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MINIMAL VARIETIES OF PI-ALGEBRAS WITH GRADED INVOLUTION: A SURVEY*

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Communicated by P. Koshlukov

To our colleague and friend Vesselin Drensky on the occasion of his 75th anniversary

ABSTRACT. This survey provides an overview of recent developments in the structure and classification of minimal varieties of associative PI-algebras equipped with graded involutions, particularly over an algebraically closed field of characteristic zero and under the action of a cyclic group of prime order.

1. Introduction. Let F be a field of characteristic zero and let $F\langle X \rangle$ denote the *free associative algebra* over F generated by a countable set $X = \{x_1, x_2, \dots\}$. A central object of study in PI-theory is the class of *T-ideals* of $F\langle X \rangle$, which are ideals invariant under all endomorphisms of $F\langle X \rangle$. There is a well-known connection between *T-ideals* of $F\langle X \rangle$ and PI-algebras: every *T-ideal* arises as the ideal of polynomial identities satisfied by some associative F -algebra A . We recall that a *polynomial identity* (or PI) of A is a nonzero

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element $f(x_1, \dots, x_n) \in F\langle X \rangle$ that vanishes under all evaluations in A . The set of all polynomial identities of A forms a two-sided ideal in $F\langle X \rangle$ denoted by $\text{Id}(A)$.

A fundamental numerical invariant associated with a PI-algebra A is the sequence of *codimensions* $\{c_n(A)\}_{n \geq 1}$, introduced by Regev in [32]. For each natural $n \geq 1$, let P_n denote the space of multilinear polynomials in $x_1, \dots, x_n$. Then the n -th codimension is defined by

$$c_n(A) := \dim_F \left(\frac{P_n}{P_n \cap \text{Id}(A)} \right).$$

The sequence $\{c_n(A)\}_{n \geq 1}$ measures the growth of the space of multilinear polynomial identities of A . Regev established a fundamental connection between the existence of polynomial identities and the growth of codimensions: an algebra A satisfies a polynomial identity if and only if the sequence of codimensions $\{c_n(A)\}_{n \geq 1}$ is exponentially bounded.

Later Giambruno and Zaicev answered in a positive way to a conjecture of Amitsur establishing (see [20] and [21]) that for any PI-algebra A over a field F of characteristic zero, the following limit exists:

$$\exp(A) := \lim_{n \rightarrow \infty} \sqrt[n]{c_n(A)}.$$

This quantity, called the *PI-exponent* of A , captures the exponential growth rate of the sequence $\{c_n(A)\}_{n \geq 1}$. Furthermore, they showed that $\exp(A)$ is a positive integer and, in many important cases, it reflects deep structural properties of A .

In the language of varieties it is well-established that the identities satisfied by an algebra A generate a variety $\mathcal{V} = \text{var}(A)$ and the growth of the codimensions of A reflects the growth of the variety itself. Then, if $\mathcal{V} = \text{var}(A)$ is the variety generated by A we have $c_n(\mathcal{V}) = c_n(A)$ and $\exp(\mathcal{V}) = \exp(A)$. An important application of the exponent is the classification of varieties by their growth on an integral scale whose steps are the minimal varieties of given exponent d : a variety is *minimal of exponent d* if it has exponent d , and every proper subvariety has strictly smaller exponent. In this context Giambruno and Zaicev in [24] characterized minimal varieties of exponent d . Previously in [23] they proved that affine varieties (i.e. generated by a finitely generated algebra) are minimal if and only if are generated by an upper block triangular matrix algebras.

This framework has been extended to algebras with additional structures, such as gradings and involutions.

We recall that an algebra A is a G -graded algebra, where G is a group, if A can be written as a direct sum of vector spaces $A = \bigoplus_{g \in G} A_g$ such that $A_g A_h \subseteq A_{gh}$, for all $g, h \in G$. The subspaces A_g are called *homogeneous components* of A of degree g and an element $a \in A_g$ is called *homogeneous of degree g* . In the study of polynomial identities of associative algebras, the connection with identities satisfied by *superalgebras* (i.e., $\mathbb{Z}_2$ -graded algebras) and, more generally, by G -graded algebras plays a fundamental role.

Let E be the *infinite dimensional Grassmann algebra*, that is the algebra with 1 generated by countably many elements $e_1, e_2, \dots$, subject to the condition $e_i e_j = -e_j e_i$. E has a natural structure of superalgebra $E = E_0 \oplus E_1$ where every monomial in the e_i 's of even (odd) length has homogeneous degree zero (one resp.).

If $B = B_0 \oplus B_1$ is a superalgebra, one can construct the algebra $G(B) = (E_0 \otimes B_0) \oplus (E_1 \otimes B_1)$ which is called the *Grassmann envelope* of B . The importance of the Grassmann envelope is highlighted in a well-known result of Kemer (see [30]) stating that every T-ideal of polynomial identities of a PI-algebra over a field of characteristic zero is determined by the identities of a Grassmann envelope of a finite dimensional $\mathbb{Z}_2$ -graded algebras. Moreover any T -ideal of identities is finitely generated. This revealed the deep structural importance of graded identities in understanding the variety of associative PI-algebras.

An *involution* $*$ on an algebra A is an antiautomorphism of A of order at most two. An algebra A endowed with an involution $*$ is called a $*$ -algebra. Given a G -graded algebra $A = \bigoplus_{g \in G} A_g$ endowed with an involution $*$, we say that $*$ is a *graded involution* if it preserves the homogeneous components of A , i.e., $A_g^* \subseteq A_g$ for all $g \in G$. A G -graded algebra endowed with a graded involution is called a $(G, *)$ -algebra.

