Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Serdica
Mathematical Journal
Сердика
Математическо списание

The attached copy is furnished for non-commercial research and education use only.
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://serdica.math.bas.bg
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **51** (2025), 197–202

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON TWO REFINEMENTS OF THE GAUSS-LUCAS THEOREM*

Hristo Sendov

Communicated by B. Draganov

Abstract. In this note, we give a new proof of a refinement of the Gauss-Lucas theorem given in [6]. The new proof uses a technique used to produce and older refinement of the same theorem in [3]. It was observed that the two refinements are complimentary to each other and we hope that this work explains why this is so.

1. Introduction. Denote by $\mathbb{C}$ the complex plane and let $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$. Consider the polynomial

$$p(z) := (z - z_1)^{k_1} \cdots (z - z_m)^{k_m}, \quad \text{where } \sum_{j=1}^{m} k_j = n, \tag{1.1}$$

with distinct zeros $z_1, \ldots, z_m$ having multiplicities $k_1, \ldots, k_m$, respectively. The Gauss-Lucas theorem, see Theorem 2.1.1 in [5], states that the critical points of

https://doi.org/10.55630/serdica.2025.51.197-202

2020 *Mathematics Subject Classification:* Primary 30C10.

Key words: geometry of polynomials, Gauss-Lucas theorem, Schur-Szegö's composition theorem, polar convexity, Specht, Krawtchouk, Dimitrov.

*The author was partially supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada. (Grant number RGPIN-2020-06425.)

$p(z)$ are in $\mathrm{conv}\{z_1,\dots,z_m\}$, where conv stands for the convex hull of the given points. Several refinements of the Gauss-Lucas theorem use the points

$$\gamma_{j,k} := \begin{cases} (k_j z_k + (n-k_j)z_j)/n & \text{if } j \neq k, \\ \infty & \text{if } j = k, \end{cases}$$

defined for all $j,k \in \{1,\dots,m\}$. Let Ω^0_{jk}, for $j \neq k$, be the closed disk having segment $[\gamma_{j,k}, z_k]$ as a diameter. The following result is Corollary 1 from [3].

Theorem 1 (Dimitrov)**.** *The non-trivial critical points of $p(z)$ are in*

$$\bigcap_{j=1}^{m} \bigcup_{k\neq j} \Omega^0_{jk}$$

Specht, see [9, Satz 2, p.89], established the following.

Theorem 2 (Specht)**.** *All non-trivial critical points of $p(z)$ are in*

$$\mathrm{conv}\{\gamma_{j,k} : 1 \leq j,k \leq m, j \neq k\}. \tag{1.2}$$

Finally, the proof of the following lemma can be found in [4].

Lemma 3 (Krawtchouk)**.** *Let $\mathcal{M}_j$ be an open, convex, circular domain. If $z_j \in \mathrm{cl}\,\mathcal{M}_j$, but $\gamma_{j,k} \notin \mathcal{M}_j$, for all $k \neq j$, then the non-trivial critical points of $p(z)$ are in $\mathcal{M}_j^c$.*

There are many domains $\mathcal{M}_j$ that satisfy the conditions of the lemma. To describe the union of all such domains, we need some notation.

Consider three distinct points u, v_1, v_2 in $\mathbb{C}^*$. Let $\mathrm{arc}_u[v_1, v_2]$ be the arc with endpoints v_1, v_2 and not containing u, from the unique circle (or line) defined by u, v_1, v_2. A set $U \subset \mathbb{C}^*$ is called *convex with respect to* $u \in \mathbb{C}^*$ if for any $v_1, v_2 \in U$, we have $\mathrm{arc}_u[v_1, v_2] \subset U$. The point u is called a *pole of* U. Define $\mathrm{conv}_u\{v_1,\dots,v_n\}$ to be the smallest set, with respect to inclusion, containing the points $v_1,\dots,v_n$ and convex with respect to u. The classical convex sets are convex with respect to ∞, that is $\mathrm{conv}\{v_1,\dots,v_n\} = \mathrm{conv}_\infty\{v_1,\dots,v_n\}$. It can be shown that $\mathrm{conv}_u\{v_1,\dots,v_n\}$ is the intersection of all closed circular domains that contain the points $v_1,\dots,v_n$ and have u on their boundary, with point u removed. Further, if T is a Möbius transformation, then

$$T(\mathrm{conv}_u\{v_1,\dots,v_n\}) = \mathrm{conv}_{T(u)}\{T(v_1),\dots,T(v_n)\}. \tag{1.3}$$

For more on polar convexity and its applications to zeros of polynomials, see [1], [2], [7], and [8].

The following result is Theorem 5.1 in [6].

