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# GRADED POLYNOMIAL IDENTITIES FOR THE LIE ALGEBRA OF $3 \times 3$ UPPER TRIANGULAR MATRICES ENDOWED WITH THE CANONICAL GRADING*

Daniela Martinez Correa, Dimas José Gonçalves, Evandro Riva

*Communicated by P. Koshlukov*

*Dedicated to Professor Vesselin Drensky on the occasion of his 75th birthday*

Abstract. Let $\mathbb{F}$ be a finite field of characteristic different from 2, and let $UT_3^{(-)} = UT_3(\mathbb{F})^{(-)}$ be the Lie algebra of $3 \times 3$ upper triangular matrices over $\mathbb{F}$ endowed with the canonical $\mathbb{Z}_3$-grading. In this paper, we describe the $T_{\mathbb{Z}_3}$-ideal $Id = Id(UT_3^{(-)})$ of all $\mathbb{Z}_3$-graded polynomial identities of $UT_3^{(-)}$ and we produce a linear basis for the corresponding relatively free graded algebra. Moreover, we prove that $Id$ satisfies the Specht property, that is, $Id$ and every $T_{\mathbb{Z}_3}$-ideal containing $Id$ is finitely generated as a $T_{\mathbb{Z}_3}$-ideal.

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*Key words:* Lie algebra, upper triangular matrices, canonical grading, graded polynomial identities, Specht problem.

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**1. Introduction.** The theory of polynomial identities (PI-theory) plays a significant role in ring theory. Key questions in the qualitative approach to PI-theory involve determining the polynomial identities of particular algebras and studying the properties of the varieties that these algebras generate. Although the most classical results in this area deal with polynomial identities for associative algebras over fields of characteristic 0, in recent years, many results in PI-theory have dealt with other classes of algebras, as well as other types of identities, but there are still very many questions to answer.

The sophisticated theory developed by Kemer in [14] was a milestone in the case of associative algebras over a field of characteristic 0. Kemer proved that, for associative algebras in characteristic 0, every T-ideal is finitely generated as a T-ideal, thus giving a positive answer to the long standing Specht problem. On the other hand, if the base field is infinite but of positive characteristic, the Specht problem can have a negative answer. The first examples for Lie algebras were obtained by Vaughan-Lee [29, 30], in characteristic 2, and by Drensky [7], for every characteristic $p > 0$. The first examples in the case of associative algebras were obtained, much later, independently by Belov [3], Grishin [12], and Shchigolev [24]. Consequently, we are led to inquire under what hypotheses or conditions the Specht problem admits a positive solution.

One of the main tools used in Kemer's theory was the study of $\mathbb{Z}_2$-graded algebras, as a result, this kind of algebra has captured the attention of researchers, developing new techniques for a deeper understanding and description of the (graded) polynomial identities of a given (graded) algebra. In [1, 27], it was proved that the Specht problem has positive answer for graded associative algebras over fields of characteristic zero when the grading group is finite. We want to highlight that Kemer's theory, while powerful for associative algebras, may encounter limitations when extended to non-associative contexts. Thus one should impose certain restrictions on the classes of algebras when studying the Specht problem for Lie or Jordan algebras. Therefore it is interesting to study the Specht problem for concrete varieties of Lie and Jordan algebras. For example, Krasilnikov showed in [19] that the variety of Lie algebras defined by $UT_n(\mathbb{F})^{(-)}$, the Lie algebra of $n \times n$ upper triangular matrices over a field $\mathbb{F}$, satisfies the Specht property when $\mathbb{F}$ is infinite of characteristic 0 or characteristic $p$ greater than or equal to $n$.

In 2017, Giambruno and Souza [8] proved that, over a field of characteristic 0, the variety of graded Lie algebras generated by $sl_2$, the Lie algebra of $2 \times 2$ traceless matrices, has the Specht property when it is endowed by any grading. In [17], Koshlukov and Yukihide found a finite basis for the graded identities of

$UT_n(\mathbb{F})^{(-)}$ when this algebras is endowed with the canonical grading by $\mathbb{Z}_n$ and the field $\mathbb{F}$ is infinite. Consequently, in [20] it was proved the Specht property for the graded identities of $UT_n(\mathbb{F})^{(-)}$, endowed with the canonical grading, when the basis field is infinite of characteristic 0 or characteristic $p \geq n$.

With respect to $UT_3(\mathbb{F})^{(-)}$, Yukihide [31] described its graded polynomial identities for the elementary gradings when the characteristic of $\mathbb{F}$ is 0.

There are some works about graded identities for Jordan algebras graded by finite abelian groups in characteristic zero. We can cite [16] and [4] where the authors studied graded identities for $UJ_2$, the Jordan algebra of $2 \times 2$ upper triangular matrices. In [5] was showed that the variety generated by $UJ_2$ has the Specht property when it is graded by any finite abelian group. Moreover, Koshlukov and Silva in [15] found the graded identities for $\mathbb{Z}_2$-gradings of the Jordan algebra of symmetric matrices of order two, then in [26] was proved the Specht property for the finite dimensional Jordan algebra of a non-degenerate symmetric bilinear form graded by $\mathbb{Z}_2$, in characteristic 0. Morais and Souza [21] showed that for any grading, the variety of graded commutative algebras generated by $(UT_2, \circ)$ has the Specht property in characteristic 2.

Now we recall some of the recent results concerning group graded Lie and Jordan algebras and their identities when the base field is finite. In [9], the authors described the set of all polynomial identities for the Jordan algebra of $2 \times 2$ upper triangular matrices over an arbitrary field. Moreover, if the base field is infinite, they proved the Specht property for these identities. Gonçalves and Salomão [10] studied the $\mathbb{Z}_2$-graded polynomial identities for the Jordan algebra of $2 \times 2$ upper triangular matrices over an arbitrary field.

In [28] and [11], it was described the identities for the associative algebra of upper triangular matrices over an arbitrary field. Recently, Morais et al. [22], exhibited a finite basis of graded identities for the variety of Lie algebras generated by the Lie algebra of $2 \times 2$ upper triangular matrices over a finite field of characteristic 2 for any grading and show that it has the Specht property. We want to highlight that this case for characteristic different from 2 was studied in [23].

Following that direction, in this paper, we investigate the graded identities of the Lie algebra of $3 \times 3$ upper triangular matrices endowed with the canonical grading over an arbitrary field and find linear basis for the corresponding relatively free algebra when the characteristic is greater than 2. Finally, we give a positive answer to the Specht Property for this graded algebra when the base field is finite with characteristic different from 2. Our methods are combinatorial in nature, we treat the case of finite and of infinite base field separately,

as the proofs in each case diverge significantly. Finally we employ the technique of partially well-ordered sets [2] in order to deduce the Specht property.

**2. Preliminaries.** Throughout this paper, $\mathbb{F}$ is a field and all algebras are over $\mathbb{F}$.

Let $UT_3 = UT_3(\mathbb{F})$ be the $3 \times 3$ upper triangular matrix algebra with entries in $\mathbb{F}$. The usual matrix multiplication $\cdot$ in $UT_3$ makes it an associative algebra. Denote by $UT_3^{(-)}$ the vector space $UT_3$ with the bracket product $[\ ,\ ]$ given by

$$[u, v] = u \cdot v - v \cdot u$$

where $u, v \in UT_3$. Then $UT_3^{(-)}$ becomes a Lie algebra.

We recall that if $(G, \cdot)$ is a group and $A$ is a Lie algebra with product $[\ ,\ ]$, then a $G$-grading on $A$ is a direct sum of subspaces

$$A = \bigoplus_{\theta \in G} A_\theta \ \text{ such that } \ [A_\theta, A_{\theta'}] \subseteq A_{\theta \cdot \theta'}$$

for all $\theta, \theta' \in G$. If $u \in A_\theta$, we write $\deg_G u = \theta$.

The $G$-gradings on $UT_3^{(-)}$ were described in [18]. Let $e_{ij}$ be the matrix unit in $UT_3^{(-)}$ whose $(i,j)$th entry equals 1 and all remaining entries equal 0, and let $\mathbb{Z}_3 = \{0, 1, 2\}$ be the usual cyclic group of order 3 with operation $+$. Observe that $UT_3^{(-)}$ is a $\mathbb{Z}_3$-graded algebra by setting

$$UT_3^{(-)} = \bigoplus_{m=0}^{2} [UT_3^{(-)}]_m \ \text{ where } \ [UT_3^{(-)}]_m = \operatorname{span}\{e_{ij} \mid j - i = m\}.$$

This grading is called canonical $\mathbb{Z}_3$-grading on $UT_3^{(-)}$.

Given an abelian group $(G, \cdot)$, for each $\theta \in G$ consider an infinite countable set $W_\theta = \{w_1^\theta, w_2^\theta, \ldots\}$ and denote $W = \bigcup_{\theta \in G} W_\theta$. Let $\mathcal{L}(W)$ be the free $G$-graded Lie algebra, freely generated by $W$. We recall that $\mathcal{L}(W)$ is the free Lie algebra, freely generated by $W$, endowed with the $G$-grading

$$\mathcal{L}(W) = \bigoplus_{\theta \in G} (\mathcal{L}(W))_\theta$$

satisfying the following conditions:

(i) $\deg_G(w_i^\theta) = \theta$ for all $i \geq 1$ and $\theta \in G$,

(ii) If $u, v$ are monomials in $\mathcal{L}(W)$, then $\deg_G([u,v]) = (\deg_G u) \cdot (\deg_G v)$.

A $T_G$-ideal of $\mathcal{L}(W)$ is a $G$-graded ideal of $\mathcal{L}(W)$ closed under all $G$-graded endomorphisms of $\mathcal{L}(W)$. For a subset $S \subseteq \mathcal{L}(W)$, we denote by $\langle S \rangle^{T_G}$ the $T_G$-ideal of $\mathcal{L}(W)$ generated by $S$, that is the minimal $T_G$-ideal of $\mathcal{L}(W)$ containing $S$. If $f \in \langle S \rangle^{T_G}$, we say that $f$ is a consequence of the polynomials in $S$.