Here we consider $(G, *)$ -algebras. For a $(G, *)$ -algebra A satisfying an ordinary polynomial identity it follows from the literature that, for instance see [25, Lemma 10.1.3], the sequence of graded codimensions $\{c_n(A)^{(G, *)}\}_{n \geq 1}$ is also exponentially bounded and the corresponding exponent exists [26], at least in the finite dimensional case. This statement naturally applies to commutative graded algebras, which can be viewed as $(G, *)$ -graded algebras equipped with the trivial involution, as well as to algebras with involution which correspond to $(G, *)$ -graded algebras with a trivial grading. Nevertheless, the result was independently established for arbitrary finite group graded PI-algebras in [1], and for PI algebras with involution in [19], both without any additional assumptions. In particular,

the latter result relies on the Representability Theorem as formulated in [2], where graded involutions of superalgebras play a central role. More precisely, any PI-algebra with involution is $*$ -PI equivalent to the Grassmann envelope $G(A) = A \hat{\otimes} E$ of a finite dimensional superalgebra A equipped by some superinvolution $*$. Notice that the involution on $G(A)$ induced by $*$ and by the parity automorphism of E is indeed a graded involution. Superalgebras endowed with a superinvolution play a natural and relevant role in the Theory of Lie and Jordan superalgebras [29], [31] and have recently become a subject of increasing interest and extensive investigations in PI-Theory (see [28], [17], [11]).

We emphasize that the existence of the corresponding exponent of varieties of algebras with additional structure has been established for finite dimensional superalgebras [8], and more generally for finite dimensional associative algebras graded by a finite group [1]. The result also holds in the cases involving the action of a Hopf algebra [26, 27] or a PI-algebra with involution [19].

In each of these settings, it becomes natural to ask for a classification of the minimal varieties with respect to the values of the corresponding exponent. The papers [10],[12] and [34] provide a complete classification of affine varieties of PI-algebras with involution in terms of generating algebras. More precisely any affine $*$ -variety is $*$ -minimal if and only if it is generated by a suitable algebra $UT^*(A_1, \dots, A_m)$ of block triangular matrices built on some sequence of finite dimensional $*$ -simple algebras $A_1, \dots, A_m$. In [14], the authors characterized minimal varieties of $\mathbb{Z}_p$ -graded PI-algebras in the affine case. The classification of minimal varieties of PI $*$ -superalgebras of finite basic rank was achieved in [15]. Most recently, the authors in [3] characterized minimal affine varieties of G -graded PI-algebras where G is a finite abelian group.

This survey focuses on varieties of $(G, *)$ -algebras where $G = C_p$ is a cyclic group of prime order p and the algebras are equipped with a graded involution. In particular, we highlight the characterization of those varieties that are minimal with respect to their $(G, *)$ -exponent.

2. Background and preliminaries. Let F be a field of characteristic zero and G a finite group. Throughout this paper all algebras are assumed to be associative and to have the same ground field F , so we will normally drop the word associative in what follows.

Let $F\langle X_G \rangle$ denote the *free associative G -graded algebra* on the countable set X_G over F . Here the set X_G decomposes as $X_G = \bigcup_{g \in G} X_g$, the union of disjoint countable sets $X_g = \{x_{1,g}, x_{2,g}, \dots\}$, where $g \in G$ and the elements of X_g

have homogeneous degree g . The free algebra $F\langle X_G \rangle$ has a natural G -grading:

$$F\langle X_G \rangle = \bigoplus_{g \in G} F_g,$$

where F_g is the subspace spanned by the monomials $x_{i_1, g_{j_1}} \cdots x_{i_t, g_{j_t}}$ of homogeneous degree $g = g_{j_1} \cdots g_{j_t}$.

Let us now consider the free algebra with involution $F\langle X_G, * \rangle$, which is a $(G, *)$ -algebra if we assume also that, for each variable $x_{i,g}$, $x_{i,g}^*$ is homogeneous of degree g . $F\langle X_G, * \rangle$ is said to be the *free $(G, *)$ -algebra* over F .

If $A = \bigoplus_{g \in G} A_g$ is a $(G, *)$ -algebra, an element

$$f = f(x_{1,g_1}, x_{1,g_1}^*, \dots, x_{n,g_n}, x_{n,g_n}^*) \in F\langle X_G, * \rangle$$

is a $(G, *)$ -polynomial identity of A if

$$f(a_{1,g_1}, a_{1,g_1}^*, \dots, a_{n,g_n}, a_{n,g_n}^*) = 0$$

for all $a_{i,g_i} \in A_{g_i}$, $i = 1, \dots, n$. In this case we write $f \equiv 0$ on A . The set of all $(G, *)$ -polynomial identities satisfied by A

$$Id_G^*(A) = \{f \in F\langle X_G, * \rangle \mid f \equiv 0 \text{ on } A\}$$

is an ideal of $F\langle X_G, * \rangle$ called the *ideal of $(G, *)$ -identities of A* . It is a T_G^* -ideal, i.e., a two-sided ideal invariant under all endomorphisms of $F\langle X_G, * \rangle$ that preserve the G -structure and commute with the graded involution $*$.

