$

- ([20, Proposition 2]) *For any $q \in (0, 1)$ fixed, the function $\theta(q, .)$ has no real zeros ≥ -5.*

- ([21, Theorem 1]) *For each $q \in (-1, 0) \cup (0, 1)$ fixed, the function θ has no zeros in the closed unit disk $\overline{\mathbb{D}_1}$.* (This result is not true in the situation when q and x are complex, see an example in [21].)

The first result of the present paper reads:

Theorem 1. *For $q \in (0, 1)$, the complex conjugate pairs with non-negative real part (if any) of $\theta(q, .)$ belong to the half-annulus $\mathcal{A} := \{\mathrm{Re}\, x \geq 0, \ 1 < |x| < 5\}$ (see Fig 1).*

The proof is given in Section 3. It uses part (2) of Theorem 2; the latter is proved independently of Theorem 1. Part (1) of Theorem 2 gives an idea how far from optimal the size of the half-annulus $\mathcal{A}$ is:

Theorem 2. (1) *There exists a sequence of values q_s of q and pairs of purely imaginary zeros of $\theta(q_s, .)$ whose modulus tends as $s \to \infty$ to a quantity which is $\geq e^{\pi/2} = 4.810477382\ldots$.*

(2) *For $q \in (0, 0.2^{1/4} = 0.6687403050\ldots]$, the function $\theta(q, .)$ has no zeros with non-negative real part.*

Theorem 2 is proved in Section 4. The number $e^{\pi/2}$ is not far from 5, see Theorem 1, while 5 is much smaller than 18, see the domain $\mathcal{E}_+$ above.

Our next theorem suggests a location of the complex zeros of θ with negative real part, a location which is much smaller than $\mathcal{E}_+$:

Theorem 3. *The complex conjugate pairs of zeros of* $\theta(q,.)$ *which are with negative real part belong to the left open half-disk of radius* 49.8 *centered at the origin.*

The theorem is proved in Section 5. The next section reminds some analytic properties of the partial theta function.

2. The zeros and the spectrum of the partial theta function.

2.1. Zeros and spectrum of θ ***for*** $q \in (0,1)$**.** We define *the spectrum* Γ of θ as the set of values of the parameter q for which $\theta(q,.)$ has a multiple zero (this notion was introduced by B. Shapiro in [22]). When q and x are real, the spectrum consists of real values of q. When q and x are complex, the spectrum contains also complex values, see [18, Proposition 8]. For $q \in (-1,0)$, properties of the spectrum of θ can be found in [16]. In this subsection we remind facts about the zeros and the spectrum of θ for $q \in (0,1)$:

Theorem 4 ([14, Theorem 1])**.** (1) *The spectrum* Γ *consists of countably-many values of* q *denoted by* $0 < \tilde{q}_1 < \tilde{q}_2 < \cdots$, *where* $\lim_{j\to\infty} \tilde{q}_j = 1^-$.

(2) *For* $\tilde{q}_N \in \Gamma$, *the function* $\theta(\tilde{q}_N,.)$ *has exactly one multiple real zero which is of multiplicity* 2 *and is the rightmost of its real zeros.*

(3) *For* $q \in (\tilde{q}_N, \tilde{q}_{N+1})$ *(we set* $\tilde{q}_0 := 0$*), the function* θ *has exactly* N *complex conjugate pairs of zeros (counted with multiplicity).*

Remarks 5. (1) All coefficients of θ as a function in x being positive for $q \in (0,1)$, θ has no positive zeros.

(2) As it was mentioned in [22], the former students A. Broms and I. Nilsson have calculated the first 25 spectral numbers with an accuracy of 12 decimal positions. Their list with 6 decimals reads:

$$
\begin{array}{ccccc}
0.309249, & 0.516959, & 0.630628, & 0.701265, & 0.749269, \\
0.783984, & 0.810251, & 0.830816, & 0.847353, & 0.860942, \\
0.872305, & 0.881949, & 0.890237, & 0.897435, & 0.903747, \\
0.909325, & 0.914291, & 0.918741, & 0.922751, & 0.926384, \\
0.929689, & 0.932711, & 0.935482, & 0.938035, & 0.940393.
\end{array}
$$

Hence for $q \in (0, \tilde{q}_1 = 0.309249\ldots]$, there are no complex zeros and for $q \in (\tilde{q}_1, \tilde{q}_2 = 0.516959\ldots)$, there is exactly one complex conjugate pair.

(3) For $q \in (0, \tilde{q}_1)$, all zeros of θ are negative, so they form a strictly decreasing sequence: $\cdots < \xi_2 < \xi_1 < 0$. The zeros of each of its derivatives w.r.t. the variable x are also real negative. For $q \in (0, \tilde{q}_1)$, it is true that:

1. At even (resp. at odd) zeros the function $\theta(q, .)$ is decreasing (resp. increasing).

2. $\theta > 0$ for $x \in (\xi_{2k+1}, \xi_{2k})$ and $\theta < 0$ for $x \in (\xi_{2k+2}, \xi_{2k+1})$ (see [14, Proposition 6]).

3. For $k \in \mathbb{N}^*$, one has $\theta(q, -q^{-k}) \in (0, q^k)$ (see [14, Proposition 9]). Hence for $q \in (0, \tilde{q}_1)$, the following inequalities hold true:

$$-q^{-2k} < \xi_{2k} < \xi_{2k-1} < -q^{-2k+1}. \tag{3}$$

(4) As q increases and takes the value $\tilde{q}_k$, $k = 1, 2, \ldots$, the zeros ξ_{2k-1} and ξ_{2k} coalesce to form a double zero y_k and then a complex conjugate pair. If one denotes by ξ_k^ℓ the real zeros of $\partial^\ell \theta / \partial x^\ell (q, .)$, $\ell = 0, 1, \ldots$, then for $q \in (0, \tilde{q}_N)$, the following zeros ξ_k^ℓ are well-defined continuous real-valued functions in q: $\cdots < \xi_k^\ell < \cdots < \xi_{2N-1}^\ell < 0$ ([14, Corollary 2]). Items 1 and 2 of part (3) of these remarks and the inequalities (3) remain valid for any $q \in (0, 1)$ for the indices j for which the corresponding zeros ξ_j are well-defined.

(5) The following asymptotic formulae can be found in [15]:

$$\tilde{q}_k = 1 - \pi/2k + (\ln k)/8k^2 + O(1/k^2) \text{ and } y_k = -e^\pi e^{-(\ln k)/4k + O(1/k)}. \tag{4}$$

(6) As q varies in $(0, 1)$, the zeros of the partial theta function do not go to or arrive from infinity. Indeed, the complex zeros remain in the domain $\mathcal{E}_+$. For the real zeros it is true that $\lim_{k \to \infty} \xi_k q^k = -1$ ([14, Theorem 4]). That is, they can be approximated by the terms of a geometric progression with ratio $1/q$. One can also use as argument the fact that for j sufficiently large, there is just one zero ξ_j such that $|q|^{-j+1/2} < |\xi_j| < |q|^{-j-1/2}$, see [18]. Similar remarks are valid in the cases $q \in (-1, 0)$ and $q \in \mathbb{D}_1$.