By using similar arguments as in [6, Proposition 4.2.3], we state the following proposition.

**Proposition 2.1.** *Let $\mathbb{F}$ be a field of cardinality $|\mathbb{F}|$. Given $f \in \mathcal{L}(W)$ and $w \in W$, write*

$$f = \sum_{i=0}^{d_w} f^{(i)},$$

*where $f^{(i)}$ is an homogeneous polynomial of degree $i$ in $w$, that is $\deg_w f^{(i)} = i$. If $d_w < |\mathbb{F}|$, then*

$$\langle f \rangle^{T_G} = \langle f^{(0)}, f^{(1)}, \ldots, f^{(d_w)} \rangle^{T_G}.$$

P r o o f. See [6, Proposition 4.2.3]. □

Let $\mathbb{F}[w_1, \ldots, w_n]$ be the free commutative associative algebra, freely generated by $w_1, \ldots, w_n$. It is the usual commutative polynomial algebra in the variables $w_1, \ldots, w_n$.

The next result is a consequence of [6, Proposition 4.2.3].

**Proposition 2.2.** *Let $\mathbb{F}$ be a field such that $|\mathbb{F}| \geq q$. Given $f \in \mathbb{F}[w_1, \ldots, w_n]$, write*

$$f(w_1, \ldots, w_n) = \sum_{d_1=0}^{q-1} \cdots \sum_{d_n=0}^{q-1} \lambda_{(d_1,\ldots,d_n)} w_1^{d_1} \cdots w_n^{d_n},$$

*where $\lambda_{(d_1,\ldots,d_n)} \in \mathbb{F}$. If $f(\alpha_1, \ldots, \alpha_n) = 0$ for all $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, then $\lambda_{(d_1,\ldots,d_n)} = 0$ for every $(d_1, \ldots, d_n)$.*

We recall that a polynomial $f\left(w_{i_1}^{\theta_1}, \ldots, w_{i_n}^{\theta_n}\right) \in \mathcal{L}(W)$ is a $G$-graded polynomial identity for a $G$-graded Lie algebra $A = \bigoplus_{\theta \in G} A_\theta$ if

$$f(u_1, \ldots, u_n) = 0$$

for all $u_j \in A_{\theta_j}$ and $j = 1, \ldots, n$. The set of all $G$-graded polynomial identities of $A$ is a $T_G$-ideal of $\mathcal{L}(W)$, denoted by $Id(A)$.

We will study the $\mathbb{Z}_3$-graded polynomial identities of $UT_3^{(-)}$ when this algebra has the canonical $\mathbb{Z}_3$-grading. For this, let

$$Y = \{y_1, y_2, \ldots\},\ X = \{x_1, x_2, \ldots\} \text{ and } Z = \{z_1, z_2, \ldots\}$$

be infinite countable sets of variables. We denote by $\mathcal{L} = \mathcal{L}(Y \cup X \cup Z)$ the free $\mathbb{Z}_3$-graded Lie algebra, freely generated by $Y \cup X \cup Z$, where

$$\deg_{\mathbb{Z}_3} y_i = 0,\ \deg_{\mathbb{Z}_3} x_i = 1 \text{ and } \deg_{\mathbb{Z}_3} z_i = 2$$

for all $i$. We denote by $Id(UT_3^{(-)})$ the $\mathbb{Z}_3$-graded T-ideal of $\mathcal{L}$ formed by all $\mathbb{Z}_3$-graded polynomial identities of $UT_3^{(-)}$ with the canonical $\mathbb{Z}_3$-grading.

If $f_1, f_2, \ldots, f_n \in \mathcal{L}$, we use the following convention for the commutators:

$$[f_1, f_2] = f_1 f_2 - f_2 f_1 \text{ and } [f_1, \ldots, f_{n-1}, f_n] = [[f_1, \ldots, f_{n-1}], f_n].$$

**3. Some graded polynomial identities for $UT_3^{(-)}$.** In this section, we will obtain some elements in $Id(UT_3^{(-)})$ for an arbitrary field $\mathbb{F}$. We recall that $UT_3^{(-)}$ is endowed with the canonical $\mathbb{Z}_3$-grading.

**Notation 3.1.** Let $I$ be the $T_{\mathbb{Z}_3}$-ideal of $\mathcal{L} = \mathcal{L}(Y \cup X \cup Z)$ generated by the polynomials

$$[y_1, y_2], \quad [x_1, z_2] \text{ and } [z_1, z_2]. \tag{1}$$

Define the equivalence relation $\equiv$ on $\mathcal{L}$ as follows: if $f, g \in \mathcal{L}$ then

$$f \equiv g \text{ if and only if } f + I = g + I.$$

**Lemma 3.2.** $I \subseteq Id(UT_3^{(-)})$.

P r o o f. The proof consists of a straightforward computation □

**Lemma 3.3.** *The following equivalences hold:*

$$[x_1, y_1, y_2] \equiv [x_1, y_2, y_1], \tag{2}$$

$$[z_1, y_1, y_2] \equiv [z_1, y_2, y_1]. \tag{3}$$

P r o o f. By Jacobi identity, we have

$$[x_1, y_1, y_2] = -[y_1, y_2, x_1] + [x_1, y_2, y_1] \equiv [x_1, y_2, y_1]$$

and

$$[z_1, y_1, y_2] = -[y_1, y_2, z_1] + [z_1, y_2, y_1] \equiv [z_1, y_2, y_1]$$

as desired. □

As consequence of the previous lemma, we have the following result:

**Lemma 3.4.** *Let $S_n$ be the symmetric group on $\{1, \ldots, n\}$ and consider $\sigma \in S_n$. Then*

$$[x_1, y_{\sigma(1)}, \ldots, y_{\sigma(n)}] \equiv [x_1, y_1, \ldots, y_n], \tag{4}$$

$$[z_1, y_{\sigma(1)}, \ldots, y_{\sigma(n)}] \equiv [z_1, y_1, \ldots, y_n]. \tag{5}$$

**Notation 3.5.** If $f \in \mathcal{L}$, $y \in Y$ and $k \geq 1$, we denote

$$[f, y^{(k)}] = [f, \underbrace{y, y, \ldots, y}_{k \text{ factors}}] \text{ and } [f, y^{(0)}] = f.$$

Regarding the notations above, the next polynomials will be used to describe the quotient vector space $\mathcal{L}/I$.

**Definition 3.6.** *Define the following polynomials*

(a) $f^i_{(k_1,\ldots,k_n)}(x_i, y_1, \ldots, y_n) = [x_i, y_1^{(k_1)}, \ldots, y_n^{(k_n)}]$,

(b) $g^i_{(k_1,\ldots,k_n)}(z_i, y_1, \ldots, y_n) = [z_i, y_1^{(k_1)}, \ldots, y_n^{(k_n)}]$,

(c) $f^{ij}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}}(x_i, x_j, y_1, \ldots, y_n) = [x_i, y_1^{(k_1)}, \ldots, y_n^{(k_n)}, x_j, y_1^{(l_1)}, \ldots, y_n^{(l_n)}]$,

*where $1 \leq i \leq j$ and $k_t, l_t \geq 0$ for all $1 \leq t \leq n$.*

For example,

$$f^5_{(2,0,1)} = f^5_{(2,0,1)}(x_5, y_1, y_2, y_3) = [x_5, y_1^{(2)}, y_2^{(0)}, y_3^{(1)}] = [x_5, y_1, y_1, y_3].$$

**Proposition 3.7.** *Let $M$ be the subset of $\mathcal{L}$ formed by all polynomials*

$$y_i, \quad f^i_{(k_1,\ldots,k_n)}, \quad g^i_{(k_1,\ldots,k_n)} \quad \text{and} \quad f^{ij}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}},$$

*where $1 \leq i \leq j$, $n \geq 1$ and $k_t, l_t \geq 0$ for all $1 \leq t \leq n$. Then the quotient vector space $\mathcal{L}/I$ is spanned by the set of all elements $h + I$ where $h \in M$.*

P r o o f. Denote $\overline{M} = \operatorname{span}\{h + I \mid h \in M\}$. Let $f$ be a monomial in $\mathcal{L}$, and $<$ the following order on $Y \cup X \cup Z$:

$$x_1 < x_2 < \cdots < z_1 < z_2 < \cdots < y_1 < y_2 < \cdots$$

Denote by $u$ the smallest variable appearing in $f$. By [6, Exercise 4.3.8], $f$ is a linear combination of monomials

$$g = [u, v_1, \ldots, v_m]$$

where $v_1, \ldots, v_m \in Y \cup X \cup Z$. If $m = 0$, then $g + I \in \overline{M}$. If $m \geq 1$ and $g \notin I$, we have four cases to study:

(1) $u \in Z$. In this case, $u = z_i$ for some $i$ and $v_1, \ldots, v_m \in Y$. If $f = f(z_i, y_1, \ldots, y_n)$ and $\deg_{y_j} f = k_j \geq 0$, then Lemma 3.4 gives

$$g + I = g^i_{(k_1,\ldots,k_n)} + I \in \overline{M}.$$

(2) $u \in X$, $\deg_u g = 1$ and $\deg_x g = 0$ for all $x \in X - \{u\}$. In this case, $u = x_i$ for some $i$ and $v_1, \ldots, v_m \in Y$. By Lemma 3.4,

$$g + I = f^i_{(k_1,\ldots,k_n)} + I \in \overline{M}$$

for some $k_1, \ldots, k_n \geq 0$.

(3) $u \in X$, $\deg_u g = 1$ and $\deg_x g = 1$ for some $x \in X - \{u\}$. In this case, $u = x_i$ and $x = x_j$ for some $i < j$, and $v_1, \ldots, v_m \in Y$. By Lemma 3.4,

$$g + I = f^{ij}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}} + I \in \overline{M}$$

for some $k_1, \ldots, k_n, l_1, \ldots, l_n \geq 0$.

(4) $u \in X$ and $\deg_u g = 2$. In this case, $u = x_i = v_t$ for some $i, t$ and $v_1, \ldots, v_{t-1}, v_{t+1}, \ldots, v_m \in Y$. By Lemma 3.4,

$$g + I = f^{ii}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}} + I \in \overline{M}$$

for some $k_1, \ldots, k_n, l_1, \ldots, l_n \geq 0$.