Let

$$P_n^{(G,*)} = \left\{ x_{\sigma(1),g_1}^{\gamma_1} \cdots x_{\sigma(n),g_n}^{\gamma_n} \mid \sigma \in S_n, \gamma_i \in \{1, *\} \right\}$$

be the *space of all multilinear $(G, *)$ -polynomials of degree n* . Since $\text{char } F = 0$, standard Vandermonde arguments and the linearization process prove that $Id_G^*(A)$ is completely determined by its multilinear polynomials. Thus the study of $Id_G^*(A)$ is equivalent to that of $P_n^{(G,*)} \cap Id_G^*(A)$ for all $n \geq 1$. As in the ordinary case, one defines the n -th $(G, *)$ -codimension of A as

$$c_n^{(G,*)}(A) = \dim_F \frac{P_n^{(G,*)}}{P_n^{(G,*)} \cap Id_G^*(A)}.$$

If A is a PI-algebra, i.e., it satisfies an ordinary polynomial identity, then

$$c_n(A) \leq c_n^{(G,*)}(A) \leq 2^n |G|^n c_n(A),$$

hence the sequence $\{c_n^{(G,*)}(A)\}_{n \geq 1}$ is exponentially bounded. If there exists

$$\exp_G^*(A) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^{(G,*)}(A)}$$

we call it the $(G, *)$ -*exponent* of A . Let's observe that in case of algebras with involution, that is algebras with trivial grading, one has

$$c_n^*(A) = c_n^{(G,*)}(A).$$

In this case Giambruno, Polcino Milies and Valenti [19] proved the following

Theorem 1. *Let A be a PI-algebra with involution $*$ over a field of characteristic zero. Then there exist constants C_1, C_2, t_1, t_2 and a positive integer d such that*

$$C_1 n^{t_1} d^n \leq c_n^*(A) \leq C_2 n^{t_2} d^n.$$

Hence the $*$ -exponent

$$\exp^*(A) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^*(A)}$$

exists and it is an integer.

The existence of the $*$ -exponent for finite dimensional algebras was proved by Giambruno and Zaicev in [22]. Moreover in [18] the authors proved that

Theorem 2. *If A is any finitely generated PI-algebra with involution $*$ over a field of characteristic zero, then there exist constants C_1, C_2 and a positive integer d and $t \in \frac{1}{2}\mathbb{Z}$ such that*

$$C_1 n^t d^n \leq c_n^*(A) \leq C_2 n^t d^n.$$

We recall that

Definition 1. *A variety $\mathcal{V}$ generated by a $*$ -algebra is said to be $*$ -minimal of $*$ -exponent d if*

$$\exp^*(\mathcal{V}) = d \quad \text{and} \quad \exp^*(\mathcal{U}) < d$$

for every proper subvariety $\mathcal{U}$ of $\mathcal{V}$.

The classification of $*$ -minimal varieties in terms of their generating algebras was given in the case of affine varieties using a result of Sviridova [34] by [10] and [12]. Recently a partial result was obtained without restriction on the variety by Di Vincenzo and Nardoza [11]. They present a class of finite dimensional algebras such that any $*$ -minimal variety is generated by the Grassmann envelope of a suitable element in the class.

When the grading is non trivial we consider the case of finite dimensional $(G, *)$ -algebra. If A is a finite dimensional $(G, *)$ -algebra, Gordienko in [26] showed that the $(G, *)$ -exponent of A

$$\exp_G^*(A) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^{(G,*)}(A)}$$

exists and is a non-negative integer.

He provided an explicit formula generalizing that for the ordinary PI-exponent.

Since F has characteristic zero, codimensions are invariant under field extension, so we may assume that F is algebraically closed. By the generalized Wedderburn-Malcev Theorem:

$$A = A_1 \oplus \cdots \oplus A_m + J,$$

where $A_1, \dots, A_m$ are $(G, *)$ -simple algebras with induced involution, and $J = J(A)$ is the Jacobson radical that is a graded ideal of A and invariant under the action of the involution $*$.

We say that a subalgebra $A_{i_1} \oplus \cdots \oplus A_{i_k}$ is *admissible* if for some permutation $(l_1, \dots, l_k)$ of $(i_1, \dots, i_k)$ we have $A_{l_1} J \cdots J A_{l_k} \neq 0$. The notion of admissible $(G, *)$ -algebra is closed linked to that of $(G, *)$ -exponent. In fact, in [26] it was proved that $\exp_G^*(A) = d$ where d is the maximal dimension of an admissible subalgebra of A . We recall that A is a $(G, *)$ -simple algebra if $A^2 \neq 0$ and it has no non-zero $(G, *)$ -ideals. In particular in this case we have the following

Remark 1. If A is a $(G, *)$ -simple algebra then $\exp_G^*(A) = \dim_F A$.

Let I be a T_G^* -ideal of $F\langle X_G, * \rangle$ and $\mathcal{V}$ the *variety* of $(G, *)$ -algebras associated to I , i.e., the class of all $(G, *)$ -algebras A such that $I \subseteq Id_G^*(A)$. Then $I = Id_G^*(\mathcal{V}) = Id_G^*(A)$ for some A , and we write $\mathcal{V} = \text{var}_G^*(A)$ and $\exp_G^*(\mathcal{V}) = \exp_G^*(A)$.

However, for varieties generated by finite dimensional $(G, *)$ -algebras we give the following

Definition 2. A variety $\mathcal{V}$ generated by a finite dimensional $(G, *)$ -algebra is said to be **minimal** of $(G, *)$ -exponent d if

$$\exp_G^*(\mathcal{V}) = d \quad \text{and} \quad \exp_G^*(\mathcal{U}) < d$$

for every proper subvariety $\mathcal{U}$ of $\mathcal{V}$ generated by a finite dimensional $(G, *)$ -algebra.

Our main purpose is to classify the finite dimensional algebras generating $(G, *)$ -minimal varieties. We emphasize that this classification would hold in full generality in the context of affine varieties if a representability theorem for finitely generated $(G, *)$ -algebras satisfying a polynomial identity were proved. More precisely, every finitely generated $(G, *)$ -algebra satisfying a polynomial identity would be equivalent to a finite dimensional $(G, *)$ -algebra. All subvarieties of an affine variety would themselves be affine and consequently generated by a finite dimensional $(G, *)$ -algebra.