Theorem 4. *The non-trivial, critical points of $p(z)$ are in*

$$\operatorname{conv}\{z_1,\dots,z_m\}\bigcap\bigcap_{j=1}^{m}\operatorname{conv}_{z_j}\{\gamma_{j,1},\dots,\gamma_{j,m}\}. \tag{1.4}$$

We need one final result. Given two polynomials

$$A(z):=\sum_{j=0}^{n}\binom{n}{j}a_jz^j \quad\text{and}\quad B(z):=\sum_{j=0}^{n}\binom{n}{j}b_jz^j,$$

the composition of A and B is defined to be the polynomial

$$C(z):=\sum_{j=0}^{n}\binom{n}{j}a_jb_jz^j.$$

The Schur-Szegö's composition theorem, see [5, p.109], gives a relation between the zeros of $C(z)$ and the zeros of $A(z)$ and $B(z)$.

Theorem 5 (Schur-Szegö's composition theorem)**.** *Suppose all the zeros of $A(z)$ lie in the closed circular domain $\mathcal{K}$. Then every zero γ of $C(z)$ is of the form*

$$\gamma=-\alpha\beta,\quad(\alpha\in\mathcal{K},B(\beta)=0).$$

If $\beta_1,\dots,\beta_n$ are the zeros of $B(z)$, then the zeros of $C(z)$ are in $\cup_{j=1}^{n}(-\beta_j\mathcal{K})$.

Theorems 1 and 4 are complementary results, meaning that neither is stronger than the other. On the other hand, Theorem 4 is stronger than Theorem 2, since the set (1.4) is contained in (1.2), see Proposition 5.1 in [6]. The proof of Theorem 1 is a clever application of Schur-Szegö's composition theorem. While the proof of Theorem 4 shows that the union of all domains $\mathcal{M}_j$ that satisfy the conditions of Lemma 3 is $(\operatorname{conv}_{z_j}\{\gamma_{j,1},\dots,\gamma_{j,m}\})^c$. The goal of this note is to present a new proof of Theorem 4, following Dimitrov's approach for Theorem 1.

2. A proof of Theorem 4 using Dimitrov's appoach. Single out a particular zero of $p(z)$ and represent it as

$$p(z)=(z-z_j)^{k_j}\prod_{v=1}^{n-k_j}(z-\zeta_v), \tag{2.1}$$

with derivative

$$p'(z)=n(z-z_j)^{k_j-1}\prod_{v=1}^{n-k_j}(z-\xi_v).$$

Lemma 2 in [3] gives a connection between the zeros and the critical points of $p(z)$: any integer v, such that $0 \le v \le n - k_j$, one has

$$\sigma_v(\xi_1 - z_j, \dots, \xi_{n-k_j} - z_j) = \frac{n-v}{n}\sigma_v(\zeta_1 - z_j, \dots, \zeta_{n-k_j} - z_j), \tag{2.2}$$

where σ_v is the v-th elementary symmetric polynomial on $n - k_j$ variables, with $\sigma_0 := 1$. Consider the monic polynomial with zeros $\zeta_1 - z_j, \dots, \zeta_{n-k_j} - z_j$:

$$f(z) := \sum_{v=0}^{n-k_j} (-1)^{n-k_j-v}\sigma_{n-k_j-v}(\zeta_1 - z_j, \dots, \zeta_{n-k_j} - z_j)z^v \tag{2.3}$$

and let

$$g(z) := \sum_{v=0}^{n-k_j} \frac{k_j + v}{n}\binom{n-k_j}{v} z^v = (z+1)^{n-k_j-1}(z + k_j/n),$$

where the last equality follows by directly calculating the coefficient of z^v on the right-hand side, see Lemma 1 in [3]. The composition of $f(z)$ and $g(z)$ is

$$\begin{aligned} h(z) &:= \sum_{v=0}^{n-k_j} (-1)^{n-k_j-v}\sigma_{n-k_j-v}(\zeta_1 - z_j, \dots, \zeta_{n-k_j} - z_j)\frac{k_j+v}{n}z^v \\ &= \sum_{v=0}^{n-k_j} (-1)^{n-k_j-v}\sigma_{n-k_j-v}(\xi_1 - z_j, \dots, \xi_{n-k_j} - z_j)z^v, \end{aligned} \tag{2.4}$$

where the last equality follows from(2.2). Let G be any closed circular domain containing the zeros -1 and $-k_j/n$ of $g(z)$. By Theorem 5, the zeros $\xi_1 - z_j, \dots, \xi_{n-k_j} - z_j$ of $h(z)$ lie in

$$\bigcup_{v=1}^{n-k_j} (z_j - \zeta_v)G.$$

So, the critical points of $p(z)$, different from z_j, lie in

$$z_j + \bigcup_{v=1}^{n-k_j} (z_j - \zeta_v)G.$$

This is the key idea in the proof of Theorem 1.