Therefore, $f + I \in \overline{M}$ and the proof is complete. □

**Lemma 3.8.** *The following equivalence holds:*

$$f^{11}_{\binom{k_1,\dots,k_m}{l_1,\dots,l_m}}([x_1,y_{m+1}],y_1,\dots,y_m)\equiv f^{11}_{\binom{k_1,\dots,k_m,1}{l_1,\dots,l_m,1}}-f^{11}_{\binom{k_1,\dots,k_m,2}{l_1,\dots,l_m,0}}.$$

P r o o f. Denote $h=f^{11}_{\binom{k_1,\dots,k_m}{l_1,\dots,l_m}}([x_1,y_{m+1}],y_1,\dots,y_m)$. By Jacobi identity, we have

$$[a,b,[c,d]]=[a,b,c,d]-[a,b,d,c].$$

By this and Lemma 3.4, we obtain

$$\begin{aligned}h=&[[x_1,y_{m+1}],y_1^{(k_1)},\dots,y_m^{(k_m)},[x_1,y_{m+1}],y_1^{(l_1)},\dots,y_m^{(l_m)}]\\=&[x_1,y_{m+1},y_1^{(k_1)},\dots,y_m^{(k_m)},x_1,y_{m+1},y_1^{(l_1)},\dots,y_m^{(l_m)}]+\\&-[x_1,y_{m+1},y_1^{(k_1)},\dots,y_m^{(k_m)},y_{m+1},x_1,y_1^{(l_1)},\dots,y_m^{(l_m)}]\\\equiv&f^{11}_{\binom{k_1,\dots,k_m,1}{l_1,\dots,l_m,1}}-f^{11}_{\binom{k_1,\dots,k_m,2}{l_1,\dots,l_m,0}},\end{aligned}$$

as desired. □

**Lemma 3.9.** *Let* $\mathbb{F}$ *be a field of* $\operatorname{char}\mathbb{F}\neq 2$. *If*

$$\begin{gathered}k_1=\dots=k_m=1,\ k_{m+1}=\dots=k_{m+n}=0,\\ l_1=\dots=l_m=0,\ l_{m+1}=\dots=l_{m+n}=1\end{gathered}$$

*and* $k_1+\dots+k_{m+n}$ *is even, then*

$$f^{11}_{\binom{k_1,\dots,k_{m+n}}{l_1,\dots,l_{m+n}}}(x_1,y_1,\dots,y_{m+n})+I$$

*is a linear combination of elements*

$$f^{11}_{\binom{k'_1,\dots,k'_{m+n}}{l'_1,\dots,l'_{m+n}}}(x_1,y_1,\dots,y_{m+n})+I$$

*such that* $k'_1+\dots+k'_{m+n}<k_1+\dots+k_{m+n}$ *and* $k'_i+l'_i=1$ *for all* $i$.

P r o o f. The proof is by induction on $t$ where $k_1+\dots+k_m=m=2t$. By Jacobi identity,

$$[x_1,y_1,y_1,x_1]=[x_1,y_1,[y_1,x_1]]+[x_1,y_1,x_1,y_1]=[x_1,y_1,x_1,y_1]. \tag{6}$$

Consider the substitution $y_1\mapsto y_1+y_2$ in (6). Then by the linearization process, we obtain

$$[x_1,y_1,y_2,x_1]+[x_1,y_2,y_1,x_1]=[x_1,y_1,x_1,y_2]+[x_1,y_2,x_1,y_1].$$

By Lemma 3.4,

$$[x_1, y_1, y_2, x_1] \equiv (1/2)[x_1, y_1, x_1, y_2] + (1/2)[x_1, y_2, x_1, y_1].$$

Hence

$$f^{11}_{\binom{1,1,0,\ldots,0}{0,0,1,\ldots,1}} \equiv (1/2) f^{11}_{\binom{1,0,0,\ldots,0}{0,1,1,\ldots,1}} + (1/2) f^{11}_{\binom{0,1,0,\ldots,0}{1,0,1,\ldots,1}}$$

and the case $t = 1$ is proved.

Suppose the result is true for $t \geq 1$ and write

$$f^{11}_{\binom{k_1,\ldots,k_{m+n}}{l_1,\ldots,l_{m+n}}} \equiv \sum \alpha_{k',l'} f^{11}_{\binom{k'_1,\ldots,k'_{m+n}}{l'_1,\ldots,l'_{m+n}}},$$

where $\alpha_{k',l'} \in \mathbb{F}, k' = (k'_1, \ldots, k'_{m+n}), l' = (l'_1, \ldots, l'_{m+n}), k'_1 + \cdots + k'_{m+n} < k_1 + \cdots + k_{m+n}$ and $k'_i + l'_i = 1$ for all $i$. We have

$$\left[ f^{11}_{\binom{k_1,\ldots,k_{m+n}}{l_1,\ldots,l_{m+n}}} \right] ([x_1, y_{m+n+1}], y_1, \ldots, y_{m+n})$$

$$\equiv \sum \alpha_{k',l'} \left[ f^{11}_{\binom{k'_1,\ldots,k'_{m+n}}{l'_1,\ldots,l'_{m+n}}} \right] ([x_1, y_{m+n+1}], y_1, \ldots, y_{m+n})$$

By Lemma 3.8,

(7)

$$f^{11}_{\binom{k_1,\ldots,k_{m+n},1}{l_1,\ldots,l_{m+n},1}} - f^{11}_{\binom{k_1,\ldots,k_{m+n},2}{l_1,\ldots,l_{m+n},0}} \equiv \sum \alpha_{k',l'} f^{11}_{\binom{k'_1,\ldots,k'_{m+n},1}{l'_1,\ldots,l'_{m+n},1}} - \sum \alpha_{k',l'} f^{11}_{\binom{k'_1,\ldots,k'_{m+n},2}{l'_1,\ldots,l'_{m+n},0}}.$$

Consider the substitution $y_{m+n+1} \mapsto y_{m+n+1} + y_{m+n+2}$ in (7). Then by the linearization process, we obtain

$$f^{11}_{\binom{k_1,\ldots,k_{m+n},1,0}{l_1,\ldots,l_{m+n},0,1}} + f^{11}_{\binom{k_1,\ldots,k_{m+n},0,1}{l_1,\ldots,l_{m+n},1,0}} - 2 f^{11}_{\binom{k_1,\ldots,k_{m+n},1,1}{l_1,\ldots,l_{m+n},0,0}}$$
$$\equiv \sum \alpha_{k',l'} f^{11}_{\binom{k'_1,\ldots,k'_{m+n},1,0}{l'_1,\ldots,l'_{m+n},0,1}} + \sum \alpha_{k',l'} f^{11}_{\binom{k'_1,\ldots,k'_{m+n},0,1}{l'_1,\ldots,l'_{m+n},1,0}} - \sum 2\alpha_{k',l'} f^{11}_{\binom{k'_1,\ldots,k'_{m+n},1,1}{l'_1,\ldots,l'_{m+n},0,0}}.$$

Therefore

$$f^{11}_{\binom{k_1,\ldots,k_{m+n},1,1}{l_1,\ldots,l_{m+n},0,0}} \equiv (1/2) \left[ f^{11}_{\binom{k_1,\ldots,k_{m+n},1,0}{l_1,\ldots,l_{m+n},0,1}} + f^{11}_{\binom{k_1,\ldots,k_{m+n},0,1}{l_1,\ldots,l_{m+n},1,0}} - \sum \alpha_{k',l'} f^{11}_{\binom{k'_1,\ldots,k'_{m+n},1,0}{l'_1,\ldots,l'_{m+n},0,1}} \right]$$

(8)
$$- (1/2) \sum \alpha_{k',l'} f^{11}_{\binom{k'_1,\ldots,k'_{m+n},0,1}{l'_1,\ldots,l'_{m+n},1,0}} + \sum \alpha_{k',l'} f^{11}_{\binom{k'_1,\ldots,k'_{m+n},1,1}{l'_1,\ldots,l'_{m+n},0,0}}.$$

Now, consider the substitution in (8):

$$y_{m+1} \mapsto y_{m+3},\ y_{m+2} \mapsto y_{m+4}, \dots, y_{m+n} \mapsto y_{m+n+2},$$
$$y_{m+n+1} \mapsto y_{m+1},\ y_{m+n+2} \mapsto y_{m+2}.$$

By applying Lemma 3.4 we finish the proof. □

**Lemma 3.10.** *Let* $\mathbb{F}$ *be a field of* $\operatorname{char}\mathbb{F} \neq 2$. *If* $k_1 + \cdots + k_m$ *is even, then*

$$f^{11}_{\binom{k_1,\dots,k_m}{l_1,\dots,l_m}}(x_1, y_1, \dots, y_m) + I$$

*is a linear combination of elements*

$$f^{11}_{\binom{k'_1,\dots,k'_m}{l'_1,\dots,l'_m}}(x_1, y_1, \dots, y_m) + I$$

*such that* $k'_1 + \cdots + k'_m$ *is odd,* $k'_1 + \cdots + k'_m < k_1 + \cdots + k_m$ *and* $k_i + l_i = k'_i + l'_i$ *for all* $i$.