This is the case when G has order 2. More precisely Giambruno, Ioppolo and La Mattina proved the following

Theorem 3 ([17, Theorem 5.3]). *Let R be a finitely generated superalgebra with graded involution. If R satisfies a polynomial identity then there exists a finite dimensional superalgebra A with graded involution such that $\text{Id}_G^*(R) = \text{Id}_G^*(A)$.*

See also the paper of Aljadeff, Giambruno and Karasik [2].

3. Elementary grading on matrix algebras and graded involutions. This section investigates graded involutions on matrix algebras $M_n(F)$ with elementary gradings. For simplicity, from now on, we will write M_n instead of $M_n(F)$. We set, for every positive integers m, n with $m \leq n$, $[m, n] := \{m, m+1, \dots, n\}$ and we denote by e_{ij} the usual (i, j) -matrix unit of M_n , $i, j \in [1, n]$. Also we indicate by I_n the $n \times n$ matrix identity of M_n and by J_n the matrix of M_n with all ones in the secondary diagonal and zeros in the other entries

$$J_n = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & & \vdots & \\ 0 & 1 & & \vdots & \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \in M_n.$$

Let us recall that a G -grading on $M_n = \bigoplus_{g \in G} (M_n)_g$ is an elementary grading if there exists an n -tuple $(g_1, \dots, g_n) \in G^n$ such that the matrix units e_{ij} , $i, j \in [1, n]$, are homogeneous and $e_{ij} \in (M_n)_g$ if and only if $g = g_i^{-1} g_j$. In an equivalent manner we have the following

Definition 3. Let n be a positive integer, and let $\mathcal{E}_n := \{e_{ij} \mid i, j \in [1, n]\}$ be the canonical basis of matrix units of M_n . For any map $\alpha : [1, n] \rightarrow G$, we write $\alpha_i = \alpha(i)$ and define $|e_{ij}|_\alpha := \alpha_i^{-1} \alpha_j$. The resulting grading on M_n is called the elementary grading induced by α .

We write (M_n, α) to denote the matrix algebra equipped with an elementary grading induced by α . Also we write $(M_n, \alpha, *)$ when $*$ is a graded involution on (M_n, α) .

Elementary gradings form the foundation for the classification of all G -gradings over M_n . Recall that over an algebraically closed field F , every G -grading on M_n , where G is a finite cyclic group, is isomorphic to an elementary grading.

In this section, if it is not explicitly stated, G will be a finite abelian group. We are interested in involutions that preserve the G -grading. An explicit description of graded involutions of graded matrix algebras was made by Bahturin and Zaicev in [7] (see also [6] and [4]). Here we will present a different classification of graded involutions on matrix algebras with elementary gradings.

We introduce an important class of involutions of transpose type.

Definition 4. Let $n \in \mathbb{N}$ and let $\gamma \in S_n$ be a permutation of $[1, n]$ such that $\gamma^2 = \text{id}$. We define an involution θ_γ on M_n by

$$e_{ij}^{\theta_\gamma} = e_{\gamma(j), \gamma(i)} \quad \text{for all } i, j \in [1, n].$$

If $(g_1, \dots, g_n) \in G^n$ determines the elementary G -grading on M_n , where G is abelian, then θ_γ is a graded involution if and only if

$$(1) \quad g_i g_{\gamma(i)} = g_j g_{\gamma(j)} \quad \text{for all } i, j \in [1, n].$$

Let $2q \leq n$ and define the permutation $\gamma_{n,q}$ on $[1, n]$ by

$$\gamma_{n,q}(i) = \begin{cases} n+1-i, & \text{if } i \in [1, q] \cup [n+1-q, n], \\ i, & \text{if } i \in [q+1, n-q]. \end{cases}$$

We denote the corresponding involution simply by $\theta_{n,q}$. The number of fixed points of $\gamma_{n,q}$ is exactly $n - 2q$.

- If $n - 2q \in \{0, 1\}$, we denote the corresponding involution by θ_n , called the *reflection involution*, given by

$$e_{ij}^{\theta_n} = e_{n+1-j, n+1-i}.$$

- If $q = 0$, then $\theta_{n,0}$ is the classical transpose involution

$$e_{ij}^t = e_{ji}.$$

- If $n = 2m$ is even, we define the *symplectic type involution* σ_n on M_n by

$$e_{ij}^{\sigma_n} = \delta_{[1,m]}(i) \delta_{[1,m]}(j) e_{n+1-j, n+1-i} \quad (i, j \in [1, 2m]),$$

where

$$\delta_{[1,m]}(i) = \begin{cases} 1, & i \in [1, m], \\ -1, & i \in [m+1, 2m]. \end{cases}$$

The description of the involution defined on any finite dimensional central simple algebra is well know (see, for example, [33, Chapter 3]).

Proposition 1. *Let F be an algebraically closed field of characteristic zero and let $*$ be an involution on M_n . Then either $*$ is of transpose type and*

$$(M_n, *) \cong (M_n, t) \cong (M_n, \theta_n)$$

or n is even, $*$ is of symplectic type and

$$(M_n, *) \cong (M_n, \sigma_n).$$

Once again, θ_n and σ_n are graded involutions on M_n with respect the elementary G -grading induced by $(g_1, \dots, g_n)$ if and only if

$$g_i g_{n+1-i} = g_j g_{n+1-j} \text{ for all } i, j \in [1, n]$$

and n is even in the symplectic case.

Now we recall the duality between the involutions on the matrix algebra M_n and the bilinear forms on the vector space $V = F^n$.

Let $\varphi : V \times V \rightarrow F$ be a non-degenerate bilinear form on $V = F^n$. For $X \in \text{End}(V)$ and $a, b \in V$, set

$$\varphi(Xa, b) = \varphi(a, Xb).$$

This relation defines an anti-automorphism $*$: $\text{End}(V) \rightarrow \text{End}(V)$.

