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WEIGHTED COMPOSITION OPERATORS ON SPACES OF FUNCTIONS WITH DERIVATIVE IN MIXED-NORM AND HARDY SPACES

Deepjyoti Borgohain

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ABSTRACT. Let φ be an analytic self-map of the open unit disk $\mathbb{D}$ in the complex plane. For any function ψ analytic in the unit disk $\mathbb{D}$, the weighted composition operator is given by $W_{\varphi,\psi}f(z) = \psi(z)f(\varphi(z))$ for $z \in \mathbb{D}$ and f analytic on $\mathbb{D}$. For each $0 < p \leq \infty$, let S_p be the space of analytic functions on $\mathbb{D}$ whose derivatives belong to the Hardy space H_p . For $0 < p, q \leq \infty$ and $\alpha > 0$ let $S(p, q, \alpha)$ be the space of functions with derivative in the mixed norm space $H(p, q, \alpha)$. In this article we deal with boundedness of the weighted composition operators from $S(p, q, \alpha)$ into S_p spaces for $0 < p, q \leq \infty$ and $\alpha > 0$.

1. Introduction. Let $\mathbb{D}$ be the open unit disk in the complex plane and $H(\mathbb{D})$ be the set of all functions analytic in the unit disk. For $0 < p \leq \infty$, the Hardy space H_p is defined as follows:

$$H_p = \left\{ f \in H(\mathbb{D}) : \|f\|_{H_p} = \lim_{r \rightarrow 1} M_p(f, r) < \infty \right\},$$

where

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$$M_p(f, r) = \begin{cases} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}, & \text{if } p \in (0, \infty); \\ \sup_{\theta \in [0, 2\pi]} \{|f(re^{i\theta})|\}, & \text{if } p = \infty. \end{cases}$$

For $0 < p \leq \infty$, we denote by S_p the space of all analytic functions f on the unit disc $\mathbb{D}$ whose derivative f' lies in H_p , endowed with the norm

$$(1.1) \quad \|f\|_{S_p} = |f(0)| + \|f'\|_{H_p}$$

S_p is a Banach space with respect to this norm for all $1 \leq p \leq \infty$. See [2],[3] for more information on this space.

For $\alpha > 0$ and $0 < p, q \leq \infty$ a function $f \in H(\mathbb{D})$ is said to belong to the mixed norm space $H(p, q, \alpha)$ if and only if

$$\|f\|_{H(p, q, \alpha)}^q = \alpha q \int_0^1 (1-r)^{\alpha q - 1} M_p^q(f, r) dr < \infty,$$

if $q < \infty$ and

$$\|f\|_{H(p, \infty, \alpha)} = \sup_{0 \leq r < 1} (1-r)^\alpha M_p(f, r) < \infty.$$

For $1 \leq p \leq \infty$ the space $H(p, q, \alpha)$ is a Banach space. For $p = q$ and replacing α by $\frac{\alpha+1}{p}$ we have $H\left(p, p, \frac{\alpha+1}{p}\right) \equiv A_\alpha^p$, the weighted Bergman space. In particular one can identify the Hardy space H_p with $H(p, \infty, 0)$. For $0 < p, q \leq \infty, 0 < \alpha < \infty$ a function analytic in the unit disk $\mathbb{D}$ belongs to $S(p, q, \alpha)$ if $f' \in H(p, q, \alpha)$. It is endowed with the norm

$$\|f\|_{S(p, q, \alpha)} = |f(0)| + \|f'\|_{H(p, q, \alpha)}.$$

For $\psi(z)$ analytic in the unit disk $\mathbb{D}$ and φ an analytic self map of the unit disk $\mathbb{D}$ the weighted composition operator $W_{\varphi, \psi}$ is defined as

$$W_{\varphi, \psi} f(z) = \psi(z) f(\varphi(z)), \quad z \in \mathbb{D}.$$

For $\psi(z) \equiv 1$, $W_{\varphi, \psi}$ is the Composition Operator C_φ and for $\varphi(z) = z$, $W_{\varphi, \psi}$ is the multiplication operator M_z . Weighted composition find its usefulness in many different ways. For example they are isometries of many Banach spaces

of analytic functions. Novinger and Oberin proved that the isometries in S_p for $1 \leq p < \infty, p \neq 2$ are given by

$$Tf(z) = \lambda_1 \left[f(0) + \int_0^z W_{\varphi, \lambda_2(\varphi')^{1/p}}(f')(\xi) d\xi \right]$$

for $f \in S_p$ where $|\lambda_1| = |\lambda_2| = 1$ and φ is a self-analytic map of $\mathbb{D}$.