Proof. Let $g = g(x_1, y_1, \dots, y_t, y_{t+1}, \dots, y_{t+s})$ be the polynomial

$$g = f_{(\overline{k_1},\dots,\overline{k_{t+s}})}^{(\overline{l_1},\dots,\overline{l_{t+s}})},\ \text{where}\ \begin{cases} t = k_1 + \cdots + k_m, \\ s = l_1 + \cdots + l_m, \\ \overline{k_1} = \cdots = \overline{k_t} = 1,\ \overline{k_{t+1}} = \cdots = \overline{k_{t+s}} = 0, \\ \overline{l_1} = \cdots = \overline{l_t} = 0,\ \overline{l_{t+1}} = \cdots = \overline{l_{t+s}} = 1. \end{cases}$$

By Lemma 3.9,

$$f^{11}_{\binom{k_1,\dots,k_m}{l_1,\dots,l_m}} = g(x_1, \underbrace{y_1, \dots, y_1}_{k_1}, \dots, \underbrace{y_m, \dots, y_m}_{k_m}, \underbrace{y_1, \dots, y_1}_{l_1}, \dots, \underbrace{y_m, \dots, y_m}_{l_m})$$

is a linear combination, modulo $I$, of polynomials

$$f^{11}_{\binom{k'_1,\dots,k'_m}{l'_1,\dots,l'_m}}(x_1, y_1, \dots, y_m)$$

such that $k'_1 + \cdots + k'_m < k_1 + \cdots + k_m$ and $k_i + l_i = k'_i + l'_i$ for all $i$. If $k'_1 + \cdots + k'_m$ is even we repeat the same argument. The proof is done. □

**Lemma 3.11.** *Let* $\mathbb{F}$ *be a field of* $\operatorname{char}\mathbb{F} \neq 2$. *Denote by* $H$ *the subset of* $\mathcal{L}$ *formed by all polynomials*

$$y_i,\ f^i_{(k_1,\dots,k_n)},\ g^i_{(k_1,\dots,k_n)}\ \text{and}\ f^{ij}_{\binom{k_1,\dots,k_n}{l_1,\dots,l_n}}$$

*where $n \geq 1$; $i \geq 1$; $j \geq i$; $k_1, \ldots, k_n, l_1, \ldots, l_n \geq 0$; and $k_1 + \cdots + k_n$ is odd if $i = j$. Then the vector space $\mathcal{L}/I$ is spanned by the set of all elements $h + I$ where $h \in H$.*

Proof. This is a direct consequence of Proposition 3.7 and Lemma 3.10. □

**Lemma 3.12.** *Let $a_i, a_{ij}, a'_{ij}, b^{(i)}_{jj} \in \mathbb{F}$, and set $k = (k_1, \ldots, k_n)$ and $l = (l_1, \ldots, l_n)$. The following sentences are true:*

*(a) If $X_1 = e_{12}$ and $Y_i = a_i e_{22}$, then*

$$f^1_{(k)}(X_1, Y_1, \ldots, Y_n) = \left( \prod_{i=1}^{n} a_i^{k_i} \right) e_{12}.$$

*(b) If $Z_1 = e_{13}$ and $Y_i = a_i e_{33}$, then*

$$g^1_{(k)}(Z_1, Y_1, \ldots, Y_n) = \left( \prod_{i=1}^{n} a_i^{k_i} \right) e_{13}.$$

*(c) If $X_1 = a_{12}e_{12} + a_{23}e_{23}$, $X_2 = a'_{12}e_{12} + a'_{23}e_{23}$ and $Y_i = \sum_{j=1}^{3} b^{(i)}_{jj} e_{jj}$, then*

$$\begin{aligned} f^{12}_{\binom{k}{l}}(X_1, X_2, Y_1, \ldots, Y_n) &= \left( a_{12}a'_{23} \prod_{i=1}^{n} (b^{(i)}_{22} - b^{(i)}_{11})^{k_i} - a'_{12}a_{23} \prod_{i=1}^{n} (b^{(i)}_{33} - b^{(i)}_{22})^{k_i} \right) \\ &\quad \times \left( \prod_{i=1}^{n} (b^{(i)}_{33} - b^{(i)}_{11})^{l_i} \right) e_{13} \end{aligned}$$

*and also*

$$\begin{aligned} f^{11}_{\binom{k}{l}}(X_1, Y_1, \ldots, Y_n) &= a_{12}a_{23} \left( \prod_{i=1}^{n} (b^{(i)}_{22} - b^{(i)}_{11})^{k_i} - \prod_{i=1}^{n} (b^{(i)}_{33} - b^{(i)}_{22})^{k_i} \right) \\ &\quad \times \left( \prod_{i=1}^{n} (b^{(i)}_{33} - b^{(i)}_{11})^{l_i} \right) e_{13}. \end{aligned}$$

Proof. The proof consists of a straightforward computation. □

**4. $Id(UT_3^{(-)})$ when $\mathbb{F}$ is infinite.** In this section, we will describe the $T_{\mathbb{Z}_3}$-ideal $Id(UT_3^{(-)})$ when $\mathbb{F}$ is an infinite field. We recall that $UT_3^{(-)}$ is endowed with the canonical $\mathbb{Z}_3$-grading.

**Lemma 4.1.** *Let $\mathbb{F}$ be an infinite field. Then the set of all elements*

$$f^{12}_{\binom{k_1,\dots,k_n}{l_1,\dots,l_n}} + Id(UT_3^{(-)}),$$

*such that $k_1,\dots,k_n,l_1,\dots,l_n \geq 0$, is a linearly independent subset of $\mathcal{L}/Id(UT_3^{(-)})$.*

Proof. Suppose that

$$f = \sum_{s=1}^{m} \alpha_s f^{12}_{\binom{k_{1,s},\dots,k_{n,s}}{l_{1,s},\dots,l_{n,s}}}(x_1, x_2, y_1, \dots, y_n) \in Id(UT_3^{(-)}),\ \alpha_s \in \mathbb{F}.$$

Given $a_{12}, a'_{23}, b^{(i)}_{22}, b^{(i)}_{33} \in \mathbb{F}$, denote $Y_i = b^{(i)}_{22}e_{22} + b^{(i)}_{33}e_{33}$, $X_1 = a_{12}e_{12}$ and $X_2 = a'_{23}e_{23}$. By Lemma 3.12,

$$0 = f(X_1, X_2, Y_1, \dots, Y_n) = \left[\sum_{s=1}^{m} \alpha_s \left( a_{12}a'_{23} \prod_{i=1}^{n} (b^{(i)}_{22})^{k_{i,s}} \prod_{i=1}^{n} (b^{(i)}_{33})^{l_{i,s}} \right)\right] e_{13}.$$

Therefore

$$\sum_{s=1}^{m} \alpha_s \left( a_{12}a'_{23} \prod_{i=1}^{n} (b^{(i)}_{22})^{k_{i,s}} \prod_{i=1}^{n} (b^{(i)}_{33})^{l_{i,s}} \right) = 0$$

for all $a_{12}, a'_{23}, b^{(1)}_{22}, \dots, b^{(n)}_{22}, b^{(1)}_{33}, \dots, b^{(n)}_{33} \in \mathbb{F}$, and Proposition 2.2 gives $\alpha_s = 0$ for every $s$. □

**Lemma 4.2.** *Let $\mathbb{F}$ be an infinite field of* $\operatorname{char}\mathbb{F} \neq 2$. *Then the set of all elements*

$$f^{11}_{\binom{k_1,\dots,k_n}{l_1,\dots,l_n}} + Id(UT_3^{(-)}),$$

*such that $k_1 + \cdots + k_n$ is odd and $k_1,\dots,k_n,l_1,\dots,l_n \geq 0$, is a linearly independent subset of $\mathcal{L}/Id(UT_3^{(-)})$.*

Proof. Suppose that

$$f(x_1, y_1, \dots, y_n) = \sum_{s=1}^{m} \alpha_s f^{11}_{\binom{k_{1,s},\dots,k_{n,s}}{l_{1,s},\dots,l_{n,s}}}(x_1, y_1, \dots, y_n) \in Id(UT_3^{(-)}),$$

where $\alpha_s \in \mathbb{F}$ and $k_{1,s} + \cdots + k_{n,s}$ is odd for all $s$. We shall prove that $\alpha_s = 0$ for all $s$.

Given $b_{22}^{(i)}, b_{33}^{(i)} \in \mathbb{F}$, denote $Y_i = b_{22}^{(i)} e_{22} + b_{33}^{(i)} e_{33}$. By Lemma 3.12, we obtain

$$\begin{aligned} 0 &= f(e_{12} + e_{23}, Y_1, \dots, Y_n) \\ &= \sum_{s=1}^{m} \alpha_s \left( \left( \prod_{i=1}^{n} (b_{22}^{(i)})^{k_{i,s}} - \prod_{i=1}^{n} (b_{33}^{(i)} - b_{22}^{(i)})^{k_{i,s}} \right) \prod_{i=1}^{n} (b_{33}^{(i)})^{l_{i,s}} \right) e_{13}. \end{aligned} \tag{9}$$

Let $\mathcal{W} = \{w_{jj}^{(i)} \mid 1 \le i \le n \text{ and } 2 \le j \le 3\}$ be a set of variables and consider the commutative polynomial algebra $\mathbb{F}[\mathcal{W}]$. For each $1 \le s \le m$ define the polynomial

$$\begin{aligned} &h_s(w_{22}^{(1)}, \dots, w_{22}^{(n)}, w_{33}^{(1)}, \dots, w_{33}^{(n)}) \\ &\qquad = \left( \prod_{i=1}^{n} (w_{22}^{(i)})^{k_{i,s}} - \prod_{i=1}^{n} (w_{33}^{(i)} - w_{22}^{(i)})^{k_{i,s}} \right) \prod_{i=1}^{n} (w_{33}^{(i)})^{l_{i,s}}, \end{aligned}$$

and define

$$h(w_{22}^{(1)}, \dots, w_{33}^{(n)}) = \sum_{s=1}^{m} \alpha_s h_s(w_{22}^{(1)}, \dots, w_{33}^{(n)}).$$

By (9),

$$h(b_{22}^{(1)}, \dots, b_{33}^{(n)}) = \sum_{s=1}^{m} \alpha_s h_s(b_{22}^{(1)}, \dots, b_{33}^{(n)}) = 0 \tag{10}$$

for all $b_{jj}^{(i)} \in \mathbb{F}$.