2.2. Katsnelson's contour. We define *Katsnelson's contour* as the union of two arcs in $\mathbb{R}^2$

$$(x, y) = (e^t \cos t, \pm e^t \sin t) \ , \ \ t \in [0, \pi],$$

see Fig. 1. In [10] the following result is to be found:

Theorem 6. *The family of functions* $f_\varepsilon(z) = \sum_{n=0}^{\infty} e^{-\varepsilon n^2} z^n$ *with* $\varepsilon > 0$ *converges as* $\varepsilon \to 0^+$ *to the function* $1/(1-z)$ *locally uniformly w.r.t.* z *in the interior of Katsnelson's contour. Outside that contour one has* $\overline{\lim_{\varepsilon \to 0^+}} |f_\varepsilon(z)| = \infty$.

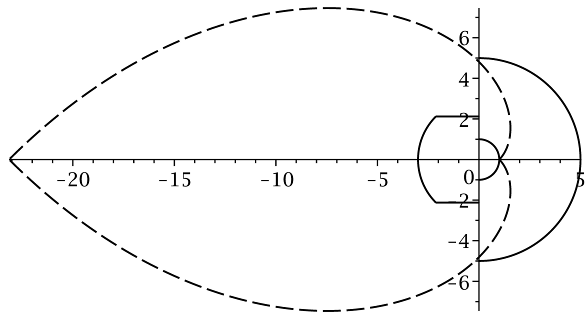

Fig. 1. Katsnelson's contour (in dashed line) and the borders of the domain $\mathcal{D}$ and the half-annulus $\mathcal{A}$ (in solid line)

Remarks 7. (1) The domain inside Katsnelson's contour is much larger than the unit disk in which the Taylor series of the function $1/(1-z)$ converges. Using the notation of the present paper one can say that as $q \to 1^-$, $\theta(q, x)$ tends to $1/(1-x)$ locally uniformly inside Katsnelson's contour. Indeed, set $q := e^{-2\varepsilon}$. Then $f_\varepsilon = \theta(q, z/\sqrt{q})$.

(2) The part of the domain inside Katsnelson's contour belonging to the right half-plane is much smaller than its part in the left half-plane. This can be compared to the fact that one finds much more precisely the possible location of the complex zeros of θ with positive than of the ones with negative real parts, see Theorems 1 and 3 and the domain $\mathcal{E}_+$.

(3) One can observe that to Katsnelson's contour belong the points $e^{(\pi/2)(1\pm i)}$, see Theorem 2.

(4) The angle between the tangent line to Katsnelson's contour and the radius-vector is everywhere equal to $\pm\pi/4$, so this line is vertical for $t = \pi/4$, i. e. at the points

$$e^{(\pi/4)(1\pm i)}, \text{ where } e^{(\pi/4)}\cos(\pi/4) = e^{(\pi/4)}\sin(\pi/4) = 1.550883197\ldots.$$

It is horizontal at the points

$$e^{(3\pi/4)(1\pm i)}, \text{ where } e^{(3\pi/4)}\cos(3\pi/4) = -e^{(3\pi/4)}\sin(3\pi/4) = -7.460488536\ldots.$$

(5) It would be interesting to know whether for any $q \in (\tilde{q}_1, 1)$, all complex conjugate pairs of zeros of $\theta(q, .)$ belong to the interior of Katsnelson's contour.

(6) For $q = 0.8$, the function $\theta_{100}(q, x) := \sum_{j=0}^{100} q^{j(j+1)/2}x^j$ (the 100th truncation of θ) has zeros

$$b_\pm := 0.6128998489\ldots \pm 2.37247194\ldots i \text{ of modulus } 2.450361061\ldots$$

lying inside Katsnelson's contour. Given that the 101th term of θ is of modulus $< 2\times 10^{-460}$ and that the next terms decrease in modulus faster than a geometric progression with ratio 3.2×10^{-10}, and finally that the value of the derivative

$$(\partial\theta/\partial x)(0.8, b_\pm) = -0.6143813197\ldots \mp 1.099995004\ldots i$$

is of modulus > 1.25, it seems likely that $\theta(0.8, .)$ has conjugate zeros close to $b_\pm$. See Remark 15 for other similar numerical results.

3. Proof of Theorem 1.

3.1. The method of proof. By the result of [21] cited before Theorem 1 the partial theta function has no zeros with $|x| \leq 1$, so we prove only that it has no zeros in the domain

$$\mathcal{F} := \{|x| \geq 5, \ -\pi/2 \leq \arg x \leq \pi/2\}.$$

We consider the function Θ^* defined with the help of the Jacobi theta function as follows:

$$\Theta^*(q, x) = \Theta(\sqrt{q}, \sqrt{q}x) = \sum_{j=-\infty}^{\infty} q^{j(j+1)/2}x^j.$$

The function Θ can be represented by the Jacobi triple product

$$\Theta(q,x^2)=\prod_{m=1}^{\infty}(1-q^{2m})(1+x^2q^{2m-1})(1+x^{-2}q^{2m-1})$$

from which one can deduce the equality

$$(5)\qquad \begin{aligned}\Theta^*(q,x) &= \prod_{m=1}^{\infty}(1-q^m)(1+xq^m)(1+q^{m-1}/x)\\ &= (1+1/x)\prod_{m=1}^{\infty}(1-q^m)(1+xq^m)(1+q^m/x).\end{aligned}$$

We set $G:=\sum_{j=-\infty}^{-1}q^{j(j+1)/2}x^j$. Thus $\theta=\Theta^*-G$. We prove that in the domain $\mathcal{F}$ one has $|\Theta^*|>|G|$ from which the theorem follows. For $|x|\geq 5$, the function G satisfies the inequality

$$(6)\qquad |G|\leq\sum_{j=1}^{\infty}5^{-j}=1/4.$$

We use the following lemma:

Lemma 8. *Suppose that* $q\in[0.5,1)$ *is fixed. Then:*

(1) *For* $\arg x\in[-\pi/2,\pi/2]$, *if* $|x|=:B\geq 5$ *is fixed, then* $|\Theta^*(q,x)|$ *is minimal for* $x=\pm Bi$.

(2) *For* $x=\pm Ri$, $R\geq 5$, $|\Theta^*(q,x)|$ *is minimal for* $x=\pm 5i$.

Proof. Part (1). Set $x:=Re^{i\varphi}$, $R>0$, $\varphi\in[-\pi/2,\pi/2]$. Using the cosine rule one obtains

$|1+q^{m-1}/x|=|1+(q^{m-1}R^{-1})e^{-i\varphi}|=\sqrt{1+q^{2m-2}R^{-2}+2q^{m-1}R^{-1}\cos(\varphi)}$ and

$$|1+q^mx| = |1+(q^mR)e^{-i\varphi}| = \sqrt{1+q^{2m}R^2+2q^mR\cos(\varphi)}$$

which quantities are minimal for $\varphi=\pm\pi/2$.

Part (2). For $m\geq 2$, the real and imaginary part of each product

$$(7)\qquad (1+q^mx)(1+q^m/x)=1+q^{2m}+iq^m(R-1/R)$$

are non-decreasing in R for $R \geq 5$ and the modulus of the product is increasing. For the square of the modulus of the triple product

$$T := (1+1/x)(1+qx)(1+q/x) = 1+q^2+q/R^2-q+i(-q/R+qR+1/R+q^2/R)$$

one gets

$$\partial(|T|^2)/\partial R = -4q^2/R^5 - 6q^2/R^3 + 2q^2R - 2/R^3 - 2q^4/R^3$$

which is positive for $R \geq 5$, $q \in [0.5, 1)$. $\square$

For $(q, x) = (0.5, \pm 5i)$, one computes the values of Θ^* and $|\Theta^*|$ numerically:

$$\begin{aligned} \Theta^*(0.5, \pm 5i) &= -1.542068340\ldots \pm 0.4429511372\ldots i, \\ |\Theta^*(0.5, \pm 5i)| &= 1.604425279\ldots > 1/4, \end{aligned} \tag{8}$$

see (6), so $\theta(0.5, .)$ has no zeros in the domain $\mathcal{F}$, see Lemma 8.