In 1978, the S_p spaces were introduced by Roan in [5] where he studied the boundedness and compactness of the composition operators C_φ , ($1 \leq p < \infty$).

Contreras and Hernández-Díaz characterized the boundedness, compactness of $W_{\varphi, \psi}$ from S_p into S_q , $1 \leq p, q \leq \infty$ in terms of weighted composition operators on Hardy spaces.

In this paper we characterize the boundedness and compactness of the operator $W_{\psi, \varphi}$ from $S(p, q, \alpha)$ to S_p spaces. Throughout this paper C is any constant which may vary for different lines.

2. Preliminary results. We collect some basic lemmas which are useful in the proof of the main results.

Lemma 2.1 (Proposition 1.4.10, [6]). *For $\gamma > 1$ one has*

$$\int_0^{2\pi} \frac{d\theta}{|1 - z|^\gamma} \leq C \frac{1}{(1 - |z|)^{\gamma-1}}, \quad z = re^{i\theta} (0 \leq r < 1)$$

Lemma 2.2 (Lemma 6, [7]). *For $\gamma > -1$ and $m > 1 + \gamma$ one has*

$$\int_0^1 (1 - \rho r)^{-m} (1 - r)^\gamma dr \leq C(1 - \rho)^{1+\gamma-m}.$$

Lemma 2.3 (Theorem 1, [1]). *If $p \geq u$, then*

$$H(p, q, \alpha) \subseteq H(u, v, \beta) \iff \begin{cases} \alpha < \beta \text{ or} \\ \alpha = \beta \text{ and } q \leq v. \end{cases}$$

Lemma 2.4 (Proposition 1, [1]). *If $f \in H(p, q, \alpha)$, $0 < p \leq \infty$, $0 < q$, $\alpha < \infty$ then*

$$|f(z)| = o((1 - |z|)^{\alpha+1/p})$$

as $|z| \rightarrow 1$. This gives a pointwise estimate:

$$(2.1) \quad |f(z)| \leq C \frac{\|f\|_{H(p,q,\alpha)}}{(1-|z|)^{\alpha+1/p}}.$$

Remark 2.5. Lemma 2.4 is not true for $q = \infty$, but the point-wise estimate holds:

$$(2.2) \quad |f(z)| \leq C \frac{\|f\|_{H(p,\infty,\alpha)}}{(1-|z|)^{\alpha+1/p}}.$$

We have the following result if $f \in S(p, q, \alpha)$.

Proposition 2.6. Suppose $f \in S(p, q, \alpha)$ for $0 < p, q \leq \infty$ and $0 < \alpha < 1$ such that $\alpha + \frac{1}{p} > 1$. Then

$$(2.3) \quad |f(z)| \leq C \frac{\|f\|_{S(p,q,\alpha)}}{(1-|z|)^{\alpha+\frac{1}{p}-1}}.$$

Proof. For all $p, q \in (0, \infty]$ and $0 < \alpha < 1$, $f \in S(p, q, \alpha)$ implies $f' \in H(p, q, \alpha)$. Hence equations (2.1) and (2.2) gives

$$|f'(z)| \leq C \frac{\|f'\|_{H(p,q,\alpha)}}{(1-|z|)^{\alpha+\frac{1}{p}}}.$$

Now for any curve $\alpha(t) = x(t) + iy(t)$, $0 \leq t \leq 1$ from 0 to $z = x + iy$ we have

$$f(z) = \int_0^z f'(w)dw = \int_0^1 f'(\alpha(t)) \alpha'(t)dt$$

It is clear that

$$(2.4) \quad |f(z)| = \left| z \int_0^1 f'(tz) dt \right| \leq |z| \int_0^1 |f'(tz)| dt$$

This gives

$$\begin{aligned} |f(z)| &\leq C \int_0^1 \frac{\|f\|_{S(p,q,\alpha)}}{(1-t|z|)^{\frac{1}{p}+\alpha}} dt \\ &\leq C \frac{\|f\|_{S(p,q,\alpha)}}{(1-|z|)^{\frac{1}{p}+\alpha-1}} \end{aligned}$$

□

Remark 2.7. If $\alpha \geq 1$ then $\alpha + \frac{1}{p} > 1$ is vacuously true. Hence the point-wise estimate in Proposition 2.6 holds for $\alpha \geq 1$ without requiring the condition $\alpha + \frac{1}{p} > 1$.