The monomial

$$\prod_{i=1}^{n} (w_{22}^{(i)})^{k_{i,s}} \prod_{i=1}^{n} (w_{33}^{(i)})^{l_{i,s}}$$

appears in $h_s(w_{22}^{(1)}, \dots, w_{33}^{(n)})$ with coefficient 2, and it appears in $h_{s'}(w_{22}^{(1)}, \dots, w_{33}^{(n)})$ with coefficient 0 if

$$(k_{1,s'}, \dots, k_{n,s'}, l_{1,s'}, \dots, l_{n,s'}) < (k_{1,s}, \dots, k_{n,s}, l_{1,s}, \dots, l_{n,s}),$$

where $<$ is the lexicographic order on $\mathbb{N}^{2n}$. Without loss of generality, we can suppose

$$(k_{1,s'}, \dots, k_{n,s'}, l_{1,s'}, \dots, l_{n,s'}) < (k_{1,s}, \dots, k_{n,s}, l_{1,s}, \dots, l_{n,s})$$

if $1 \leq s' < s \leq m$. Thus,

$$\prod_{i=1}^{n}(w_{22}^{(i)})^{k_{i,m}} \prod_{i=1}^{n}(w_{33}^{(i)})^{l_{i,m}}$$

appears in $h(w_{22}^{(1)}, \ldots, w_{33}^{(n)})$ with coefficient $2\alpha_m$ and, by (10) and Proposition 2.2, we obtain $\alpha_m = 0$. Now we can repeat the same argument with

$$h(w_{22}^{(1)}, \ldots, w_{33}^{(n)}) = \sum_{s=1}^{m-1} \alpha_s h_s(w_{22}^{(1)}, \ldots, w_{33}^{(n)})$$

and, after some steps, we will obtain $\alpha_s = 0$ for all $s$. □

**Theorem 4.3.** *If $\mathbb{F}$ is an infinite field of* $\operatorname{char}\mathbb{F} \neq 2$*, then $Id(UT_3^{(-)}) = I$. Moreover, a basis for the quotient vector space $\mathcal{L}/Id(UT_3^{(-)})$ is given by the set $\overline{H} = \{h + Id(UT_3^{(-)}) \mid h \in H\}$, where $H$ is defined as in Lemma 3.11.*

Proof. By Lemma 3.2 we have $I \subseteq Id(UT_3^{(-)})$.

It follows from Lemma 3.11 that $\mathcal{L}/Id(UT_3^{(-)}) = \operatorname{span}\overline{H}$. We shall prove that $\overline{H}$ is a linearly independent set. Let

$$f = \sum_{h \in H} \alpha_h h \in Id(UT_3^{(-)}),\ \alpha_h \in \mathbb{F}.$$

Since $\mathbb{F}$ is infinite, Proposition 2.1 says that every multihomogeneous component of $f$ belongs to $Id(UT_3^{(-)})$. Therefore, we may assume that $f$ is multihomogeneous and, without loss of generality,

$$f = \alpha y_1, \tag{11}$$

or

$$f = \alpha f^1_{(k_1,\ldots,k_n)}(x_1, y_1, \ldots, y_n), \tag{12}$$

or

$$f = \alpha g^1_{(k_1,\ldots,k_n)}(z_1, y_1, \ldots, y_n), \tag{13}$$

or

$$f = \sum_{k,l} \alpha_{k,l} f^{12}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}}, \tag{14}$$

or

$$f = \sum_{k,l} \alpha_{k,l} f^{11}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}}, \tag{15}$$

where $\alpha, \alpha_{k,l} \in \mathbb{F}$, $k = (k_1, \ldots, k_n)$ and $l = (l_1, \ldots, l_n)$.

Consider the cases (11), (12) and (13). Since $f \in Id(UT_3^{(-)})$, we obtain

$$0 = \alpha e_{11},\ 0 = \alpha f^1_{(k_1,\ldots,k_n)}(e_{12}, e_{22}, \ldots, e_{22}) = \alpha e_{12} \text{ and}$$

$$0 = \alpha g^1_{(k_1,\ldots,k_n)}(e_{13}, e_{33}, \ldots, e_{33}) = \alpha e_{13}.$$

Therefore, $\alpha = 0$.

Now consider the cases (14) and (15). By Lemmas 4.1 and 4.2, we obtain $\alpha_{k,l} = 0$ for all $k, l$.

Therefore, $\overline{H}$ is a basis for the vector space $\mathcal{L}/Id(UT_3^{(-)})$. Moreover, by $I \subseteq Id(UT_3^{(-)})$ and Lemma 3.11, we obtain $I = Id(UT_3^{(-)})$. □

**5. $Id(UT_3^{(-)})$ when $\mathbb{F}$ is finite.** In this section, we will describe the $T_{\mathbb{Z}_3}$-ideal $Id(UT_3^{(-)})$ when $\mathbb{F}$ is a finite field. We recall that $UT_3^{(-)}$ is endowed with the canonical $\mathbb{Z}_3$-grading.

Throughout this section, $\mathbb{F}$ is a finite field with $q$ elements. Since $\mathbb{F} - \{0\}$ is a multiplicative group with $q-1$ elements, we have $\alpha^{q-1} = 1$ for every $\alpha \in \mathbb{F} - \{0\}$, and so

$$\alpha^q = \alpha \text{ for every } \alpha \in \mathbb{F}. \tag{16}$$

Let $\mathbb{F}\langle W\rangle$ be the free associative algebra, freely generated by $W$, over $\mathbb{F}$. If $\cdot$ is the product of $\mathbb{F}\langle W\rangle$, denote by $\mathbb{F}\langle W\rangle^{(-)}$ the vector space $\mathbb{F}\langle W\rangle$ with the bracket product $[\ ,\ ]$ given by

$$[f, g] = f \cdot g - g \cdot f$$

for all $f, g \in \mathbb{F}\langle W\rangle$. We recall that $\mathcal{L}$ is the Lie subalgebra of $\mathbb{F}\langle W\rangle^{(-)}$ generated by $W$. If $g \in \mathbb{F}\langle W\rangle$ and $n \geq 1$, we write

$$g^n = \underbrace{g \cdot g \cdot g \cdot \ldots \cdot g}_{n \text{ factors}}$$

**Lemma 5.1.** *Let $\mathbb{F}$ be a finite field with $q$ elements. If $f, g \in \mathcal{L}$ then*

$$[f, g^{(q)}] = [f, g^q].$$

P r o o f. See [25, Lemma 2.5]. □

**Notation 5.2.** Let $I_q$ be the $T_{\mathbb{Z}_3}$-ideal of $\mathcal{L}$ generated by the polynomials

$$[y_1, y_2], \quad [x_1, z_2], \quad [z_1, z_2], \quad [x_1, y_1^{(q)}] - [x_1, y_1] \quad \text{and} \quad [z_1, y_1^{(q)}] - [z_1, y_1]. \tag{17}$$

**Lemma 5.3.** *Let $\mathbb{F}$ be a finite field with $q$ elements. Then $I \subseteq I_q \subseteq Id(UT_3^{(-)})$.*

P r o o f. If $Y_1 \in (UT_3^{(-)})_0$ and $X_1 \in (UT_3^{(-)})_1$, then Lemma 5.1 and (16) imply

$$[X_1, Y_1^{(q)}] = [X_1, Y_1^q] = [X_1, Y_1].$$

Thus $[x_1, y_1^{(q)}] - [x_1, y_1] \in Id(UT_3^{(-)})$. Analogously, $[z_1, y_1^{(q)}] - [z_1, y_1] \in Id(UT_3^{(-)})$. By these two identities, Notation 3.1 and Lemma 3.2, we obtain $I \subseteq I_q \subseteq Id(UT_3^{(-)})$ as desired. □

**Lemma 5.4.** *Let $\mathbb{F}$ be a finite field with $q$ elements. Denote by $H_q$ the subset of $\mathcal{L}$ formed by all polynomials*

$$y_i, \ f^i_{(k_1,\dots,k_n)}, \ g^i_{(k_1,\dots,k_n)} \ \text{and} \ f^{ij}_{\binom{k_1,\dots,k_n}{l_1,\dots,l_n}}$$

*where $n \geq 1$; $i \geq 1$; $j \geq i$; $0 \leq k_1, \dots, k_n, l_1, \dots, l_n \leq q-1$. Moreover, if $\operatorname{char} \mathbb{F} \neq 2$ and $i = j$, then $k_1 + \cdots + k_n$ is odd. Then the vector space $\mathcal{L}/I_q$ is spanned by the set of all elements $h + I_q$ where $h \in H_q$.*

P r o o f. Since $[x_1, y_1^{(q)}] - [x_1, y_1] \in I_q$ and $[z_1, y_1^{(q)}] - [z_1, y_1] \in I_q$ we have

$$[x_1, y_1^{(q)}] + I_q = [x_1, y_1] + I_q \quad \text{and} \quad [z_1, y_1^{(q)}] + I_q = [z_1, y_1] + I_q.$$

Thus, if $k_t \geq q$ or $l_t \geq q$ then

$$\begin{aligned}
f^i_{(k_1,\dots,k_t,\dots,k_n)} + I_q &= f^i_{(k_1,\dots,k_t-q+1,\dots,k_n)} + I_q, \\
g^i_{(k_1,\dots,k_t,\dots,k_n)} + I_q &= g^i_{(k_1,\dots,k_t-q+1,\dots,k_n)} + I_q, \\
f^{ij}_{\binom{k_1,\dots,k_t,\dots,k_n}{l_1,\dots,l_t,\dots,l_n}} + I_q &= f^{ij}_{\binom{k_1,\dots,k_t-q+1,\dots,k_n}{l_1,\dots,l_t,\dots,l_n}} + I_q, \\
f^{ij}_{\binom{k_1,\dots,k_t,\dots,k_n}{l_1,\dots,l_t,\dots,l_n}} + I_q &= f^{ij}_{\binom{k_1,\dots,k_t,\dots,k_n}{l_1,\dots,l_t-q+1,\dots,l_n}} + I_q.
\end{aligned}$$

Now we can apply these four equalities several times and use Proposition 3.7 and Lemma 3.11. The proof is complete. □

**Lemma 5.5.** *Let $\mathbb{F}$ be a finite field with $q$ elements. Then the set of all elements*

$$f^1_{(k_1,\dots,k_n)} + Id(UT_3^{(-)}),$$

*such that $0 \le k_1, \dots, k_n \le q-1$, is a linearly independent subset of $\mathcal{L}/Id(UT_3^{(-)})$.*