Proposition 9. *For $q \in [0.5, 1)$, one has $d(|\Theta^*(q, 5i)|)/dq > 0$ (hence $d(|\Theta^*(q, -5i)|)/dq > 0$).*

The proposition is proved in the next subsection. It implies that for $(q, x) \in [0.5, 1) \times \mathcal{F}$,

$$|\Theta^*(q, x)| - |G(q, x)| \geq |\Theta^*(0.5, 5i)| - 1/4 > 0,$$

so θ has no zeros in $\mathcal{F}$ for $q \in [0.5, 1)$. On the other hand, for $q \in (0, 0.5]$, it has at most one complex conjugate pair of zeros, see Remarks 5, and when it does, the real part of the pair is negative, see part (2) of Theorem 2. This proves Theorem 1.

3.2. Proof of Proposition 9. We prove that $d(|\Theta^*(q, 5i)|^2)/dq > 0$. Making use of equality (7) with $R = 5$ one sees that

$$\begin{aligned} W_m := |(1 + q^m 5i)(1 - q^m i/5)|^2 &= (1 + q^{2m})^2 + (5 - 1/5)^2 q^{2m} \\ &= 1 + (626/25)q^{2m} + q^{4m}. \end{aligned} \tag{9}$$

It is clear that (see (5))

$$d(|\Theta^*(q,5i)|^2)/dq = |\Theta^*|^2(S+W), \text{ where}$$

$$\begin{aligned} S &:= \sum_{m=1}^{\infty}(d(1-q^m)^2/dq)/(1-q^m)^2 \\ &= -\sum_{m=1}^{\infty} 2mq^{m-1}/(1-q^m), \\ W &:= \sum_{m=1}^{\infty}(dW_m/dq)/W_m \\ &= \sum_{m=1}^{\infty}((626/25)+2q^{2m})\cdot 2mq^{2m-1}/(1+(626/25)q^{2m}+q^{4m}), \end{aligned} \tag{10}$$

so $S<0$ and $W>0$. We first estimate the quantity $|S|$.

Proposition 10. *One has* $|S| \leq \pi^2/(3q\ln^2 q)$.

P r o o f. For $x \geq 0$, we define the function f as

$$f(x) = \begin{cases} 2xq^x/q(1-q^x) & \text{for} \quad x>0, \\ -2/(q\ln q) & \text{for} \quad x=0. \end{cases}$$

Lemma 11. *The function f is continuous and decreasing for $x \geq 0$, with* $\lim_{x\to\infty} f(x) = 0^+$.

P r o o f. One finds that $\lim_{x\to 0^+} f(x) = f(0)$, so f is continuous at 0. One has

$$f' = 2q^x g/q(1-q^x)^2 \ , \ \text{where } g := 1 + x\ln q - q^x.$$

Clearly $g' = (\ln q)(1-q^x)$, with $\ln q < 0$ and $1-q^x \geq 0$, so $g(0) = g'(0) = 0$ and $g' \leq 0$, with equality only for $x=0$. Therefore for $x>0$, $g(x)<0$ and $f'<0$.

For the denominator one notices that for $x=0$, there is the equivalence $(1-q^x)^2 \sim ((\ln q)x)^2$. As $g'' = -(\ln^2 q)q^x$, one obtains $f'(0) = -1/q$. Using the limits

$$\lim_{x\to\infty} xq^x = 0^+ \ \text{ and } \ \lim_{x\to\infty}(1-q^x) = 1^- \ ,$$

one concludes that $\lim_{x\to\infty} f = 0^+$. □

The lemma implies that $|S| = f(1) + f(2) + \cdots \leq I := \int_0^\infty f(x)dx$. We set $a := -\ln q$. Integration by parts yields

$$I = [2x\ln(1-q^x)/aq]_0^\infty - (2/aq)\int_0^\infty \ln(1-q^x)dx.$$

The first term to the right is interpreted as the difference of the limits at the upper and lower bounds. Both these limits equal 0. The second term is interpreted as

$$-(2/aq)\lim_{\alpha\to 0^+}\lim_{\omega\to\infty}\int_\alpha^\omega \ln(1-q^x)dx\ .$$

To find this term we set

$$\ln(1-q^x) = -\sum_{j=1}^{\infty} q^{jx}/j$$

(the series converges for $x > 0$), thus

$$\int_0^\infty \ln(1-q^x)dx = \lim_{\alpha\to 0^+}\lim_{\omega\to\infty}\sum_{j=1}^{\infty}[q^{jx}]_\alpha^\omega/(aj^2) = -\sum_{j=1}^{\infty} 1/(aj^2) = -\pi^2/(6a)$$

and $I = 2\pi^2/(6qa^2) = \pi^2/(3qa^2)$. Therefore $|S| \leq \pi^2/(3qa^2) = \pi^2/(3q\ln^2 q)$. □

Proposition 12. *For $q \in [0.5, 1)$, one has $W > 4.46/(q\ln^2 q)$.*

P r o o f. We remind that the quantities W_m are defined in (9). We set $V(y) := 1 + (626/25)y + y^2 = (y + 1/25)(y + 25)$ hence $V'(y) = (626/25) + 2y$ and $h(x) := ((626/25) + 2q^x)xq^{x-1}/(1 + (626/25)q^x + q^{2x}) = \psi(x)\phi(q^x)$, where

$$\begin{cases} \phi(y) & := & V'(y)/V(y) & = & (\ln V(y))' \text{ and} \\ \psi(x) & := & xq^{x-1}\ . \end{cases}$$

As

$$\phi'(y) = -2(y^2 + (626/25)y + (195313/625))/(V(y))^2,$$

(the numerator has no real roots), one has $\phi'(y) < 0$ for $y > 0$. The function q^x (with $q \in (0,1)$) being decreasing, the superposition $\phi(q^x)$ is continuous and increasing for $x \geq 0$, with $\phi(q^0) = 1$.

Lemma 13. *The derivative h' has a single zero $\zeta_0/\ln(q)$,*
$\zeta_0 = -2.685347089\ldots$.