Corollary 2.8. Suppose $1 \leq p < \infty$, $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 < \alpha < 1$ such that $\alpha + \frac{1}{p} > 1$ then $\|f\|_{H(p,q,\alpha)} \leq C\|f\|_{S(p,q,\alpha)}$.

Remark 2.9. It can be easily verified that if $1 \leq p < \infty$, $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 < \alpha < 1$ such that $\alpha + \frac{1}{p} > 1$ there is another equivalent norm for $S(p, q, \alpha)$ defined by $\|f\|_{S(p,q,\alpha)} = \|f\|_{H(p,q,\alpha)} + \|f'\|_{H(p,q,\alpha)}$

The following theorem characterizes some important properties of $S(p, q, \alpha)$.

Theorem 2.10. Suppose $1 \leq p < \infty$, $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 < \alpha < 1$ such that $\alpha + \frac{1}{p} > 1$. Then

- (a) $S(p, q, \alpha)$ is a Banach space under the norm $\|f\|_{S(p,q,\alpha)} = |f(0)| + \|f'\|_{H(p,q,\alpha)}$.
- (b) The set of polynomials is dense in $S(p, q, \alpha)$.

Proof.

- (a) Let $\{f_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $S(p, q, \alpha)$. Then for any $\epsilon > 0$ there exist $M \in \mathbb{N}$ such that $n, m \geq M$ implies that $\|f_n - f_m\|_{S(p,q,\alpha)} < \epsilon$. We have

$$\|f_n - f_m\|_{H(p,q,\alpha)} \leq C\|f_n - f_m\|_{S(p,q,\alpha)} < C\epsilon$$

Hence $\{f_n\}_{n \in \mathbb{N}}$ is also a Cauchy sequence in $H(p, q, \alpha)$. Since $H(p, q, \alpha)$ is a Banach space, therefore both sequences $\{f_n\}_{n \in \mathbb{N}}$ and $\{f'_n\}_{n \in \mathbb{N}}$ converges to analytic functions f, g in $H(p, q, \alpha)$ respectively. Clearly $g = f'$ and hence $\{f_n\}_{n \in \mathbb{N}}$ converges to f in $S(p, q, \alpha)$. Hence $S(p, q, \alpha)$ is a Banach space.

- (b) Suppose $f \in S(p, q, \alpha)$. Then $f' \in H(p, q, \alpha)$. Since polynomials are dense in $H(p, q, \alpha)$, therefore there exist a sequence of polynomials $\{p_n(z)\}_{n=1}^{\infty}$ in $H(p, q, \alpha)$ such that $\lim_{n \rightarrow \infty} \|p_n - f'\|_{H(p,q,\alpha)} = 0$. Define

$$P_n(z) = f(0) + \int_0^z p_n(w)dw.$$

Then $P_n(z)$ is a polynomial and

$$\begin{aligned} \lim_{n \rightarrow \infty} \|P_n - f\|_{S(p,q,\alpha)} &= \lim_{n \rightarrow \infty} [|P_n(0) - f(0)| + \|P'_n - f'\|_{H(p,q,\alpha)}] \\ &= \lim_{n \rightarrow \infty} \|p_n - f'\|_{H(p,q,\alpha)} = 0. \end{aligned} \quad \square$$

3. Boundedness of the weighted composition operator $W_{\varphi,\psi}$.

In this section we characterize the boundedness of the operator from $S(p, q, \alpha)$ spaces to S_p spaces. First we prove the boundedness of the operator $W_{\varphi,\psi} : S(p, q, \alpha) \rightarrow H_p$ spaces.

Theorem 3.1. *Suppose $1 \leq p < \infty$, $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 < \alpha < 1$ such that $\alpha + \frac{1}{p} > 1$. Suppose $f \in S(p, q, \alpha)$ and let $\psi(z) \in H(\mathbb{D})$ and $\varphi(z)$ is an analytic self-map of $\mathbb{D}$. Then the operator $W_{\varphi,\psi} : S(p, q, \alpha) \rightarrow H_p$ is bounded if and only if*

$$(3.1) \quad \sup_{0 \leq r < 1} \int_0^{2\pi} \frac{|\psi(re^{i\theta})|^p}{(1 - |\varphi(re^{i\theta})|^2)^{\alpha p + 1 - p}} d\theta < \infty.$$