P r o o f. Suppose that

$$f(x_1, y_1, \dots, y_n) = \sum_{j=1}^{m} \alpha_j f^1_{(k_{1,j},\dots,k_{n,j})}(x_1, y_1, \dots, y_n) \in Id(UT_3^{(-)}),$$

where $\alpha_j \in \mathbb{F}$ and $0 \le k_{1,j}, \dots, k_{n,j} \le q-1$ for every $j$. We shall prove that $\alpha_j = 0$ for every $j$. Given $a_1, \dots, a_n \in \mathbb{F}$, denote $Y_i = a_i e_{22}$ and $X_1 = e_{12}$. By Lemma 3.12,

$$0 = f(X_1, Y_1, \dots, Y_n) = \left[\sum_{j=1}^{m} \alpha_j \left(\prod_{i=1}^{n} a_i^{k_{i,j}}\right)\right] e_{12}.$$

Therefore

$$\sum_{j=1}^{m} \alpha_j \left(\prod_{i=1}^{n} a_i^{k_{i,j}}\right) = 0$$

for all $a_1, \dots, a_n \in \mathbb{F}$, and Proposition 2.2 gives $\alpha_j = 0$ for every $j$. □

**Lemma 5.6.** *Let $\mathbb{F}$ be a finite field with $q$ elements. Then the set of all elements*

$$g^1_{(k_1,\dots,k_n)} + Id(UT_3^{(-)}),$$

*such that $0 \le k_1, \dots, k_n \le q-1$, is a linearly independent subset of $\mathcal{L}/Id(UT_3^{(-)})$.*

P r o o f. Suppose that

$$g(z_1, y_1, \dots, y_n) = \sum_{j=1}^{m} \alpha_j g^1_{(k_{1,j},\dots,k_{n,j})}(z_1, y_1, \dots, y_n) \in Id(UT_3^{(-)}),$$

where $\alpha_j \in \mathbb{F}$ and $0 \le k_{1,j}, \dots, k_{n,j} \le q-1$ for every $j$. We shall prove that $\alpha_j = 0$ for every $j$. Given $a_1, \dots, a_n \in \mathbb{F}$, denote $Y_i = a_i e_{33}$ and $Z_1 = e_{13}$. By Lemma 3.12,

$$0 = g(Z_1, Y_1, \dots, Y_n) = \left[\sum_{j=1}^{m} \alpha_j \left(\prod_{i=1}^{n} a_i^{k_{i,j}}\right)\right] e_{13}.$$

Therefore

$$\sum_{j=1}^{m} \alpha_j \left( \prod_{i=1}^{n} a_i^{k_{i,j}} \right) = 0$$

for all $a_1, \ldots, a_n \in \mathbb{F}$, and Proposition 2.2 gives $\alpha_j = 0$ for every $j$. □

**Lemma 5.7.** *Let $\mathbb{F}$ be a finite field with $q$ elements. Then the set of all elements*

$$f^{12}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}} + Id(UT_3^{(-)}),$$

*such that $0 \leq k_1, \ldots, k_n, l_1, \ldots, l_n \leq q-1$, is a linearly independent subset of $\mathcal{L}/Id(UT_3^{(-)})$.*

Proof. The proof is similar to the proof of Lemma 4.1. □

**Lemma 5.8.** *Let $\mathbb{F}$ be a finite field with $q$ elements and* $\operatorname{char}\mathbb{F} \neq 2$. *Then the set of all elements*

$$f^{11}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}} + Id(UT_3^{(-)}),$$

*such that $k_1 + \cdots + k_n$ is odd and $0 \leq k_1, \ldots, k_n, l_1, \ldots, \ l_n \leq q-1$, is a linearly independent subset of $\mathcal{L}/Id(UT_3^{(-)})$.*

Proof. Suppose that

$$f(x_1, y_1, \ldots, y_n) = \sum_{s=1}^{m} \alpha_s f^{11}_{\binom{k_{1,s},\ldots,k_{n,s}}{l_{1,s},\ldots,l_{n,s}}}(x_1, y_1, \ldots, y_n) \in Id(UT_3^{(-)}),$$

where $\alpha_s \in \mathbb{F}$ and $k_{1,s} + \cdots + k_{n,s}$ is odd for all $s$. We shall prove that $\alpha_s = 0$ for all $s$.

Without loss of generality, we can suppose

$$(k_{1,s'}, \ldots, k_{n,s'}, l_{1,s'}, \ldots, l_{n,s'}) < (k_{1,s}, \ldots, k_{n,s}, l_{1,s}, \ldots, l_{n,s})$$

if $1 \leq s' < s \leq m$, where $<$ is the lexicographic order on $\mathbb{N}^{2n}$.

Given $b_{22}^{(i)}, b_{33}^{(i)} \in \mathbb{F}$, denote $Y_i = b_{22}^{(i)} e_{22} + b_{33}^{(i)} e_{33}$. By Lemma 3.12, we obtain

$$\begin{aligned} 0 &= f(e_{12} + e_{23}, Y_1, \ldots, Y_n) \\ &= \sum_{s=1}^{m} \alpha_s \left( \left( \prod_{i=1}^{n} (b_{22}^{(i)})^{k_{i,s}} - \prod_{i=1}^{n} (b_{33}^{(i)} - b_{22}^{(i)})^{k_{i,s}} \right) \prod_{i=1}^{n} (b_{33}^{(i)})^{l_{i,s}} \right) e_{13} \end{aligned}$$

and so $w = 0$, where

$$w = \sum_{s=1}^{m} \alpha_s \left( \left( \prod_{i=1}^{n} (b_{22}^{(i)})^{k_{i,s}} - \prod_{i=1}^{n} (b_{33}^{(i)} - b_{22}^{(i)})^{k_{i,s}} \right) \prod_{i=1}^{n} (b_{33}^{(i)})^{l_{i,s}} \right). \tag{18}$$

Since $(b_{33}^{(i)})^q = b_{33}^{(i)}$ we can write (18) as

$$w = \sum_{d,e} \beta_{d,e} (b_{22}^{(1)})^{d_1} \cdots (b_{22}^{(n)})^{d_n} (b_{33}^{(1)})^{e_1} \cdots (b_{33}^{(n)})^{e_n}, \tag{19}$$

where $\beta_{d,e} \in \mathbb{F}$, $d = (d_1, \ldots, d_n)$, $e = (e_1, \ldots, e_n)$ and $0 \leq d_1, \ldots, d_n, e_1, \ldots, e_n \leq q-1$. If $d = (k_{1,m}, \ldots, k_{n,m})$ and $e = (l_{1,m}, \ldots, l_{n,m})$, then $\beta_{d,e} = 2\alpha_m$. Since $w = 0$ for all $b_{22}^{(1)}, \ldots, b_{22}^{(n)}, b_{33}^{(1)}, \ldots, b_{33}^{(n)} \in \mathbb{F}$, Proposition 2.2 gives $2\alpha_m = 0$ and so $\alpha_m = 0$. In particular,

$$f(x_1, y_1, \ldots, y_n) = \sum_{s=1}^{m-1} \alpha_s f^{11}_{\binom{k_{1,s},\ldots,k_{n,s}}{l_{1,s},\ldots,l_{n,s}}}(x_1, y_1, \ldots, y_n) \in Id(UT_3^{(-)})$$

and we can use the same argument to prove that $\alpha_{m-1} = 0, \ \ldots, \ \alpha_1 = 0$. □

**Theorem 5.9.** *If $\mathbb{F}$ is a finite field with $q$ elements and* $\operatorname{char} \mathbb{F} \neq 2$*, then $Id(UT_3^{(-)}) = I_q$. Moreover, a basis for the quotient vector space $\mathcal{L}/Id(UT_3^{(-)})$ is given by the set $\overline{H_q} = \{h + Id(UT_3^{(-)}) \mid h \in H_q\}$, where $H_q$ is defined as in Lemma 5.4.*

Proof. By Lemma 5.3 we have $I_q \subseteq Id(UT_3^{(-)})$.

It follows from Lemma 5.4 that $\mathcal{L}/Id(UT_3^{(-)}) = \operatorname{span} \overline{H_q}$. We shall prove that $\overline{H_q}$ is a linearly independent set. Let

$$f = \sum_{h \in H_q} \alpha_h h \in Id(UT_3^{(-)}), \ \alpha_h \in \mathbb{F}.$$

Since $\deg_{x_i} f \leq 2 < q$ and $\deg_{z_i} f \leq 1 < q$ for every $i$, by Proposition 2.1 we can suppose, without loss of generality,

$$f = \alpha y_1, \tag{20}$$

or

$$f = \sum_{k} \alpha_k f^1_{(k_1,\ldots,k_n)}(x_1, y_1, \ldots, y_n), \tag{21}$$

or

$$f = \sum_k \alpha_k g^1_{(k_1,\ldots,k_n)}(z_1, y_1, \ldots, y_n), \tag{22}$$

or

$$f = \sum_{k,l} \alpha_{k,l} f^{12}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}}, \tag{23}$$

or

$$f = \sum_{k,l} \alpha_{k,l} f^{11}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}}, \tag{24}$$

where $\alpha, \alpha_{k,l} \in \mathbb{F}$, $k = (k_1, \ldots, k_n)$ and $l = (l_1, \ldots, l_n)$.

Consider the cases (20), (21) and (22). Since $\deg_{y_i} f \leq q-1$ for every $i$, by Proposition 2.1 we have

$$\alpha_k f^1_{(k_1,\ldots,k_n)}(x_1, y_1, \ldots, y_n),\ \alpha_k g^1_{(k_1,\ldots,k_n)}(z_1, y_1, \ldots, y_n) \in Id(UT_3^{(-)}).$$

Thus

$$0 = \alpha e_{11},\ 0 = \alpha_k f^1_{(k_1,\ldots,k_n)}(e_{12}, e_{22}, \ldots, e_{22}) = \alpha_k e_{12} \text{ and}$$

$$0 = \alpha_k g^1_{(k_1,\ldots,k_n)}(e_{13}, e_{33}, \ldots, e_{33}) = \alpha_k e_{13}$$

imply $\alpha = \alpha_k = 0$.