Proof. Set $t := (\ln q)x$, so $q^x = e^t$. With V as above, one has $dh/dx = (\ln q)(dh/dt)$ and

$$dh/dt \;=\; 2e^tU(t)/((q\ln q)(V(e^t))^2)\ , \ \text{ where}$$

$$625U(t) := 625e^{3t} + 7825te^{2t} + 23475e^{2t} + 1250te^t + 196563e^t + 7825t + 7825.$$

The derivative dU/dt, where

$$625dU/dt = 1875e^{3t} + 15650te^{2t} + 54775e^{2t} + 1250te^t + 197813e^t + 7825,$$

is easily shown to take only positive values (the minimal value of te^t and $2te^{2t}$ is $-e^{-1} > -0.37$). As

$$\lim_{t\to-\infty} U(t) = -\infty \ \text{ and } \ \lim_{t\to\infty} U(t) = \infty,$$

one concludes that U has a unique zero ζ_0. Numerical computation yields $\zeta_0 = -2.685347089\ldots$. □

We remind that the terms of the sum W (see (10)) equal $h_m := h(x)|_{x=2m}$. We define the index $m_0 \in \mathbb{N}$ by the condition

$$2m_0 \le \zeta_0/(\ln q) < 2(m_0+1).$$

Lemma 13 implies that the sum W is minorized by the difference

$$I_* - 2h(\zeta_0/(\ln q)), \quad I_* := \int_0^\infty h(x)dx.$$

Indeed, for $m \le m_0$ (resp. for $m \ge m_0+1$), the quantity h_m can be represented as (the surface of) a rectangle with base the interval $[2(m-1), 2m]$ (resp. $[2m, 2(m+1)]$), of length 2, and height h_m. For $m \le m_0$ (resp. for $m \ge m_0+1$), the union of these rectangles forms a figure containing the surface between the graph of the function $h(x)$ and the abscissa-axis for $x \in [0, 2m_0]$ (resp. for $x \in [2(m_0+1), \infty)$). Not covered by these rectangles is the figure representing $\int_{2m_0}^{2(m_0+1)} h(x)dx < 2h(\zeta_0/(\ln q))$.

The change of variables $t = |\ln q|x$ transforms the integral I_* into $(1/(q\ln^2 q))I_\dagger$, with

$$I_\dagger \;:=\; \int_0^\infty t\left(((626/25)e^{-t} + 2e^{-2t})/(1 + (626/25)e^{-t} + e^{-2t})\right) dt = \kappa_\dagger,$$

$$\kappa_\dagger \;:=\; 6.82551484\ldots.$$

On the other hand

$$\begin{aligned} &2h(\zeta_0/(\ln q)) \\ &\quad = 2(1/(q|\ln q|))\left(t((626/25)e^{-t}+2e^{-2t})/(1+(626/25)e^{-t}+e^{-2t})\,|_{t=|\zeta_0|}\right) \\ &\quad = 2r_0\cdot(|\ln q|/(q\ln^2 q)),\ \ r_0 := 1.699895161\ldots, \end{aligned}$$

so for $q\in[0.5,1)$, one has

$$|\ln q|\le|\ln 0.5| = 0.6931471806\ldots,\quad 2h(\zeta_0/(\ln q))\le 2r_0\cdot(|\ln 0.5|/(q\ln^2 q)),$$

$$2r_0\cdot|\ln 0.5| < 2.358 \qquad\qquad \text{and}$$

$$W\ge I_* - 2h(\zeta_0/(\ln q)) > (\kappa_\dagger - 2r_0\cdot|\ln 0.5|)/(q\ln^2 q) > 4.46/(q\ln^2 q). \qquad \Box$$

Recall that $W>0$ and $S<0$, see (10). Propositions 10 and 12 imply that for $q\in[0.5,1)$, the following inequalities hold true:

$$W > 4.46/(q\ln^2 q) > \pi^2/(3q\ln^2 q)\ge|S|\ (\text{with }\pi^2/3 = 3.28\ldots).$$

Hence $d(|\Theta^*(q,5i)|^2)/dp>0$.

4. Proof of Theorem 2.

4.1. Proof of part (1). We use the following representation of θ:

$$\theta(q,x) = \theta(q^4,x^2/q)+qx\theta(q^4,qx^2). \tag{11}$$

We set $v := q^4$, so

$$\theta(q,x) = \theta(v,v^{-1/4}x^2)+v^{1/4}x\theta(v,v^{1/4}x^2). \tag{12}$$

The function θ has a zero on the imaginary axis if and only if for $x\in i\mathbb{R}$, the functions $\theta(v,v^{-1/4}x^2)$ and $\theta(v,v^{1/4}x^2)$ have a common real zero. Indeed, if one sets $x := iy$, $y\in\mathbb{R}$, then

$$\begin{aligned} &\mathrm{Re}(\theta(q,x)) = \psi_1(v,y) := \theta(v,-v^{-1/4}y^2)\ \text{ and} \\ &\mathrm{Im}(\theta(q,x)) = \psi_2(v,y) := v^{1/4}y\theta(v,-v^{1/4}y^2). \end{aligned} \tag{13}$$

The positive zeros of $\psi_1(v,.)$ and $\psi_2(v,.)$ equal

(14) $\chi_k := v^{1/8}(-\xi_k^*)^{1/2}$ and $\mu_k := v^{-1/8}(-\xi_k^*)^{1/2}$ respectively, so $\chi_k = v^{1/4}\mu_k$,

where ξ_k^* stand for the zeros of $\theta(v,.)$. We use the following result (see [14, Theorem 4]): one has $\lim_{k\to\infty} \xi_k q^k = -1$. Hence

(15) $$\lim_{k\to\infty} \xi_k^* v^k = -1, \quad \lim_{k\to\infty} \chi_k^2 v^{k-1/4} = 1 \text{ and } \lim_{k\to\infty} \mu_k^2 v^{k+1/4} = 1.$$

Thus for k sufficiently large,

(16) $$\chi_{k-1} < \mu_{k-1} < \chi_k < \mu_k.$$

Suppose that for $v = v_0 \in (0,1)$ and for $k \geq 2k_1$, the inequalities (16) hold true. Fix $k_2 \geq k_1$ and start increasing v. When v takes the value $\tilde{q}_{k_2}$, one has

$$\chi_{2k_2-1} = \chi_{2k_2} < \mu_{2k_2-1} = \mu_{2k_2}.$$

Hence for some $v = v_{k_2}^* \in (v_0, \tilde{q}_{k_2})$, it is true that

$$\chi_{2k_2-1} < \mu_{2k_2-1} = \chi_{2k_2} < \mu_{2k_2}.$$

i. e. the functions ψ_1 and ψ_2 have a common real zero.

Further we use the notation $\xi_k^*(v)$ to denote the values of the zeros ξ_k^* as functions of the parameter v. In particular, we write $\xi_{2k_2}^*(\tilde{q}_k)$ for $\xi_{2k_2}^*(v)|_{v=\tilde{q}_k}$. We also consider a sequence of indices k_2 and the corresponding double positive zeros of ψ_1 and ψ_2 obtained for $v = \tilde{q}_{k_2}$. The sequences of these zeros tend to $e^{\pi/2}$ as $k_2 \to \infty$, see part (5) of Remarks 5. Indeed, it is true that

$$\lim_{k\to\infty} |\xi_{2k}^*(\tilde{q}_k)| = e^{\pi} \text{ and } \lim_{k\to\infty} \tilde{q}_k = 1^-, \text{ so } \lim_{k_2\to\infty} \chi_{2k_2}(\tilde{q}_{k_2}) = e^{\pi/2} \text{ (see (14))}.$$

Similarly $\lim_{k_2\to\infty} \mu_{2k_2}(\tilde{q}_{k_2}) = e^{\pi/2}$. One has (see (3))

$$|\xi_{2k_2}^*(v_{k_2}^*)| > (v_{k_2}^*)^{-2k_2+1} > (\tilde{q}_{k_2})^{-2k_2+1}.$$

The right-hand side tends to e^{π} as $k_2 \to \infty$. This can be deduced from (3) and (4), see Remarks 5. Hence the sequence of numbers $|\xi_{2k_2}^*(v_{k_2}^*)|$ has an accumulation point which is $\geq e^{\pi}$ (we admit the possibility this point to be ∞).