If B is a fixed basis of V and Φ is the matrix of φ in B , then in matrix form one has

$$X^* = \Phi^{-1} X^t \Phi,$$

where X^t is the transpose of X and t denotes the transpose involution on $M_n(F)$. Conversely, by the Noether–Skolem theorem, any involution $*$ of $M_n(F)$ is of the above form with $\Phi^t = \pm \Phi$, and Φ is uniquely determined by $*$ up to a scalar factor.

In particular:

- If $*$ = θ_n , then $\Phi = J_n$;
- If $n = 2m$ and $*$ = σ_n , then $\Phi = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}$ where J_m is the $m \times m$ exchange (anti-diagonal-ones) matrix.

The following result expresses the relevance of the graded involution

Proposition 2 (see, [9, Proposition 5]). *Let $(M_n, \alpha, *)$ be a matrix algebra over an algebraically closed field of characteristic zero F , with elementary G -grading α and graded involution $*$ defined by a non-degenerate bilinear form φ .*

- (a) *If φ is symmetric then, for some integer q such that $0 \leq 2q \leq n$, $(M_n, \alpha, *)$ is isomorphic as $(G, *)$ -algebra to $(M_n, \tilde{g}, \theta_{n,q})$ where the elementary G -grading is defined by a n -tuple $\tilde{g} = (g_1, \dots, g_n) \in G^n$ satisfying*

$$g_i g_{\gamma_{n,q}(i)} = g_j g_{\gamma_{n,q}(j)} \text{ for all } i, j \in [1, n].$$

Moreover:

- *if $|G|$ is odd, then $(M_n, \alpha, *)$ is isomorphic as a $(G, *)$ -algebra to $(M_n, \tilde{g}, \theta_n)$;*
 - *if $|G| = 2$, that is M_n is a $*$ -superalgebra, then either $(M_n, \alpha, *)$ is isomorphic to the $*$ -superalgebra $(M_{q,q}, \theta_{2q})$ for some q such that $n = 2q$ or $(M_n, \alpha, *)$ is isomorphic to (M_{m_1, m_2}, t) for some $m_1 \geq m_2$ such that $n = m_1 + m_2$.*
- (b) *If φ is skew-symmetric then $n = 2m$ is an even integer and $(M_n, \alpha, *)$ is isomorphic as $(G, *)$ -algebra to $(M_n, \tilde{g}, \sigma_n)$ where the elementary G -grading is defined by the n -tuple $\tilde{g} = (g_1, \dots, g_n) \in G^n$ satisfying*

$$g_i g_{n+1-i} = g_j g_{n+1-j} \text{ for all } i, j \in [1, n].$$

Let us recall that given a G -graded algebra A then the opposite algebra A^{op} has a natural G -grading defined by $A_g^{op} = A_g$, for all $g \in G$, because A and A^{op} have the same structure as vector space over the field F and G is abelian. Clearly $C = A \oplus A^{op}$ has an induced G -grading, given by $C_g = A_g \oplus A_g^{op}$ and in this case the exchange involution

$$(a, b)^{exc} = (b, a), \quad \text{for all } (a, b) \in C$$

is a graded involution on $A \oplus A^{op}$.

When A is a matrix algebra with an elementary grading α we can realize $(A \oplus A^{op}, \alpha, exc)$ in two more convenient ways. More precisely:

Definition 5. If $\alpha : [1, n] \rightarrow G$ is any map, we present the map α as a G -word (or n -tuple) of length n , that is $\alpha := (\alpha_1, \dots, \alpha_n)$, then we define $\alpha^{-1} := (\alpha_n^{-1}, \dots, \alpha_1^{-1})$ and we consider the concatenation $\bar{\alpha} = (\alpha \mid \alpha^{-1})$ of the words α and α^{-1} , that is

$$\bar{\alpha} = (\alpha_1, \dots, \alpha_n, \alpha_n^{-1}, \dots, \alpha_1^{-1}).$$

We say that $\bar{\alpha}$ is the $*$ -closure of α .

Clearly $\bar{\alpha}_i \bar{\alpha}_{2n+1-i} = 1_G$ for all $i \in [1, 2n]$ and so θ_{2n} is a graded involution on $(M_{2n}, \bar{\alpha})$. Now let $(M_n \oplus M_n, \bar{\alpha}, \theta_{2n})$ denote the $(G, *)$ -subalgebra of $n \times n$ block diagonal matrices of $(M_{2n}, \bar{\alpha}, \theta_{2n})$ and let us consider the linear map $\psi_n : M_n \oplus M_n^{op} \rightarrow M_{2n}$ defined by:

$$\psi_n(a, b) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}^{\theta_{2n}} = \begin{pmatrix} a & 0 \\ 0 & b^{\theta_n} \end{pmatrix},$$

for all $(a, b) \in M_n \oplus M_n^{op}$.

We obtain the following:

Lemma 1. The map ψ_n induces an isomorphism as $(G, *)$ -algebras from

$$(M_n \oplus M_n^{op}, \alpha, exc) \quad \text{to} \quad (M_n \oplus M_n, \bar{\alpha}, \theta_{2n}).$$

On the other hand we can consider the graded algebra $(M_n, \alpha) \oplus (M_n, \alpha^{-1})$ and define a G -graded involution $\tilde{\vartheta}_n$ on it by

$$(a, b)^{\tilde{\vartheta}_n} = (b^{\theta_n}, a^{\theta_n}).$$

We denote this $(G, *)$ -algebra by $(M_n \oplus M_n, \bar{\alpha}, \tilde{\vartheta}_n)$. Clearly, considering the map

$$(a, b) \mapsto (a, b^{\theta_n})$$

we obtain

Lemma 2. *The $(G, *)$ -algebras $(M_n \oplus M_n^{op}, \alpha, exc)$ and $(M_n \oplus M_n, \bar{\alpha}, \tilde{\nu}_n)$ are isomorphic.*

4. Classification of finite dimensional simple $(\mathbb{C}_p, *)$ -algebras.

In this section we present a classification of finite dimensional simple $(\mathbb{C}_p, *)$ -algebras over an algebraically closed field F of characteristic zero, where p is a prime and $\mathbb{C}_p = \langle \varepsilon \rangle$ denotes the cyclic group of order p .