Now consider the cases (23) and (24). By Lemmas 5.7 and 5.8, we obtain $\alpha_{k,l} = 0$ for all $k, l$.

Therefore, $\overline{H_q}$ is a basis for the vector space $\mathcal{L}/Id(UT_3^{(-)})$. Moreover, by $I_q \subseteq Id(UT_3^{(-)})$ and Lemma 5.4, we obtain $I_q = Id(UT_3^{(-)})$. □

**6. Specht Property for $Id(UT_3^{(-)})$.** From now on, $\mathbb{F}$ is a finite field with $q$ elements. In this section, we will prove that $I_q = Id(UT_3^{(-)})$ satisfies the Specht property when $\operatorname{char}\mathbb{F} \neq 2$, that is, every $T_{\mathbb{Z}_3}$-ideal containing $I_q$ is finitely generated as a $T_{\mathbb{Z}_3}$-ideal.

One of the most important tools in this part will be the theory of partially well ordered sets. Recall that a partially ordered set $(P, \leq)$ is called a partially well-ordered set if given an arbitrary nonempty subset $Q \subseteq P$, there exist $q_1, \ldots, q_n \in Q$ with the following property: if $x \in Q$, then $q_i \leq x$ for some $1 \leq i \leq n$.

Partially well-ordered sets were first studied systematically from an algebraic point of view by Higman [13]. Later on the principal results from [13] have been rediscovered several times by different authors.

Now, we state some know results about partially well-ordered sets. For more details about the following results, we refer the reader to [2, Section 5.2].

**Proposition 6.1.** *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be partially well-ordered sets. If $P = P_1 \times P_2$, define the relation $\leq$ on $P$ as follows: if $p_1, p_1' \in P_1$ and $p_2, p_2' \in P_2$, then*

$$(p_1, p_2) \leq (p_1', p_2') \quad \text{if and only if} \quad p_1 \leq_1 p_1' \quad \text{and} \quad p_2 \leq_2 p_2'.$$

*Then $(P, \leq)$ is a partially well-ordered set.*

P r o o f. See [2, Corollary 5.11] □

Consider the usual order $\leq$ on the non-negative integers $\mathbb{N}_0$ and let $m \geq 1$. By Proposition 6.1,

$$\mathbb{N}_0^m = \underbrace{\mathbb{N}_0 \times \cdots \times \mathbb{N}_0}_{m \text{ factors}}$$

is a partially well-ordered set with the order $\leq_m$ given by

$$(n_1, \ldots, n_m) \leq_m (n_1', \ldots, n_m') \text{ if and only if } n_i \leq n_i'$$

for every $i \in \{1, \ldots, m\}$.

**Proposition 6.2.** *Let $D(\mathbb{N}_0^m)$ be the set of all finite sequences $(a_1, \ldots, a_r)$ such that $a_1, \ldots, a_r \in \mathbb{N}_0^m$ and $r \geq 1$. Define the following order $\preceq$ on $D(\mathbb{N}^m)$: $(a_1, \ldots, a_r) \preceq (b_1, \ldots, b_s)$ if and only if there exists an injection $\psi : \mathbb{N} \to \mathbb{N}$ such that:*

(*i*) *$\psi$ preserves the order, that is, if $u \leq v$ then $\psi(u) \leq \psi(v)$;*

(*ii*) *$\psi(r) \leq s$;*

(*iii*) *$a_i \leq_m b_{\psi(i)}$ for every $i \in \{1, \ldots, r\}$.*

*Then $(D(\mathbb{N}_0^m), \preceq)$ is a partially well-ordered set.*

P r o o f. It is a consequence of [2, Theorem 5.12]. □

For each commutator $f = f(x_1, y_1, \ldots, y_n) = [x_1, y_1^{(k_1)}, \ldots, y_n^{(k_n)}]$, we associate the following finite sequence of non-negative integers

$$V_f = (k_1, \ldots, k_n). \tag{25}$$

**Lemma 6.3.** *Let $\mathbb{F}$ be a finite field with $q$ elements. Let $(D(\mathbb{N}_0), \preceq)$ be the partially well-ordered set from Proposition 6.2. Consider the following commutators*

$$f(x_1, y_1, \dots, y_n) = [x_1, y_1^{(k_1)}, \dots, y_n^{(k_n)}],$$

$$g(x_1, y_1, \dots, y_m) = [x_1, y_1^{(k'_1)}, \dots, y_m^{(k'_m)}],$$

*where $0 \leq k_1, \dots, k_n, k'_1, \dots, k'_m \leq q-1$. If $V_f \preceq V_g$, then $g \in \langle f \rangle^{T_{\mathbb{Z}_3}} + I_q$.*

Proof. Since $V_f \preceq V_g$, there exists a subsequence $(k'_{i_1}, \dots, k'_{i_n})$ of $V_g$ such that $k_j \leq k'_{i_j}$ for all $j \in \{1, \dots, n\}$. By Lemma 3.4, we have

$$\begin{aligned} g + I_q &= [x_1, y_{i_1}^{(k_1)}, \dots, y_{i_n}^{(k_n)}, y_{i_1}^{(k'_1-k_1)}, \dots, y_{i_n}^{(k'_n-k_n)}, y_{j_1}^{(k'_{j_1})}, \dots, y_{j_{m-n}}^{(k'_{j_{m-n}})}] + I_q \\ &= [f(x_1, y_{i_1}, \dots, y_{i_n}), y_{i_1}^{(k'_1-k_1)}, \dots, y_{i_n}^{(k'_n-k_n)}, y_{j_1}^{(k'_{j_1})}, \dots, y_{j_{m-n}}^{(k'_{j_{m-n}})}] + I_q, \end{aligned}$$

where $\{j_1, \dots, j_{m-n}\} = \{1, \dots, m\} \setminus \{i_1, \dots, i_n\}$. Therefore $g \in \langle f \rangle^{T_{\mathbb{Z}_3}} + I_q$. □

**Lemma 6.4.** *Let $\mathbb{F}$ be a finite field with $q$ elements. If $A$ is a subset of the set*

$$\{f^1_{(k_1,\dots,k_n)} \mid k_1, \dots, k_n, n \geq 0\}$$

*then there exist $f_1, \dots, f_t \in A$ such that*

$$\langle A \rangle^{T_{\mathbb{Z}_3}} + I_q = \langle f_1, \dots, f_t \rangle^{T_{\mathbb{Z}_3}} + I_q.$$

Proof. Let $(D(\mathbb{N}_0), \preceq)$ be the partially well-ordered set from Proposition 6.2, and set $Q = \{V_f \mid f \in A\}$. Since $Q \subseteq D(\mathbb{N}_0)$, there exist $q_1, \dots, q_t \in Q$ with the following property: if $x \in Q$, then $q_i \leq x$ for some $1 \leq i \leq t$. Let $f_i \in A$ such that $V_{f_i} = q_i$. By Lemma 6.3, we have

$$\langle A \rangle^{T_{\mathbb{Z}_3}} + I_q = \langle f_1, \dots, f_t \rangle^{T_{\mathbb{Z}_3}} + I_q$$

as desired. □

**Lemma 6.5.** *Let $\mathbb{F}$ be a finite field with $q$ elements. If $B$ is a subset of the set*

$$\{g^1_{(k_1,\dots,k_n)} \mid k_1, \dots, k_n, n \geq 0\}$$

*then there exist $g_1, \dots, g_t \in B$ such that*

$$\langle B \rangle^{T_{\mathbb{Z}_3}} + I_q = \langle g_1, \dots, g_t \rangle^{T_{\mathbb{Z}_3}} + I_q.$$

P r o o f. It is similar to the proof of Lemma 6.4. □

For each $f = [x_1, y_1^{(k_1)}, \dots, y_n^{(k_n)}, x_2, y_1^{(l_1)}, \dots, y_n^{(l_n)}]$, we associate the following finite sequence of elements in $\mathbb{N}_0^2$:

$$S_f = ((k_1, l_1), \dots, (k_n, l_n)). \tag{26}$$

**Proposition 6.6.** *Let $\mathbb{F}$ be a finite field with $q$ elements. Let $(D(\mathbb{N}_0^2), \preceq)$ be the partially well-ordered set from Proposition 6.2. Consider the following commutators*

$$f(x_1, x_2, y_1, \dots, y_n) = [x_1, y_1^{(k_1)}, \dots, y_n^{(k_n)}, x_2, y_1^{(l_1)}, \dots, y_n^{(l_n)}],$$

$$g(x_1, x_2, y_1, \dots, y_m) = [x_1, y_1^{(k'_1)}, \dots, y_m^{(k'_m)}, x_2, y_1^{(l'_1)}, \dots, y_m^{(l'_m)}],$$

*where $0 \le k_1, \dots, k_n, l_1, \dots, l_n, k'_1, \dots, k'_m, l'_1, \dots, l'_m \le q-1$. If $S_f \preceq S_g$, then $g \in \langle f \rangle^{T_{\mathbb{Z}_3}} + I_q$.*

P r o o f. Since $S_f \preceq S_g$, there exists a subsequence $((k'_{i_1}, l'_{i_1}), \dots, (k'_{i_n}, l'_{i_n}))$ of $S_g$ such that $(k_j, l_j) \le_2 (k'_{i_j}, l'_{i_j})$ for every $j \in \{1, \dots, n\}$. Therefore $k_j \le k'_{i_j}$ and $l_j \le l'_{i_j}$ for every $j \in \{1, \dots, n\}$. Then

$$\tilde{f} = f(x_1, x_2, y_{i_1}, \dots, y_{i_n}) = [x_1, y_{i_1}^{(k_1)}, \dots, y_{i_n}^{(k_n)}, x_2, y_{i_1}^{(l_1)}, \dots, y_{i_n}^{(l_n)}] \in \langle f \rangle^{T_{\mathbb{Z}_3}}.$$