Recall that $\chi_k = v^{1/8}(-\xi_k^*)^{1/2}$. We show below that one can choose the sequence of indices k_2 such that $\lim_{k_2\to\infty} v_{k_2}^* = 1^-$. Hence there exists an accumulation point of the sequence of points $\chi_{2k_2}(v_{k_2}^*)$ (i. e. of the common zeros of ψ_1 and ψ_2) which is $\geq e^{\pi/2}$.

Now about the choice of the sequence of indices k_2. We denote it by k_2^1, k_2^2, ... When k_2^1 is chosen, then one first increases v up to the value $\tilde{q}_{k_2^1}$ and then chooses the index k_2^2 such that for $v = \tilde{q}_{k_2^1}$ and for $k = 2k_2^2$, the inequalities (16) hold true. Thus $v^*_{k_2^2} > \tilde{q}_{k_2^1} > v^*_{k_2^1}$, the sequence $v^*_{k_2^s}$ is minorized by the sequence $\tilde{q}_{k_2^{s-1}}$ and the latter tends to 1^-. This proves part (1) of Theorem 2, with $q_s = (v^*_{k_2^s})^{1/4}$.

4.2. Proof of part (2) of Theorem 2. We need the following lemma:

Lemma 14. *With the notation of Remarks 5 and for* $q \in (0, 0.2]$, *one has*

$$\xi_{2k+1} \in (-q^{-2k-1.2}, -q^{-2k-1}) \text{ and } \xi_{2k+2} \in (-q^{-2k-2}, -q^{-2k-1.8}), \quad k = 0, 1, \ldots .$$

P r o o f. For $k = 0$, we consider the functions $\tau_1(q) := \theta(q, -q^{-1.2})$ and $\tau_2(q) := \theta(q, -q^{-1.8})$:

$$\tau_1 = 1 - q^{-0.2} + A \ , \quad A := q^{0.6} - q^{2.4} + q^{5.2} - q^9 + \cdots \ ,$$

$$\tau_2 = 1 - q^{-0.8} + B \ , \quad B := q^{-0.6} - q^{0.6} + q^{2.8} - q^6 + \cdots \ .$$

For $q = 0.2$, the sums of the Leibniz series A and B equal $0.3599499830\ldots$ and $2.256770928\ldots$ respectively and one finds that

$$\tau_1(0.2) = -0.0197796780\ldots < 0 \text{ and } \tau_2(0.2) = -0.367127390\ldots < 0.$$

It is easy to show that both functions τ_1 and τ_2 are increasing for $q \in (0, 0.2]$, so one has

$$\xi_1 \in (-q^{-1.2}, -q^{-1}) \text{ and } \xi_2 \in (-q^{-2}, -q^{-1.8}),$$

see 2. and 3. of part (3) of Remarks 5 and the inequalities (3). Recall that (see (2))

$$\theta(q, x) = 1 + qx\theta(q, qx) = 1 + qx + q^3x^2\theta(q, q^2x).$$

Then

$$\theta(q, -q^{-2k-1.2}) = 1 - q^{-2k-0.2} + q^{0.6-4k}\theta(q, -q^{-2(k-1)-1.2}) \qquad \text{and}$$

$$\theta(q, -q^{-2k-1.8}) = 1 - q^{-2k-0.8} + q^{-0.6-4k}\theta(q, -q^{-2(k-1)-1.8}),$$

so if for $q \in (0, 0.2]$,

$$\theta(q, -q^{-2(k-1)-1.2}) < 0 \text{ and } \theta(q, -q^{-2(k-1)-1.8}) < 0,$$

then $\theta(q, -q^{-2k-1.2}) < 0$ and $\theta(q, -q^{-2k-1.8}) < 0$. Thus by induction on k and using (3) the lemma is proved. □

Next we use equations (11), (12) and (13) and the corresponding notation. The zeros of θ depend continuously on the parameter q hence on v. For $q \in (0, \tilde{q}_1]$, all zeros are real negative. If for some value $v_\dagger \in (0, 0.2]$ of v, the function θ has complex conjugate zeros with positive real part, then for some $v = v_1 \in (0, v_\dagger)$, it has a pair of purely imaginary zeros and the functions $\psi_1(v_1, .)$ and $\psi_2(v_1, .)$ have a common zero.

However for $v \in (0, 0.2]$, such a common zero is impossible. Indeed, if one denotes as in the previous subsection by $\pm\chi_j$ and $\pm\mu_j$ the real zeros of the functions $\psi_1(v, .)$ and $\psi_2(v, .)$, where $0 < \chi_j < \chi_{j+1}$ and $0 < \mu_j < \mu_{j+1}$, and by $-\xi_j^*$ the zeros of $\theta(v, .)$, then one has (14).

One cannot have $\chi_j = \mu_s$ for $s \geq j$, because $v \in (0, 1)$ and $\mu_s \geq \mu_j = v^{-1/4}\chi_j > \chi_j$. This is also impossible for $s \leq j - 3$, because $\mu_s \leq \mu_{j-3}$ and by (3) one has

$$\begin{aligned}\mu_{j-3} &= v^{-1/8}\sqrt{-\xi_{j-3}^*} < v^{-1/8-(j-2)/2} = v^{-j/2+7/8} < v^{-j/2+5/8} \\ &= v^{1/8-(j-1)/2} < v^{1/8}\sqrt{-\xi_j^*} = \chi_j.\end{aligned}$$

For $s = j - 2$, if j is odd, then (see (3))

$$\mu_{j-2} = v^{-1/8}\sqrt{-\xi_{j-2}^*} < v^{-1/8-(j-1)/2} < v^{1/8-j/2} < v^{1/8}\sqrt{-\xi_j^*} = \chi_j.$$

If j is even, then

$$\mu_{j-2} = v^{-1/8}\sqrt{-\xi_{j-2}^*} < v^{-1/8-(j-2)/2} < v^{1/8-(j-1)/2} < v^{1/8}\sqrt{-\xi_j^*} = \chi_j.$$

In the remaining case $s = j - 1$ we use the condition $v \in (0, 0.2]$ and Lemma 14. If j is odd, then

$$\mu_{j-1} = v^{-1/8}\sqrt{-\xi_{j-1}^*} < v^{-1/8-(j-1)/2} < v^{1/8-j/2} < v^{1/8}\sqrt{-\xi_j^*} = \chi_j.$$

If j is even, then

$$\mu_{j-1} = v^{-1/8}\sqrt{-\xi_{j-1}^*} < v^{-1/8-(j-1+0.2)/2} < v^{1/8-(j-0.2)/2} < v^{1/8}\sqrt{-\xi_j^*} = \chi_j.$$

Thus for $v \in (0, 0.2]$, i.e., for $q \in (0, 0.2^{1/4} = 0.6687403050\ldots]$, there is no purely imaginary zero of θ. The zeros of θ depending continuously on q and being all real negative for $q \in (0, \tilde{q}_1)$, for $q \in (0, 0.2^{1/4}]$, they are either real negative or complex with negative real part.