We begin by recalling the classification of finite dimensional simple algebras graded by $\mathbb{C}_p$, which is a special case of the following structure theorem for graded algebras due to Bahturin, Sehgal, and Zaicev [5].

Theorem 4. *Let A be a finite dimensional algebra over an algebraically closed field F graded by an arbitrary group G . If the characteristic of F is zero or does not divide the order of any finite subgroup of G , then A is a G -graded simple algebra if and only if A is isomorphic to the tensor product $M_n(F) \otimes D$ where*

- (1) $D \equiv F^\sigma[H]$ is the skew-group algebra;
- (2) H is a finite subgroup of G ;
- (3) $\sigma : H \times H \rightarrow F$ is a 2-cocycle on H .

The grading with respect to H on $F^\sigma[H]$ is canonical, and the G -grading on $M_n(F^\sigma[H]_h)$ is defined by a family $(g_1, \dots, g_n) \in G^n$ so that $\deg(e_{ij} \otimes x_h) = g_i^{-1} h g_j$ for each matrix unit e_{ij} and each homogeneous element $x_h \in F^\sigma[H]_h$.

Let $F[\mathbb{C}_p]$ denote the group algebra. This algebra can be realized as a graded subalgebra of M_p with the elementary grading induced by the tuple

$$\tilde{\varepsilon} = (1_G, \varepsilon, \varepsilon^2, \dots, \varepsilon^{p-1}).$$

Let $\eta := (1 \ 2 \ \dots \ p)$ be the cyclic permutation in S_p , and define D to be the graded subalgebra of M_p generated over F by the elements

$$(2) \quad c_{\varepsilon^{h-1}} := \sum_{i=0}^{p-1} e_{\eta^i(1), \eta^i(h)}, \quad h \in [1, p].$$

Every element of D has the form

$$a_1 c_{1_G} + a_2 c_\varepsilon + \cdots + a_p c_{\varepsilon^{p-1}} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ a_p & a_1 & \cdots & \cdots & a_{p-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_3 & & \ddots & \ddots & a_2 \\ a_2 & a_3 & \cdots & a_p & a_1 \end{pmatrix} \in M_p,$$

with $a_1, \dots, a_p \in F$. Clearly, $D \cong F[\mathbb{C}_p]$ as graded algebras.

The algebra $M_n(D)$ inherits a natural $\mathbb{C}_p$ -grading, induced from the grading on D and the trivial grading on M_n . Explicitly, if

$$\tilde{1}_G = (\underbrace{1_G, \dots, 1_G}_n),$$

then

$$M_n(D) \cong D \otimes M_n \subseteq M_{pn} \cong M_p \otimes M_n,$$

with the elementary grading induced by

$$\tilde{\varepsilon} \otimes \tilde{1}_G = (\underbrace{1_G, \varepsilon, \dots, \varepsilon^{p-1}, \dots, 1_G, \varepsilon, \dots, \varepsilon^{p-1}}_{n \text{ times}}).$$

We denote both this elementary grading on M_{pn} and the natural one in $M_n(D)$ by $\tilde{\varepsilon}_n$. Now we have

Proposition 3 ([14], Proposition 3.1). *Let F be algebraically closed of characteristic zero, and let $\mathbb{C}_p = \langle \varepsilon \rangle$ be the cyclic group of prime order p . If A is a finite dimensional $\mathbb{C}_p$ -simple algebra, then A is isomorphic to one of the following $\mathbb{C}_p$ -graded algebras:*

- (1) (M_n, α) , for some elementary grading α ;
- (2) $(M_n(D), \tilde{\varepsilon}_n)$.

We next classify the graded involutions on $(M_n(D), \tilde{\varepsilon}_n)$. Define F -linear maps on $M_n(D)$ by

$$(de_{ij})^{\bar{\theta}_n} := de_{ij}^{\theta_n}, \quad \text{and if } n \text{ is even,} \quad (de_{ij})^{\bar{\sigma}_n} := de_{ij}^{\sigma_n},$$

for all $d \in D$. These are graded involutions on $(M_n(D), \tilde{\varepsilon}_n)$, induced from θ_{pn} and σ_{pn} on $(M_{pn}, \tilde{\varepsilon}_n)$. Up to isomorphism, these are the only possibilities. More precisely, we have:

Proposition 4. *Let F be algebraically closed of characteristic zero. For any graded involution $*$ on $(M_n(D), \tilde{\varepsilon}_n)$ we have either*

$$(M_n(D), \tilde{\varepsilon}_n, *) \cong (M_n(D), \tilde{\varepsilon}_n, \bar{\theta}_n),$$

or else

$$(M_n(D), \tilde{\varepsilon}_n, *) \cong (M_n(D), \tilde{\varepsilon}_n, \bar{\sigma}_n),$$

and in this case n is even.

We now summarize the classification of finite dimensional $(\mathbb{C}_p, *)$ -simple algebras.