Proof. Suppose (3.1) holds. Then Proposition 2.6 gives

$$\begin{aligned} M_p^p(W_{\varphi,\psi}f, r) &= \frac{1}{2\pi} \int_0^{2\pi} |W_{\varphi,\psi}f(re^{i\theta})|^p d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} |\psi(re^{i\theta})f(\varphi(re^{i\theta}))|^p d\theta \\ &\leq C \|f\|_{S(p,q,\alpha)}^p \frac{1}{2\pi} \int_0^{2\pi} \frac{|\psi(re^{i\theta})|^p}{(1 - |\varphi(re^{i\theta})|^2)^{\alpha p + 1 - p}} d\theta \leq C \|f\|_{S(p,q,\alpha)}^p \end{aligned}$$

Hence

$$\|W_{\varphi,\psi}f\|_{H_p}^p = \sup_{0 \leq r < 1} M_p^p(W_{\varphi,\psi}f, r) \leq C \|f\|_{S(p,q,\alpha)}^p.$$

Therefore $W_{\varphi,\psi} : S(p, q, \alpha) \rightarrow H_p$ is bounded.

Conversely suppose $W_{\varphi,\psi} : S(p, q, \alpha) \rightarrow H_p$ is bounded and for any $w \in \mathbb{D}$ define the test function

$$f_w(z) = \frac{(1 - |w|^2)}{(1 - \bar{w}z)^{\alpha + \frac{1}{p}}}$$

Then

$$f'_w(z) = - \left(\alpha + \frac{1}{p} \right) \frac{\overline{w}(1 - |w|^2)}{(1 - \overline{w}z)^{\alpha + \frac{1}{p} + 1}}$$

We have

$$\begin{aligned} M_p^p(f'_w, r) &= \frac{1}{2\pi} \int_0^{2\pi} |f'_w(re^{i\theta})|^p d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} C \frac{(1 - |w|^2)^p}{|1 - \overline{w}re^{i\theta}|^{\alpha p + 1 + p}} d\theta \end{aligned}$$

Lemma 2.1 gives

$$M_p(f'_w, r) \leq C \frac{(1 - |w|^2)}{(1 - |w|r)^{\alpha + 1}}.$$

Hence using Lemma 2.2 we have

$$\begin{aligned} \|f'_w\|_{H(p, q, \alpha)} &\leq C \int_0^1 (1 - r)^{\alpha q - 1} \frac{(1 - |w|^2)^q}{(1 - |w|r)^{\alpha q + q}} dr \\ &= C(1 - |w|^2)^q \int_0^1 (1 - |w|r)^{-(\alpha q + q)} (1 - r)^{\alpha q - 1} dr < \infty. \end{aligned}$$

Therefore $f'_w \in H(p, q, \alpha)$ for $1 \leq p < \infty$ and hence $f_w \in S(p, q, \alpha)$. Boundedness of $W_{\varphi, \psi} : S(p, q, \alpha) \rightarrow H_p$ gives for $1 \leq p < \infty$

$$\begin{aligned} \|W_{\varphi, \psi} f_w\|_{H_p} &\leq C \|f_w\|_{S(p, q, \alpha)} < \infty \\ \sup_{0 \leq r < 1} \left[\frac{1}{2\pi} \int_0^{2\pi} |W_{\varphi, \psi} f_w(re^{i\theta})|^p d\theta \right]^{1/p} &< \infty \\ \sup_{0 \leq r < 1} \int_0^{2\pi} |\psi(z) f_w(\varphi(re^{i\theta}))|^p d\theta &< \infty \\ \sup_{0 \leq r < 1} \int_0^{2\pi} \frac{|\psi(re^{i\theta})|^p (1 - |w|^2)^p}{|1 - \overline{w}\varphi(re^{i\theta})|^{\alpha p + 1}} d\theta &< \infty \end{aligned}$$

Taking $w = \varphi(z)$ gives us (3.1). $\square$

In the following we have a result similar to the previous theorem. We omit the proof as it is similar to the previous proof.

Theorem 3.2. *Let $0 < p, q < \infty$ and $\alpha > 0$. Let $\psi(z) \in H(\mathbb{D})$ and $\varphi(z)$ be an analytic self-map of $\mathbb{D}$. Then the operator $W_{\varphi, \psi} : H(p, q, \alpha) \rightarrow H_p$ is bounded if and only if*

$$(3.2) \quad \sup_{0 \leq r < 1} \int_0^{2\pi} \frac{|\psi(re^{i\theta})|^p}{(1 - |\varphi(re^{i\theta})|^2)^{\alpha p + 1}} d\theta < \infty.$$

Since $(W_{\varphi, \psi} f)' = \psi' f \circ \varphi + \psi \varphi' f' \circ \varphi$, therefore the study of $W_{\varphi, \psi} : S(p, q, \alpha) \rightarrow S_p$ is equivalent to the study of operators $W_{\varphi, \psi'} : S(p, q, \alpha) \rightarrow H_p$ and $W_{\varphi, \psi \varphi'} : H(p, q, \alpha) \rightarrow H_p$. Obviously, the following result follows.