Remark 15. For $q = 0.726$, 0.727 and $\tilde{q}_1^{1/4} = 0.7457222066\ldots$, the truncation θ_{100} of θ (see part (6) of Remarks 7) has the following complex conjugate pairs of zeros respectively (they belong to the interior of Katsnelson's contour):

$$-0.004522146605\ldots \pm 2.911439535\ldots i, \; 0.005050176876\ldots \pm 2.904960208\ldots i$$

$$\text{and} \qquad 0.1780767569\ldots \pm 2.779382065\ldots i.$$

These are the zeros closest to the imaginary axis. This makes one suppose that Theorem 2 could hold true with 0.726 instead of $0.2^{1/4}$ in its formulation.

5. Proof of Theorem 3.

5.1. Beginning of the proof of Theorem 3. We remind that y_s denotes the double zero of $\theta(\tilde{q}_s, .)$, see Section 2. By [19, Theorem 5], for $s \geq 15$, one has $y_s > -38.9$. Any real zero of θ is < -5 (see [20, Proposition 2]). Hence for $s \geq 15$, all double zeros of θ belong to the interval $(-38.9, -5)$.

Suppose that $q \in [\tilde{q}_s, \tilde{q}_{s+1})$. Then:

1) for $x \in [-q^{-2s-1}, -q^{-2s+1}]$, one has $\theta(q, x) \geq 0$, with equality only for $(q, x) = (\tilde{q}_s, y_s)$;

2) there exists $x_\dagger \in (-q^{-2s-2}, -q^{-2s-1})$ such that $\theta(q, x_\dagger) < 0$;

3) this means that $x_\dagger/q \in (-q^{-2s-3}, -q^{-2s-2})$ and $\theta(q, x_\dagger/q) = 1 + qx_\dagger\theta(q, x_\dagger) > 1$, see (2).

Lemma 16. *Suppose that for $q \in (0,1)$ fixed and for $x = x_0 < -5$, one has $|\theta(q, x_0)| \geq 1$. Then one has $\theta(q, z) \neq 0$ for $|z| = |x_0|$.*

The lemma is proved in Subsection 5.3. It implies that for $q \in [\tilde{q}_s, \tilde{q}_{s+1})$, no complex zero of θ crosses the circumference $\mathcal{C}(0, q^{-2s-3})$ centered at 0 and of radius q^{-2s-3}. As q grows from $\tilde{q}_s$ to $\tilde{q}_{s+1}$, the quantity q^{-2s-3} decreases. Therefore to find a disk centered at 0 and containing all complex zeros of θ it suffices to find a majoration of the quantities $\tilde{q}_s^{-2s-3}$.

From the fourth to the 25th spectral value (see part (2) of Remarks 5), the corresponding quantities $\tilde{q}_s^{-2s-3}$ are < 49.6 (this can be checked directly). For $s \geq 26$, one knows that

$$|y_s| < 38.9, \;\; y_s \in (-q^{-2s}, -q^{-2s+1}) \text{ and } \tilde{q}_s \in (\tilde{q}_{25}, 1) \;, \;\; \text{with } \tilde{q}_{25} = 0.940393\ldots.$$

That's why for $s \geq 26$,

$$\tilde{q}_s^{-2s-3} = \tilde{q}_s^{-2s+1} \cdot \tilde{q}_s^{-4} < |y_s| \cdot \tilde{q}_s^{-4} < 38.9 \cdot (\tilde{q}_{25})^{-4} < 38.9 \cdot 0.940393^{-4}$$
$$= 49.74\ldots < 49.8.$$

Thus the inequality $\tilde{q}_s^{-2s-3} < 49.8$ is proved for $s \geq 4$. The quantities q^{-2s-3} being decreasing, we have proved the following lemma:

Lemma 17. *For $s \geq 4$ and $q \in [\tilde{q}_s, \tilde{q}_{s+1})$, no complex zero of θ belongs to $\mathcal{C}(0, q^{-2s-3})$.*

In particular, this means that the complex conjugate pairs born from the double zeros y_4, y_5, y_6, ... lie inside the circumference $\mathcal{C}(0, 49.8)$.

5.2. Completion of the proof of Theorem 3. For the remaining three conjugate pairs of zeros (born from the double zeros y_1, y_2 and y_3) we modify our method by considering not only integer, but also half-integer negative powers of q.

Notation 18. We denote by c_j the complex conjugate pair born from the double zero y_j for $q = \tilde{q}_j$, and by c_j^+ (resp. c_j^-) the zero of this pair with positive (resp. negative) imaginary part.

The pair c_3. One finds numerically that $q^{-15/2} < 49.8$ if and only if $q > 0.59\ldots$ and that $\theta(q, -q^{-15/2}) < -1$ if $q < 0.66393\ldots$. The pair c_3 is born for $q = \tilde{q}_3 = 0.630628\ldots$. Hence for $q \in (\tilde{q}_3, 0.663] =: \mathcal{I}$, the pair c_3 is inside the circumference $\mathcal{C}(0, q^{-15/2})$ which in turn is inside the circumference $\mathcal{C}(0, 49.8)$.

One has $q^{-19/2} < 49.8$ if and only if $q > 0.66274\ldots$; it is true that $\theta(q, -q^{-19/2}) < -1$ if $q < 0.72344\ldots$. Thus for $q \in [0.6628, 0.72] =: \mathcal{J}$, the pair c_3 is inside $\mathcal{C}(0, q^{-19/2})$ which is inside $\mathcal{C}(0, 49.8)$.

One has $q^{-21/2} < 49.8$ if and only if $q > 0.6892\ldots$, and $\theta(q, -q^{-21/2}) > 1$ if $q < 0.7465\ldots$. Therefore for $q \in [0.6893, 0.7465] =: \mathcal{K}$, the pair c_3 is inside $\mathcal{C}(0, q^{-21/2})$ which is inside $\mathcal{C}(0, 49.8)$. One notices that $\mathcal{I} \cap \mathcal{J} \neq \emptyset \neq \mathcal{J} \cap \mathcal{K}$.

But the circumference $\mathcal{C}(0, q^{-21/2})$ is inside $\mathcal{C}(0, q^{-11})$ which is inside $\mathcal{C}(0, 49.8)$ for $q \in [\tilde{q}_4, \tilde{q}_5)$ and $\tilde{q}_4 = 0.7012\ldots < 0.7465$, see Lemma 17 with $s = 4$; the lemma implies that the pair c_3 is inside $\mathcal{C}(0, q^{-2s-3})$ for $q \in [\tilde{q}_s, \tilde{q}_{s+1})$, $s \geq 4$. Thus for $q \in [\tilde{q}_4, 1)$ (hence for $q \in (\tilde{q}_3, 1)$), the pair c_3 is inside $\mathcal{C}(0, 49.8)$.

The pair c_2. This pair is born for $q = \tilde{q}_2 = 0.516959\ldots$. The conditions $q^{-11/2} < 49.8$ and $\theta(q, -q^{-11/2}) < -1$ are fulfilled for $q > 0.4913\ldots$ and $q < 0.5721\ldots$ respectively. Hence for $q \in (\tilde{q}_2, 0.5721] \subset [0.4914, 0.5721]$, the pair c_2 is inside $\mathcal{C}(0, q^{-11/2})$ which is inside $\mathcal{C}(0, 49.8)$.

Next, the inequalities $q^{-13/2} < 49.8$ and $\theta(q, -q^{-13/2}) > 1$ hold true for $q > 0.5481\ldots$ and $q < 0.6249\ldots$ respectively, so for $q \in [0.5482, 0.6249]$, the pair

c_2 is inside $\mathcal{C}(0, q^{-13/2})$ which is inside $\mathcal{C}(0, 49.8)$.