Theorem 5 ([9], Theorem 2). *Let A be a finite dimensional $(\mathbb{C}_p, *)$ -simple algebra over an algebraically closed field F of characteristic zero. Then A is isomorphic to one of the following:*

- (a) (M_n, α, θ_n) or (M_n, α, σ_n) , where $\alpha : [1, n] \rightarrow \mathbb{C}_p$ is an elementary grading such that

$$\alpha_i \alpha_{n+1-i} = \alpha_j \alpha_{n+1-j}, \quad \forall i, j \in [1, n];$$

- (b) $(M_n(D), \tilde{\varepsilon}_n, \bar{\theta}_n)$ or $(M_n(D), \tilde{\varepsilon}_n, \bar{\sigma}_n)$;

- (c) $(M_n \oplus M_n^{op}, \alpha, \text{exc})$, with grading induced by α and exchange involution;

- (d) $(M_n(D) \oplus M_n(D)^{op}, \tilde{\varepsilon}_n, \text{exc})$, with grading induced by $\tilde{\varepsilon}_n$ and exchange involution.

Remark 2. All algebras in Theorem 5 can also be defined over fields F that are not algebraically closed. In that case, they are called *classical* finite dimensional $(\mathbb{C}_p, *)$ -simple algebras over F .

5. The algebra $UT_{\mathbb{C}_p}^*(A_1, \dots, A_m)$ and the main results. In this section, we introduce the $(\mathbb{C}_p, *)$ -algebra $UT_{\mathbb{C}_p}^*(A_1, \dots, A_m)$ of upper block triangular matrices associated with a sequence $(A_1, \dots, A_m)$ of classical finite dimensional $(\mathbb{C}_p, *)$ -simple algebras.

Our main result establishes that a variety of $(\mathbb{C}_p, *)$ -algebras, generated by a finite dimensional $(\mathbb{C}_p, *)$ -algebra, is minimal with a fixed exponent if and only if it is generated by such an upper block triangular matrix algebra, endowed with an appropriate elementary grading and a compatible graded involution.

We begin by recalling some fundamental definitions and concepts required for this construction.

Let A be a classical finite dimensional $(\mathbb{C}_p, *)$ -simple algebra. Then, for some positive integer n , we assume that:

$$A = S \quad \text{or} \quad A = S \oplus S^{\text{op}},$$

where $S \in \{M_n, M_n(D)\}$ and $M_n(D)$ is a graded subalgebra of M_{np} defined in the previous section. In this context, we say that S is the *constituent* of A and n is the *size* of A . The *extended size* s_A of A is defined by:

$$s_A := \begin{cases} n, & \text{if } S = M_n, \\ pn, & \text{if } S = M_n(D). \end{cases}$$

Note that S is always a graded subalgebra of M_{s_A} with respect to an elementary grading $\alpha_S : [1, s_A] \rightarrow \mathbb{C}_p$.

Proposition 5. *Let A be a classical finite dimensional $(\mathbb{C}_p, *)$ -simple algebra, with constituent S , extended size $s = s_A$, and grading $\alpha = \alpha_S$. Then A can be embedded as a $(\mathbb{C}_p, *)$ -subalgebra of $(M_{2s}, \bar{\alpha}, \theta_{2s})$ via the map φ_A defined as:*

- If $A = S \oplus S^{\text{op}}$, then

$$\varphi_A(a, b) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}^{\theta_{2s}} = \begin{pmatrix} a & 0 \\ 0 & b^{\theta_s} \end{pmatrix}.$$

- If $A = S$, then

$$\varphi_A(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a^* & 0 \\ 0 & 0 \end{pmatrix}^{\theta_{2s}} = \begin{pmatrix} a & 0 \\ 0 & a^{*\theta_s} \end{pmatrix}.$$

We can express the embedding more elegantly by viewing $M_{2s} \cong M_s \otimes M_2$, so that:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a \otimes e_{11} + b \otimes e_{22} = a \otimes e_{11} + b \otimes e_{11}^{\theta_2}, \quad \forall a, b \in M_s.$$

Generalizing this construction to a direct sum we obtain the following

Proposition 6. *Let $A_1, \dots, A_m$ be classical finite dimensional $(\mathbb{C}_p, *)$ -simple algebras, with constituents $S_i \subseteq M_{s_i}$, grading words α^i on M_{s_i} , and $(\mathbb{C}_p, *)$ -embeddings $\varphi_i : A_i \rightarrow R_i = M_{s_i} \oplus M_{s_i}$.*

Let $s := s_1 + \dots + s_m$, and define the concatenated word $\alpha := (\alpha^1 | \dots | \alpha^m)$. Endow M_{2s} with the grading $\bar{\alpha}$ and involution θ_{2s} .

Define

$$\zeta : R_1 \oplus \dots \oplus R_m \longrightarrow M_{2s}, \text{ by } \zeta \left(\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}, \dots, \begin{pmatrix} a_m & 0 \\ 0 & b_m \end{pmatrix} \right) = \sum_{i=1}^m a_i \otimes e_{ii} + \sum_{i=1}^m b_i \otimes e_{ii}^{\theta_{2m}}.$$

Then the map

$$\varphi(u_1, \dots, u_m) := \zeta(\varphi_1(u_1), \dots, \varphi_m(u_m))$$

is a $(\mathbb{C}_p, *)$ -embedding of $A_1 \oplus \dots \oplus A_m$ into $(M_{2s}, \bar{\alpha}, \theta_{2s})$.