Theorem 3.3. *Suppose $1 \leq p < \infty$, $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 < \alpha < 1$ such that $\alpha + \frac{1}{p} > 1$. Let $\psi(z) \in H(\mathbb{D})$ and $\varphi(z)$ be an analytic self-map of $\mathbb{D}$. Then the operator $W_{\varphi, \psi} : S(p, q, \alpha) \rightarrow S_p$ is bounded if and only if*

$$(3.3) \quad \sup_{0 \leq r < 1} \int_0^{2\pi} \frac{|\psi'(re^{i\theta})|^p}{(1 - |\varphi(re^{i\theta})|^2)^{\alpha p + 1 - p}} d\theta < \infty.$$

and

$$(3.4) \quad \sup_{0 \leq r < 1} \int_0^{2\pi} \frac{|\psi(re^{i\theta}) \varphi'(re^{i\theta})|^p}{(1 - |\varphi(re^{i\theta})|^2)^{\alpha p + 1}} d\theta < \infty.$$

Proof. Suppose (3.3) and (3.4) holds. Then we have

$$\begin{aligned} \|W_{\varphi, \psi} f\|_{S_p} &= \|\psi' f \circ \varphi + \psi \varphi' f' \circ \varphi\|_{H_p} \\ &\leq \|\psi' f \circ \varphi\|_{H_p} + \|\psi \varphi' f' \circ \varphi\|_{H_p} \\ &\leq C \sup_{0 \leq r < 1} \left[\int_0^{2\pi} \frac{|\psi'(z)|^p \|f\|_{S(p, q, \alpha)}^p}{(1 - |\varphi(z)|^2)^{\alpha p + 1 - p}} d\theta \right]^{1/p} \\ &\quad + C \sup_{0 \leq r < 1} \left[\int_0^{2\pi} \frac{|\psi(z) \varphi'(z)|^p \|f\|_{S(p, q, \alpha)}^p}{(1 - |\varphi(z)|^2)^{\alpha p + 1}} d\theta \right]^{1/p} \\ &\leq \|f\|_{S(p, q, \alpha)} \end{aligned}$$

Hence $W_{\varphi, \psi} : S(p, q, \alpha) \rightarrow S_p$ is bounded.

Conversely suppose $W_{\varphi, \psi} : S(p, q, \alpha) \rightarrow S_p$ is bounded. Taking the test function f_w from proof of Theorem 3.1 we have that $f_w \in S(p, q, \alpha)$ and

$$\|W_{\varphi,\psi}f_w\|_{S_p}^p \leq C\|f_w\|_{S(p,q,\alpha)}$$

$$\|\psi'f_w\circ\varphi\|_{H_p} - \|\psi\varphi'f_w'\circ\varphi\|_{H_p} \leq C\|f_w\|_{S(p,q,\alpha)}$$

$$\int_0^{2\pi} \frac{|\psi'(z)|^p}{(1-|\varphi(z)|^2)^{\alpha p+1-p}} d\theta - \int_0^{2\pi} \frac{|w\psi(z)\varphi'(z)|^p}{(1-|\varphi(z)|^2)^{\alpha p+1}} d\theta \leq C\|f_w\|_{S(p,q,\alpha)}$$

which gives

$$(3.5) \quad \int_0^{2\pi} \frac{|\psi'(z)|^p}{(1-|\varphi(z)|^2)^{\alpha p+1-p}} d\theta \leq C\|f_w\|_{S(p,q,\alpha)} + \int_0^{2\pi} \frac{|w\psi(z)\varphi'(z)|^p}{(1-|\varphi(z)|^2)^{\alpha p+1}} d\theta$$

Again, for $w \in \mathbb{D}$ define

$$g_w(z) = \frac{(1-|w|^2)^2}{(1-\bar{w}z)^{\alpha+\frac{1}{p}+1}} - \frac{(1-|w|^2)}{(1-\bar{w}z)^{\alpha+\frac{1}{p}}}$$