Now, instead of considering a closed circular domain G containing the zeros of $g(z)$, we consider a closed circular domain containing the zeros of $f(z)$. Let Ω be an *unbounded* closed circular domain containing $\zeta_1 - z_j, \dots, \zeta_{n-k_j} - z_j$, such that the origin is on the boundary of Ω. By Theorem 5, the points $\xi_1 - z_j, \dots, \xi_{n-k_j} - z_j$ lie in

$$\Omega \bigcup \frac{k_j}{n}\Omega = \frac{k_j}{n}\Omega.$$

In other words, $\xi_1, \dots, \xi_{n-k_j} \in z_j + k_j\Omega/n$. The set $\mathrm{conv}_0\{\zeta_1 - z_j, \dots, \zeta_{n-k_j} - z_j, \infty\}$ is the intersection of all unbounded closed circular domains that contain the points $\zeta_1 - z_j, \dots, \zeta_{n-k_j} - z_j$ and have the origin on their boundary, with the origin removed from that intersection. One sees that the points $\xi_1 - z_j, \dots, \xi_{n-k_j} - z_j$ lie in

$$\frac{k_j}{n}\mathrm{conv}_0\{\zeta_1 - z_j, \dots, \zeta_{n-k_j} - z_j, \infty\} = \mathrm{conv}_0\Big\{\frac{k_j}{n}(\zeta_1 - z_j), \dots, \frac{k_j}{n}(\zeta_{n-k_j} - z_j), \infty\Big\}.$$

Thus, using (1.3) the critical points $\xi_1, \dots, \xi_{n-k_j}$, different from z_j, lie in

$$\mathrm{conv}_{z_j}\Big\{z_j + \frac{k_j}{n}(\zeta_1 - z_j), \dots, z_j + \frac{k_j}{n}(\zeta_{n-k_j} - z_j), \infty\Big\} = \mathrm{conv}_{z_j}\{\gamma_{j,1}, \dots, \gamma_{j,m}\}.$$

Thus, the non-trivial critical points of $p(z)$ lie in

$$\bigcap_{j=1}^{m} \mathrm{conv}_{z_j}\{\gamma_{j,1}, \dots, \gamma_{j,m}\}.$$

Combining with the Gauss-Lucas theorem, concludes the proof of Theorem 4. □

Remark 6. Let Ω be a *bounded* closed circular domain containing $\zeta_1 - z_j, \dots, \zeta_{n-k_j} - z_j$, and such that the origin is on the boundary of Ω. Then the points $\xi_1 - z_j, \dots, \xi_{n-k_j} - z_j$ are in $\Omega \cup k_j\Omega/n = \Omega$ and $\xi_1, \dots, \xi_{n-k_j} \in z_j + \Omega$. Intersecting such domains does not lead to a further improvement of Theorem 4, since

$$\mathrm{conv}\{z_1, \dots, z_m\} = \mathrm{conv}\{\zeta_1, \dots, \zeta_{n-k_j}, z_j\} \subset z_j + \Omega.$$

REFERENCES

[1] Sh. Bhatt, H. Sendov. On Polar Convexity in Finite-Dimensional Euclidean Spaces. *Canadian Journal of Mathematics* (2024), 37 pp, `https://doi.org/10.4153/S0008414X24000671`.

[2] Sh. Bhatt, H. S. Sendov. Gauss-Lucas theorems for multivariate polynomials and their polar derivatives. Submitted, 2024, 27 pp.

[3] D. K. Dimitrov. A refinement of the Gauss-Lucas theorem. *Proceedings of the American Mathematical Society* **126**, *7* (1998), 2065–2070.

[4] M. Krawtchouk. Note sur la distribution des racines des polynomes dérivés. *L'Enseignement Mathématique*, **25** (1926), 74–76.

[5] Q. Rahman, G. Schmeisser. Analytic theory of polynomials. London Math. Soc. Monogr. (N.S.), vol. **26**. Oxford, The Clarendon Press, Oxford University Press, 2002.

[6] H. Sendov. Refinements of the Gauss-Lucas theorem using rational lemniscates and polar convexity. *Proc. Amer. Math. Soc.*, **149**, *12* (2021), 5179–5193.

[7] Bl. Sendov, H. Sendov. Sets in the complex plane mapped into convex ones by Möbius transformations. *J. Convex Anal.* **27**, *3* (2020), 791–810.

[8] Bl. Sendov, H. Sendov, C. Wang. Polar convexity and critical points of polynomials. *J. Convex Anal.* **26**, *2* (2019), 635–660.

[9] W. Specht. Eine Bemerkung zum Satze von Gauß-Lucas. *Jber. Deutsch. Math.-Verein.* **62** (1959), 85–92.

Department of Statistical and Actuarial Sciences
Department of Mathematics
Western University
1151 Richmond Str.
London, ON, N6A 5B7 Canada
e-mail: hsendov@uwo.ca

Received March 24, 2025
Accepted April 29, 2025