In the same way as for the pair c_3 one shows that for $q \in [0.6, 0.663]$, the pair c_2 is inside $\mathcal{C}(0, q^{-15/2})$ and for $q \in [0.6628, 0.72]$, it is inside $\mathcal{C}(0, q^{-19/2})$. In both cases these circumferences are inside $\mathcal{C}(0, 49.8)$. And then one repeats the last two paragraphs of the proof that the pair c_3 is inside $\mathcal{C}(0, 49.8)$ for $q \in (\tilde{q}_3, 1)$.

The pair c_1. We set $\rho_0 := 2^{11/2} = 45.2548\ldots$. We show first that for $q \in (\tilde{q}_1, 1/2]$, one has $|c_1^{\pm}| < \rho_0$. Suppose that for some $q \in (\tilde{q}_1, 1/2]$, one has $|c_1^{\pm}| \geq \rho_0$. Recall that (see [12])

$$\theta(q, x) = \prod_{j=1}^{\infty}(1 - x/\xi_j) = \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j ,$$

where ξ_j are the zeros of $\theta(q, .)$ counted with multiplicity. Then the coefficient of x^2 equals

$$r_2 := q^3 = \sum_{1 \leq k < m} 1/\xi_k \xi_m =: s_2 .$$

For $q \in (\tilde{q}_1, 1/2]$, one obtains $\tilde{q}_1^3 = 0.02957\ldots \leq r_2 \leq 1/8 = 0.125$. We remind that for the real zeros of θ the following inequalities hold true (see (3)):

$$-q^{2j} < \xi_{2j} < \xi_{2j-1} < -q^{2j-1} .$$

In the sum s_2 we set $\xi_1 := c_1^+$ and $\xi_2 := c_1^-$, so

$$s_2 = 1/|c_1^+|^2 + ((1/c_1^+ + 1/c_1^-)\sum_{k=3}^{\infty} 1/\xi_k) + \sum_{3 \leq k < m} 1/\xi_k \xi_m . \qquad (17)$$

One obtains a majoration of the rightmost sum by setting $\xi_{2j} = \xi_{2j-1} = -q^{2j-1}$, $j = 2, 3, \ldots$. In this case the sum equals

$$\begin{aligned}
\phi(q) &:= \sum_{j=1}^{\infty} q^{2j+1}(q^{2j+1} + 2q^{2j+3} + 2q^{2j+5} + \cdots) \\
&= -\sum_{j=1}^{\infty}(q^{2j+1})^2 + 2\sum_{j=1}^{\infty} q^{2j+1}(q^{2j+1} + q^{2j+3} + q^{2j+5} + \cdots) \\
&= -q^6/(1 - q^4) + 2\sum_{j=1}^{\infty} q^{4j+2}/(1 - q^2) = q^6(1 + q^2)/((1 - q^2)(1 - q^4)) .
\end{aligned}$$

The function ϕ is increasing on the interval $[0.3, 0.5]$, with $\phi(0.3) = 0.0008803\ldots$ and $\phi(0.5) = 0.0277\ldots$.
With regard to (17), for $|c_1^\pm| \geq \rho_0$, one obtains $1/|c_1^+|^2 \leq 1/\rho_0^2 = 0.000488\ldots$.

The product $(1/c_1^+ + 1/c_1^-)\sum_{k=3}^{\infty} 1/\xi_k$ is maximal for $c_1^+ = c_1^- = -\rho_0$ (so $1/c_1^+ + 1/c_1^- = 2^{-9/2}$) and $\xi_{2j} = \xi_{2j-1} = -q^{2j-1}$, $j = 2, 3, \ldots$, in which case it equals

$$\delta(q) := 2^{-9/2}(2q^3/(1-q^2)).$$

The difference $q^3 - \phi(q) - 1/\rho_0^2 - \delta(q)$ is positive on $[0.3, 0.5]$; its minimal value $0.0230\ldots$ is attained for $q = 0.3$. Hence the equality $r_2 = s_2$ is impossible for $|c_1^\pm| \geq \rho_0$.

Thus for $q \in (\tilde{q}_1, 1/2]$, the pair c_1 is inside the circumference $\mathcal{C}(0, q^{-11/2})$. In the same way as for the pair c_2 one shows that for $q \in [0.4914, 0.5721]$, the pair c_1 is inside $\mathcal{C}(0, q^{-11/2})$ and for $q \in [0.5482, 0.6249]$, it is inside $\mathcal{C}(0, q^{-13/2})$; as this was shown for the pair c_3 one proves that the pair c_1 is inside $\mathcal{C}(0, q^{-15/2})$ for $q \in [0.6, 0.663]$ and inside $\mathcal{C}(0, q^{-19/2})$ for $q \in [0.6628, 0.72]$; hence in all these cases the pair c_1 is inside $\mathcal{C}(0, 49.8)$. Then by Lemma 17 this pair is inside $\mathcal{C}(0, 49.8)$ for $q \in (\tilde{q}_1, 1)$.

5.3. Proof of Lemma 16. We remind that for $m \in \mathbb{N}^*$, one has $\theta(q, -q^{-m}) \in (0, q^m)$ (see [14, Proposition 9]), so $|\theta(q, -q^{-m})| < 1$. Hence if $|\theta(q, x)| \geq 1$, then $x \neq -q^{-m}$. We use the Jacobi triple product, see the definition of the function Θ^* in Subsection 3.1. Consider the product

$$T_m(q, z) := (1 + q^m z)(1 + q^m/z) = q^m(q^{-m} + z)(1/z)(z + q^m)$$

for $|z| = |x_0|$. The modulus $|q^{-m} + z|$ is minimal for $z = x_0$. Indeed, we saw that $x_0 \neq -q^{-m}$. Consider the circumference $\mathcal{C}_1 := \mathcal{C}(0, |x_0|)$.

If the point $-q^{-m}$ is inside $\mathcal{C}_1$ (i. e. $|x_0| > q^{-m}$), then the circumferences $\mathcal{C}_1$ and $\mathcal{C}_2 := \mathcal{C}(-q^{-m}, |x_0| - q^{-m})$ are tangent at x_0 and $\mathcal{C}_2$ is inside $\mathcal{C}_1$. Hence for $|z| = |x_0|$ and $z \neq x_0$, the point z is outside $\mathcal{C}_2$, so

$$|z + q^{-m}| > |x_0 + q^{-m}|.$$

The circumference $\mathcal{C}_3 := \mathcal{C}(-q^m, |x_0| - q^m)$ is also inside $\mathcal{C}_1$ and tangent to it at x_0, therefore

$$|z + q^m| > |x_0 + q^m| \text{ and } |T_m(q, z)| > |T_m(q, x_0)|.$$

If the point $-q^{-m}$ is outside $\mathcal{C}_1$ (i. e. $|x_0| < q^{-m}$), then the inequality $|z+q^m| > |x_0+q^m|$ is proved in the same way. To show that $|z+q^{-m}| > |x_0+q^{-m}|$, we consider the tangent line $\mathcal{L}$ to $\mathcal{C}_1$ at x_0 and the point $z' \in \mathcal{L}$ such that $\mathrm{Im}\, z' = \mathrm{Im}\, z$. Then

$$|z + q^{-m}| > |z' + q^{-m}| > |x_0 + q^{-m}|$$

and again $|T_m(q,z)| > |T_m(q,x_0)|$. This means that $|\Theta^*(q,z)| > |\Theta^*(q,x_0)|$.