Then $g_w \in S(p, q, \alpha)$, $g_w(w) = 0$ and $g_w'(w) = \frac{\bar{w}}{(1-|w|^2)^{\alpha+\frac{1}{p}}}$. The boundedness of $W_{\varphi,\psi} : S(p, q, \alpha) \rightarrow S_p$ gives

$$\|W_{\varphi,\psi}g_w\|_{S_p}^p \leq C\|g_w\|_{S(p,q,\alpha)}$$

$$\|\psi'g_w\circ\varphi\|_{H_p} - \|\psi\varphi'g_w'\circ\varphi\|_{H_p} \leq C\|g_w\|_{S(p,q,\alpha)}$$

Taking $w = \varphi(z)$ gives

$$(3.6) \quad \int_0^{2\pi} \frac{|w\psi(z)\varphi'(z)|^p}{(1-|\varphi(z)|^2)^{\alpha p+1}} d\theta \leq C\|g_w\|_{S(p,q,\alpha)} < \infty$$

(3.5) and (3.6) proves (3.3). Note that (3.3) is the same as saying that $W_{\varphi,\psi'} : S(p, q, \alpha) \rightarrow H_p$ is bounded. Let $f \in H(p, q, \alpha)$ and let $g(z) = \int_0^z f(w)dw$. This implies $g \in S(p, q, \alpha)$. We have

$$\begin{aligned} \|W_{\varphi,\psi\varphi'}f\|_{H_p} &= \|\psi\varphi'f\circ\varphi\|_{H_p} \\ &= \|\psi\varphi'f\circ\varphi + \psi'g\circ\varphi - \psi'g\circ\varphi\|_{H_p} \\ &\leq \|(\psi g\circ\varphi)'\|_{H_p} + \|\psi'g\circ\varphi\|_{H_p} \end{aligned}$$

$$\begin{aligned}
&= \|W_{\varphi,\psi}g\|_{S_p} + \|W_{\varphi,\psi'}g\|_{H_p} \leq C_1\|g\|_{S(p,q,\alpha)} + C_2\|g\|_{S(p,q,\alpha)} \\
&\leq C\|f\|_{H(p,q,\alpha)},
\end{aligned}$$

where C_1, C_2 and C are constants. Hence $W_{\varphi,\psi\varphi'} : H(p, q, \alpha) \rightarrow H_p$ is bounded. This together with Theorem 3.2 proves (3.4). $\square$

Remark 3.4. From Remark 2.7 it follows that Theorem 3.1 and Theorem 3.3 holds for all $\alpha \geq 1$ without requiring the condition $\alpha + \frac{1}{p} > 1$.

4. Compactness of the weighted composition operator $W_{\varphi,\psi}$.

In this section we study the compactness of the operator $W_{\varphi,\psi} : S(p, q, \alpha) \rightarrow S_p$. The following criterion for the compactness follows by standard arguments (see, e.g., the proofs of the corresponding results in [8]). Hence we omit its proof.

Theorem 4.1. Suppose $1 \leq p < \infty$, $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 < \alpha < 1$ such that $\alpha + \frac{1}{p} > 1$. Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Suppose that $W_{\varphi,\psi} : H(p, q, \alpha) \rightarrow H_p(W_{\varphi,\psi} : S(p, q, \alpha) \rightarrow H_p)$ is bounded. Then, $W_{\varphi,\psi} : H(p, q, \alpha) \rightarrow H_p(W_{\varphi,\psi} : S(p, q, \alpha) \rightarrow H_p)$ is compact if and only if for any bounded sequence $\{f_n\}_{n=0}^\infty$ in $H(p, q, \alpha)(S(p, q, \alpha))$, such that $f_n \rightarrow 0$ uniformly on any compact subset of $\mathbb{D}$ as $n \rightarrow \infty$, $\|W_{\varphi,\psi}f_n\|_{H_p} \rightarrow 0$.

Theorem 4.2. Suppose $1 \leq p < \infty$, $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 < \alpha < 1$ such that $\alpha + \frac{1}{p} > 1$. Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Suppose that $W_{\varphi,\psi} : S(p, q, \alpha) \rightarrow H_p$ is bounded. Then, $W_{\varphi,\psi} : S(p, q, \alpha) \rightarrow H_p$ is compact if and only if

$$(4.1) \quad \lim_{|\varphi(z)| \rightarrow 1} \int_0^{2\pi} \frac{|\psi(z)|^p}{(1 - |\varphi(z)|^2)^{\alpha p + 1 - p}} d\theta = 0.$$