Definition 6. *Let $A_1, \dots, A_m$ be classical finite dimensional $(\mathbb{C}_p, *)$ -simple algebras. Let φ be the embedding from Proposition 6, and for $1 \leq i, j \leq m$, let $V_{ij} := M_{s_i \times s_j}$.*

Define:

$$U_{ij} := V_{ij} \otimes e_{ij}, \quad \overline{U_{ij}} := U_{ij}^{\theta_{2s}} = V_{ji} \otimes e_{ij}^{\theta_{2m}},$$

$$V := \bigoplus_{1 \leq i < j \leq m} (U_{ij} \oplus \overline{U_{ij}}),$$

and set

$$UT_{\mathbb{C}_p}^*(A_1, \dots, A_m) := \varphi(A_1 \oplus \dots \oplus A_m) \oplus V.$$

A direct verification shows that $UT_{\mathbb{C}_p}^*(A_1, \dots, A_m)$ is a $(\mathbb{C}_p, *)$ -subalgebra of M_{2s} , with semisimple part $\varphi(A_1 \oplus \dots \oplus A_m)$ and Jacobson radical V .

Now, putting together the results of [14] and [9] we have

Theorem 6. *Let $\mathcal{V}$ be a variety of $(\mathbb{C}_p, *)$ -algebras generated by some finite dimensional $(\mathbb{C}_p, *)$ -algebra, and suppose that its $(\mathbb{C}_p, *)$ -exponent satisfies*

$$\exp_{\mathbb{C}_p}^*(\mathcal{V}) = d.$$

Then there exist classical finite dimensional $(\mathbb{C}_p, *)$ -simple algebras $A_1, \dots, A_m$ such that

$$UT_{\mathbb{C}_p}^*(A_1, \dots, A_m) \in \mathcal{V}, \quad \dim_F(A_1 \oplus \dots \oplus A_m) = d.$$

In particular, if $\mathcal{V}$ is minimal, then $\mathcal{V} = \text{var}_{\mathbb{C}_p}^*(UT_{\mathbb{C}_p}^*(A_1, \dots, A_m))$.

According to Theorem 6, every minimal variety of $(\mathbb{C}_p, *)$ -algebras of $(\mathbb{C}_p, *)$ -exponent d is generated by an upper block triangular matrix algebra of the form

$$UT_{\mathbb{C}_p}^*(A_1, \dots, A_m),$$

where $A_1, \dots, A_m$ are classical finite dimensional $(\mathbb{C}_p, *)$ -simple algebras satisfying

$$\dim_F(A_1 \oplus \dots \oplus A_m) = d.$$

In order to obtain a full characterization of such varieties, it remains to establish that for every sequence $A_1, \dots, A_m$ of $(\mathbb{C}_p, *)$ -simple algebras with $\dim_F(A_1 \oplus \dots \oplus A_m) = d$, the algebra $UT_{\mathbb{C}_p}^*(A_1, \dots, A_m)$ generates a minimal variety.

We have the following:

Theorem 7 ([9], Theorem 5). *Let $\mathbf{A} = UT_{\mathbb{C}_p}^*(A_1, \dots, A_m)$ and $\mathbf{B} = UT_{\mathbb{C}_p}^*(B_1, \dots, B_n)$, then*

$$\mathbf{A} \cong \mathbf{B} \Leftrightarrow Id_{\mathbb{C}_p}^*(\mathbf{A}) = Id_{\mathbb{C}_p}^*(\mathbf{B}).$$

Now, we are able to state the main result of this paper

Theorem 8. *Let $\mathcal{V}$ be a variety of $(\mathbb{C}_p, *)$ -algebras generated by a finite dimensional $(\mathbb{C}_p, *)$ -algebra. Hence, $\mathcal{V}$ is minimal of exponent d if and only if $\mathcal{V} = \text{var}_{\mathbb{C}_p}^*(\mathbf{A})$ where $\mathbf{A} = UT_{\mathbb{C}_p}^*(A_1, \dots, A_m)$ with $A_1, \dots, A_m$ classical finite dimensional $(\mathbb{C}_p, *)$ -simple algebras satisfying $\dim_F(A_1 \oplus \dots \oplus A_m) = d$.*

Proof. Let $A_1, \dots, A_m$ be a sequence of classical finite dimensional $(\mathbb{C}_p, *)$ -simple algebras satisfying $\dim_F(A_1 \oplus \dots \oplus A_m) = d$ and $\mathbf{A} = UT_{\mathbb{C}_p}^*(A_1, \dots, A_m)$. We consider $\mathcal{U}_{\mathbb{C}_p}^*$ a subvariety of $\text{var}_{\mathbb{C}_p}^*(\mathbf{A})$ generated by a finite dimensional $(\mathbb{C}_p, *)$ -algebra such that $\exp_{\mathbb{C}_p}^*(\mathbf{A}) = \exp_{\mathbb{C}_p}^*(\mathcal{U}_{\mathbb{C}_p}^*)$. By Theorem 6, we get that there exist classical finite dimensional $(\mathbb{C}_p, *)$ -simple algebras $B_1, \dots, B_n$ such that, $\mathbf{B} = UT_{\mathbb{C}_p}^*(B_1, \dots, B_n)$, $Id_{\mathbb{C}_p}^*(\mathcal{U}_{\mathbb{C}_p}^*) \subseteq Id_{\mathbb{C}_p}^*(\mathbf{B})$ and $\exp_{\mathbb{C}_p}^*(\mathcal{U}_{\mathbb{C}_p}^*) = \exp_{\mathbb{C}_p}^*(\mathbf{B})$.

It is possible to prove that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic as $(\mathbb{C}_p, *)$ -algebras. Consequently, by the previous Theorem, $\text{Id}_{\mathbb{C}_p}^*(\mathbf{A}) = \text{Id}_{\mathbb{C}_p}^*(\mathbf{B})$ and so $\text{var}_{\mathbb{C}_p}^*(\mathbf{A}) = \mathcal{U}_{\mathbb{C}_p}^*$.

Conversely the result follows from Theorem 6. $\square$

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