Recall that $\theta = \Theta^* - G$. For $x < -5$, the series of G is a Leibniz series with a negative first term, so $0 < -G < 1/5$. This is why the inequality $\theta(q,x_0) \leq -1$ implies $\Theta^*(q,x_0) < -1$ hence $|\Theta^*(q,z)| > 1$ for $|z| = |x_0|$. As $|G(q,z)| < \sum_{j=1}^{\infty} 1/5^j = 1/4$, one obtains $|\theta(q,z)| > 3/4$ and $\theta(q,.)$ has no zero on $\mathcal{C}_1$.

If $\theta(q,x_0) \geq 1$, then $\Theta^*(q,x_0) > 4/5$, so $|\Theta^*(q,z)| > 4/5$ and $|\theta(q,z)| > 4/5 - 1/4 = 11/20$ for $|z| = |x_0|$.

REFERENCES

[1] G. E. Andrews, B. C. Berndt. Ramanujan's lost notebook. Part II. New York, Springer, 2009.

[2] B. C. Berndt, B. Kim. Asymptotic expansions of certain partial theta functions. *Proc. Amer. Math. Soc.* **139**, *11* (2011), 3779–3788.

[3] K. Bringmann, A. Folsom, A. Milas. Asymptotic behavior of partial and false theta functions arising from Jacobi forms and regularized characters. J. Math. Phys. **58**, *1* (2017), 011702, 19 pp.

[4] K. Bringmann, A. Folsom, R. C. Rhoades. Partial theta functions and mock modular forms as q-hypergeometric series. *Ramanujan J.* **29**, *1–3* (2012), 295–310, `http://arxiv.org/abs/1109.6560`.

[5] T. Creutzig, A. Milas, S. Wood. On regularised quantum dimensions of the singlet vertex operator algebra and false theta functions. *Int. Math. Res. Not. IMRN 5* (2017), 1390–1432.

[6] R. Flores, J. González-Meneses. On the growth of Artin-Tits monoids and the partial theta function. *J. Combin. Theory Ser. A* **190** (2022), Paper No. 105623, 39 pp.

[7] G. H. Hardy. On the zeros of a class of integral functions. *Messenger of Mathematics* **34** (1904), 97–101.

[8] J. I. Hutchinson. On a remarkable class of entire functions. *Trans. Amer. Math. Soc.* **25**, *3* (1923), 325–332.

[9] O. M. Katkova, T. Lobova, A. M. Vishnyakova. On power series having sections with only real zeros. *Comput. Methods Funct. Theory* **3**, *1–2* (2003), 425–441.

[10] V. Katsnelson. On summation of the Taylor series of the function $1/(1-z)$ by the theta summation method. Complex analysis and dynamical systems VI. Part 2, 141–157. Contemp. Math., vol. **667**, Israel Math. Conf. Proc. Providence, RI, AMS, 2016, `https://doi.org/10.48550/arXiv.1402.4629`.

[11] V. P. Kostov. On the multiple zeros of a partial theta function. *Funktsional. Anal. i Prilozhen.* **50**, *2* (2016), 84–88 (in Russian); English translation in *Funct. Anal. Appl.* **50**, *2* (2016), 153–156.

[12] V. P. Kostov. A property of a partial theta function, *C. R. Acad. Bulgare Sci.* **67**, *10* (2014) 1319–1326.

[13] V. P. Kostov. On the complex conjugate zeros of the partial theta function. *Funktsional. Anal. i Prilozhen.* **53**, *2* (2019), 87–91 (in Russian); English translation in *Funct. Anal. Appl.* **53**, *2* (2019), 149–152.

[14] V. P. Kostov. On the zeros of a partial theta function. *Bull. Sci. Math.* **137**, *8* (2013), 1018–1030.

[15] V. P. Kostov. On the double zeros of a partial theta function. *Bull. Sci. Math.* **140**, *4* (2016) 98–111.

[16] V. P. Kostov. On a partial theta function and its spectrum. *Proc. Roy. Soc. Edinburgh Sect. A* **146**, *3* (2016), 609–623.

[17] V. P. Kostov. A domain containing all zeros of the partial theta function. *Publ. Math. Debrecen* **93**, *1–2* (2018), 189–203.

[18] V. P. Kostov. A separation in modulus property of the zeros of a partial theta function. *Anal. Math.* **44**, *4* (2018), 501–519.

[19] V. P. Kostov. On the zero set of the partial theta function. *Serdica Math. J.* **45**, *3* (2019), 225–258.

[20] V. P. Kostov. A domain free of the zeros of the partial theta function. *Mat. Stud.* **58**, *2* (2022), 142–158.

[21] V. P. Kostov. No zeros of the partial theta function in the unit disk. *Annual of Sofia University "St. Kliment Ohridski"*, Faculty of Mathematics and Informatics **111** (2024), 129–137, `https://doi.org/10.60063/gsu.fmi.111.129-137`.

[22] V. P. Kostov, B. Shapiro. Hardy-Petrovitch-Hutchinson's problem and partial theta function. *Duke Math. J.* **162**, *5* (2013), 825–861, `https://doi.org/10.48550/arXiv.1106.6262`.

[23] D. S. Lubinsky, E. B. Saff. Convergence of Padé approximants of partial theta functions and the Rogers-Szegő polynomials. Constr. Approx. **3**, *4* (1987), 331–361.

[24] E. T. Mortenson. On the dual nature of partial theta functions and Appell-Lerch sums. *Adv. Math.* **264** (2014), 236–260.

[25] I. V. Ostrovskii. On zero distribution of sections and tails of power series. Israel Math. Conf. Proc. vol. **15**, 2001, 297–310.

[26] M. Petrovitch. Une classe remarquable de séries entières. Atti del IV Congresso Internazionale dei Matematici, Roma 6–11 Aprile 1908, vol. II (Ser. 1), 1909, 36–43.

[27] T. Prellberg. The combinatorics of the leading root of the partial theta function. arXiv:1210.0095 [math.CO], 2018, `https://doi.org/10.48550/arXiv.1210.0095`, `http://arxiv.org/pdf/1210.0095.pdf`.

[28] A. Sokal. The leading root of the partial theta function. *Adv. Math.* **229**, *5* (2012), 2603–2621, `https://doi.org/10.1016/j.aim.2012.01.012`.

[29] L. H. Sun. An extension of the Andrews-Warnaar partial theta function identity. Adv. in Appl. Math. **115** (2020), 101985, 20 pp.

[30] J. Wang, X. Ma. On the Andrews-Warnaar identities for partial theta functions. Adv. in Appl. Math. **97** (2018), 36–53.

[31] S. O. Warnaar. Partial theta functions. I. Beyond the lost notebook. *Proc. London Math. Soc. (3)* **87**, *2* (2003), 363–395.

[32] C. Wei. Partial theta function identities from Wang and Ma's conjecture. *J. Difference Equ. Appl.* **26**, *4* (2020), 532–539.

Université Côte d'Azur
Centre National de la Recherche Scientifique
Laboratoire de Mathematiques "Jean-Alexandre Dieudonne"
06108 Nice, France
e-mail: `vladimir.kostov@unice.fr`

Received February 2, 2025
Accepted March 23, 2026