Proof. Suppose $W_{\varphi,\psi} : S(p, q, \alpha) \rightarrow H_p$ is compact. Then it is also bounded. Suppose (4.1) does not hold. Then there is a sequence $\{z_k\}_{k \in \mathbb{N}}$ satisfying $|\varphi(z_k)| \rightarrow 1$ as $k \rightarrow \infty$ and $\delta > 0$ such that

$$(4.2) \quad \int_0^{2\pi} \frac{|\psi(z_k)|^p}{(1 - |\varphi(z_k)|^2)^{\alpha p + 1 - p}} d\theta \geq \delta, \quad k \in \mathbb{N}.$$

Let $g_k(z) = f_{\varphi(z_k)}(z)$, $k \in \mathbb{N}$ where $f_w(z) = \frac{(1 - |w|^2)}{(1 - \overline{w}z)^{\alpha+1/p}} \in S(p, q, \alpha)$. Clearly $\lim_{k \rightarrow \infty} g_k = 0$ uniformly on compact subsets of $\mathbb{D}$. Theorem 4.1 gives

$$\lim_{k \rightarrow \infty} \|W_{\varphi, \psi} g_k\|_{H_p} = 0.$$

On the other hand we have using (4.2)

$$\begin{aligned} \|W_{\varphi, \psi} g_k\|_{H_p}^p &= \sup_{r \in [0, 1)} \int_0^{2\pi} |W_{\varphi, \psi} g_k(re^{i\theta})|^p \frac{d\theta}{2\pi} \\ &\geq \int_0^{2\pi} \frac{|\psi(z_k)|^p}{(1 - |\varphi(z_k)|^2)^{\alpha p + 1 - p}} \frac{d\theta}{2\pi} \geq C\delta > 0. \end{aligned}$$

Conversely suppose $W_{\varphi, \psi} : S(p, q, \alpha) \rightarrow H_p$ is bounded and (4.1) holds. Then for every $\epsilon > 0$, there exist $r \in (0, 1)$ such that for $r < |\varphi(z)| < 1$

$$(4.3) \quad \int_0^{2\pi} \frac{|\psi(re^{i\theta})|^p}{(1 - |\varphi(re^{i\theta})|^2)^{\alpha p + 1 - p}} d\theta < \epsilon.$$

Let $h(z) \equiv 1$. Then $\|h\|_{S(p, q, \alpha)} = 1$. Boundedness of $W_{\varphi, \psi} : S(p, q, \alpha) \rightarrow H_p$ gives $\|\psi\|_{H_p} < \infty$. Suppose $\{h_k\}_{k \in \mathbb{N}}$ is a sequence in $S(p, q, \alpha)$ bounded by say M and converging to zero uniformly on compact subsets of $\mathbb{D}$ as $k \rightarrow \infty$. For $r < |\varphi(z)| < 1$ we obtain

$$(4.4) \quad \|W_{\varphi, \psi} h_k\|_{H_p}^p \leq \|h_k\|_{S(p, q, \alpha)}^p \int_0^{2\pi} \frac{|\psi(z)|^p}{(1 - |\varphi(z)|^2)^{\alpha p + 1 - p}} \frac{d\theta}{2\pi} < MC\epsilon.$$

For $|\varphi(z)| \leq r$ we have

$$\begin{aligned} (4.5) \quad \|W_{\varphi, \psi} h_k\|_{H_p}^p &\leq \|h_k\|_{S(p, q, \alpha)}^p \int_0^{2\pi} \frac{|\psi(z)|^p}{(1 - |\varphi(z)|^2)^{\alpha p + 1 - p}} \frac{d\theta}{2\pi} \\ &\leq \|\psi\|_{H_p}^p \int_0^{2\pi} |h_k(z)|^p \frac{d\theta}{2\pi}. \end{aligned}$$

Hence (4.4) and (4.5) gives $\|W_{\varphi, \psi} h_k\|_{H_p}^p \rightarrow 0$ as $k \rightarrow \infty$. Hence the operator must be compact. $\square$

Below we mention a result on compactness of $W_{\varphi, \psi} : H(p, q, \alpha) \rightarrow H_p$. The proof is similar to the proof of the previous result and hence is omitted here.

Theorem 4.3. *Let $0 < p, q < \infty$ and $\alpha > 0$. Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Suppose that $W_{\varphi, \psi} : H(p, q, \alpha) \rightarrow H_p$ is bounded. Then, $W_{\varphi, \psi} : H(p, q, \alpha) \rightarrow H_p$ is compact if and only if*

$$(4.6) \quad \lim_{|\varphi(z)| \rightarrow 1} \int_0^{2\pi} \frac{|\psi(z)|^p}{(1 - |\varphi(z)|^2)^{\alpha p + 1}} d\theta = 0.$$

In the following result we give conditions for the operator

$$W_{\varphi, \psi} : S(p, q, \alpha) \rightarrow S_p$$

to be compact.

Theorem 4.4. *Suppose $1 \leq p < \infty$, $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 < \alpha < 1$ such that $\alpha + \frac{1}{p} > 1$. Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Suppose that $W_{\varphi, \psi} : S(p, q, \alpha) \rightarrow S_p$ is bounded. Then, $W_{\varphi, \psi} : S(p, q, \alpha) \rightarrow S_p$ is compact if and only if*

$$(4.7) \quad \lim_{|\varphi(z)| \rightarrow 1} \int_0^{2\pi} \frac{|\psi'(z)|^p}{(1 - |\varphi(z)|^2)^{\alpha p + 1 - p}} d\theta = 0.$$

and

$$(4.8) \quad \lim_{|\varphi(z)| \rightarrow 1} \int_0^{2\pi} \frac{|\psi(z)\varphi'(z)|^p}{(1 - |\varphi(z)|^2)^{\alpha p + 1}} d\theta = 0.$$

Proof. Suppose $W_{\varphi, \psi} : S(p, q, \alpha) \rightarrow S_p$ is compact. Then it is bounded. Let $\{z_k\}_{k \in \mathbb{N}} \in \mathbb{D}$ such that $|\varphi(z_k)| \rightarrow 1$ as $k \rightarrow \infty$. Let $f_k(z) = f_{\varphi(z_k)}(z)$, $k \in \mathbb{N}$ where f_w is as defined Theorem 4.2 and 3.1. Then $\|f_k\|_{S(p, q, \alpha)} < \infty$ and $f_k \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \rightarrow \infty$. Hence $\lim_{k \rightarrow \infty} \|W_{\varphi, \psi} f_k\|_{S_p} = 0$. Since

$$\int_0^{2\pi} \frac{|\psi'(z_k)|^p}{(1 - |\varphi(z_k)|^2)^{\alpha p + 1 - p}} d\theta - \int_0^{2\pi} \frac{|\varphi(z_k)\psi(z_k)\varphi'(z_k)|^p}{(1 - |\varphi(z_k)|^2)^{\alpha p + 1}} d\theta \leq \|W_{\varphi, \psi} f_k\|_{S_p}^p$$

therefore

$$(4.9) \quad \lim_{k \rightarrow \infty} \int_0^{2\pi} \frac{|\psi'(z_k)|^p}{(1 - |\varphi(z_k)|^2)^{\alpha p + 1 - p}} d\theta = \lim_{k \rightarrow \infty} \int_0^{2\pi} \frac{|\varphi(z_k)\psi(z_k)\varphi'(z_k)|^p}{(1 - |\varphi(z_k)|^2)^{\alpha p + 1}} d\theta$$

Let $g_k(z) = g_{\varphi(z_k)}(z)$, $k \in \mathbb{N}$, where g_w is as defined in Theorem 3.1. Then $\|g_k\|_{S(p,q,\alpha)} < \infty$ and $\lim_{k \rightarrow \infty} g_k = 0$ on compact subsets of $\mathbb{D}$. By Lemma 4.1 we have $\lim_{k \rightarrow \infty} \|W_{\varphi,\psi} g_k\|_{S_p} = 0$. Since

$$\int_0^{2\pi} \frac{|\varphi(z_k)\psi(z_k)\varphi'(z_k)|^p}{(1 - |\varphi(z_k)|^2)^{\alpha p+1}} d\theta \leq \|W_{\varphi,\psi} g_k\|_{S_p}^p$$

we have

$$(4.10) \quad \lim_{k \rightarrow \infty} \int_0^{2\pi} \frac{|\varphi(z_k)\psi(z_k)\varphi'(z_k)|^p}{(1 - |\varphi(z_k)|^2)^{\alpha p+1}} d\theta = 0$$

(4.9) and (4.10) proves the result. Conversely suppose $W_{\varphi,\psi} : S(p,q,\alpha) \rightarrow S_p$ is bounded and (4.7) and (4.8) holds. This implies $W_{\varphi,\psi'} : S(p,q,\alpha) \rightarrow H_p$ and $W_{\varphi,\psi\varphi'} : H(p,q,\alpha) \rightarrow H_p$ are bounded and by Theorem 4.3 and Theorem 4.4 are compact. It is now easy to show that $W_{\varphi,\psi} : S(p,q,\alpha) \rightarrow S_p$ is compact. $\square$

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Department of Mathematics

B. P. Chaliha College, Assam, India

e-mail: deep1a2b@gmail.com

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