By making the substitution $x_1 \mapsto [x_1, y_{i_1}^{(k'_{i_1}-k_1)}, \dots, y_{i_n}^{(k'_{i_n}-k_n)}, y_{j_1}^{(k'_{j_1})}, \dots, y_{j_{m-n}}^{(k'_{j_{m-n}})}]$ on $\tilde{f}$, where $\{j_1, \dots, j_{m-n}\} = \{1, \dots, m\} \setminus \{i_1, \dots, i_n\}$, we obtain a new polynomial $\overline{f} \in \langle f \rangle^{T_{\mathbb{Z}_3}}$. By Lemma 3.4,

$$[\overline{f}, y_{i_1}^{(l'_{i_1}-l_1)}, \dots, y_{i_n}^{(l'_{i_n}-l_n)}, y_{j_1}^{(l'_{j_1})}, \dots, y_{j_{m-n}}^{(l'_{j_{m-n}})}] + I_q = g + I_q.$$

Therefore $g \in \langle f \rangle^{T_{\mathbb{Z}_3}} + I_q$. □

**Definition 6.7.** *Consider the following commutators*

$$f(x_1, x_2, y_1, \dots, y_n) = [x_1, y_1^{(k_1)}, \dots, y_n^{(k_n)}, x_2, y_1^{(l_1)}, \dots, y_n^{(l_n)}],$$

$$g(x_1, x_2, y_1, \dots, y_n) = [x_1, y_1^{(k'_1)}, \dots, y_n^{(k'_n)}, x_2, y_1^{(l'_1)}, \dots, y_n^{(l'_n)}],$$

*where $0 \le k_1, \dots, k_n, l_1, \dots, l_n, k'_1, \dots, k'_n, l'_1, \dots, l'_n \le q-1$. We write $f <_{dl} g$ if*

(a) *either* $\sum_{i=1}^{n} k_i + l_i < \sum_{i=1}^{n} k'_i + l'_i$

(*b*) *or* $\sum_{i=1}^{n} k_i + l_i = \sum_{i=1}^{n} k'_i + l'_i$ *and* $(k_1, \ldots, k_n, l_1, \ldots, l_n) <_l (k'_1, \ldots, k'_n, l'_1, \ldots, l'_n)$.

*Here,* $<_l$ *is the lexicographic order on* $\mathbb{N}_0^{2n}$.

**Remark 6.8.** $<_{dl}$ is a linear order on the set

$$\{f^{12}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}}(x_1, x_2, y_1, \ldots, y_n) \mid 0 \leq k_1, \ldots, k_n, l_1, \ldots, l_n \leq q-1\},$$

when $n$ is fixed.

**Definition 6.9.** *Given a polynomial*

$$f(x_1, x_2, y_1, \ldots, y_n) = \sum_{i=1}^{m} \alpha_i f^{12}_{\binom{k_{1,i},\ldots,k_{n,i}}{l_{1,i},\ldots,l_{n,i}}}(x_1, x_2, y_1, \ldots, y_n), \ \alpha_i \in \mathbb{F},$$

*we denote by* $\mu(f)$ *its leading monomial with respect to* $<_{df}$*:*

$$\mu(f) := \max_{<_{df}} \{f^{12}_{\binom{k_{1,i},\ldots,k_{n,i}}{l_{1,i},\ldots,l_{n,i}}}(x_1, x_2, y_1, \ldots, y_n) \mid 1 \leq i \leq m \text{ and } \alpha_i \neq 0\}.$$

**Lemma 6.10.** *If* $f^{12}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}} <_{dl} f^{12}_{\binom{k'_1,\ldots,k'_n}{l'_1,\ldots,l'_n}}$, *then we obtain the following sentences:*

(*a*) $f^{12}_{\binom{k_1,\ldots,k_n,0}{l_1,\ldots,l_n,1}} <_{dl} f^{12}_{\binom{k'_1,\ldots,k'_n,0}{l'_1,\ldots,l'_n,1}}$,

(*b*) $f^{12}_{\binom{k_1,\ldots,k_i,\ldots,k_n}{l_1,\ldots,l_i+1,\ldots,l_n}} <_{dl} f^{12}_{\binom{k'_1,\ldots,k'_i,\ldots,k'_n}{l'_1,\ldots,l'_i+1,\ldots,l'_n}}$,

(*c*) $f^{12}_{\binom{k_1,\ldots,k_n,1}{l_1,\ldots,l_n,0}} <_{dl} f^{12}_{\binom{k'_1,\ldots,k'_n,1}{l'_1,\ldots,l'_n,0}}$,

(*d*) $f^{12}_{\binom{k_1,\ldots,k_i+1,\ldots,k_n}{l_1,\ldots,l_i,\ldots,l_n}} <_{dl} f^{12}_{\binom{k'_1,\ldots,k'_i+1,\ldots,k'_n}{l'_1,\ldots,l'_i,\ldots,l'_n}}$.

P r o o f. The proof is straightforward. □

**Lemma 6.11.** *Let $\mathbb{F}$ be a finite field with $q$ elements. Let $(D(\mathbb{N}_0^2), \preceq)$ be the partially well-ordered set from Proposition* 6.2. *Consider the following polynomials*

$$f(x_1, x_2, y_1, \dots, y_n) = \sum_{i=1}^{t} \alpha_i f^{12}_{\binom{k_{1,i},\dots,k_{n,i}}{l_{1,i},\dots,l_{n,i}}}(x_1, x_2, y_1, \dots, y_n),\ \alpha_i \in \mathbb{F},$$

$$g(x_1, x_2, y_1, \dots, y_m) = \sum_{j=1}^{s} \beta_j f^{12}_{\binom{k'_{1,j},\dots,k'_{m,j}}{l'_{1,j},\dots,l'_{m,j}}}(x_1, x_2, y_1, \dots, y_m),\ \beta_j \in \mathbb{F}.$$

*If $S_{\mu(f)} \preceq S_{\mu(g)}$, then there exists a polynomial $h \in \langle f \rangle^{T_{\mathbb{Z}_3}} + I_q$ such that*

$$h = h(x_1, x_2, y_1, \dots, y_m) = \sum_{j=1}^{r} \gamma_j f^{12}_{\binom{k''_{1,j},\dots,k''_{m,j}}{l''_{1,j},\dots,l''_{m,j}}}(x_1, x_2, y_1, \dots, y_m),\ \gamma_j \in \mathbb{F},$$

*and $\mu(h) = \mu(g)$.*

P r o o f. We will use similar arguments as in the proof of Proposition 6.6. Denote

$$\mu(f) = [x_1, y_1^{(k_1)}, \dots, y_n^{(k_n)}, x_2, y_1^{(l_1)}, \dots, y_n^{(l_n)}],$$
$$\mu(g) = [x_1, y_1^{(k'_1)}, \dots, y_m^{(k'_m)}, x_2, y_1^{(l'_1)}, \dots, y_m^{(l'_m)}].$$

Since $S_{\mu(f)} \preceq S_{\mu(g)}$ there exists a subsequence $((k'_{i_1}, l'_{i_1}), \dots, (k'_{i_n}, l'_{i_n}))$ of $S_{\mu(g)}$ such that $k_j \le k'_{i_j}$ and $l_j \le l'_{i_j}$ for every $j \in \{1, \dots, n\}$. Then

$$\tilde{f} = f(x_1, x_2, y_{i_1}, \dots, y_{i_n}) \in \langle f \rangle^{T_{\mathbb{Z}_3}}.$$

Note that

$$\mu(\tilde{f}) = \mu(f)(x_1, x_2, y_{i_1}, \dots, y_{i_n}).$$

By making the substitution $x_1 \mapsto [x_1, y_{i_1}^{(k'_{i_1}-k_1)}, \dots, y_{i_n}^{(k'_{i_n}-k_n)}, y_{j_1}^{(k'_{j_1})}, \dots, y_{j_{m-n}}^{(k'_{j_{m-n}})}]$ on $\tilde{f}$, where $\{j_1, \dots, j_{m-n}\} = \{1, \dots, m\} \setminus \{i_1, \dots, i_n\}$, we obtain a new polynomial $\overline{f} \in \langle f \rangle^{T_{\mathbb{Z}_3}}$. Moreover,

$$\widehat{f} = [\overline{f}, y_{i_1}^{(l'_{i_1}-l_1)}, \dots, y_{i_n}^{(l'_{i_n}-l_n)}, y_{j_1}^{(l'_{j_1})}, \dots, y_{j_{m-n}}^{(l'_{j_{m-n}})}] \in \langle f \rangle^{T_{\mathbb{Z}_3}}.$$

By Lemma 3.4, there exists

$$h(x_1, x_2, y_1, \dots, y_m) = \sum_{i=1}^{r} \gamma_i f^{12}_{\binom{k''_{1,i},\dots,k''_{m,i}}{l''_{1,i},\dots,l''_{m,i}}}(x_1, x_2, y_1, \dots, y_m)$$

such that

$$\widehat{f} + I_q = h + I_q.$$

By Lemma 6.10, $\mu(h) = \mu(g)$. □

**Proposition 6.12.** *Let $\mathbb{F}$ be a finite field with $q$ elements. If $J$ is a $T_{\mathbb{Z}_3}$-ideal of $\mathcal{L}$ such that $I_q \subseteq J$ and*

$$A^{12} = span\{f^{12}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}} \mid 0 \leq k_1, \ldots, k_n, l_1, \ldots, l_n \leq q-1,\ n \geq 0\} \cap J,$$

*then there exists a finite subset $\widetilde{A}_{12}$ of $A^{12}$ such that $A^{12} \subseteq \langle \widetilde{A}_{12} \rangle^{T_{\mathbb{Z}_3}} + I_q$.*

P r o o f. Consider the sets

$$M^{12} = \{\mu(f) |\ f \in A^{12}\},$$

$$C^{12} = \{S_{\mu(f)} | f \in A^{12}\}.$$

Since $C^{12} \subseteq D(\mathbb{N}_0^2)$, by Proposition 6.2 there exist $f_1, \ldots, f_t \in A^{12}$ such that $S_{\mu(f_1)}, \ldots, S_{\mu(f_t)}$ are minimal elements of $C^{12}$. By Proposition 6.6, we obtain

$$M^{12} \subseteq \langle \mu(f_1), \ldots, \mu(f_t) \rangle^{T_{\mathbb{Z}_3}} + I_q.$$

Now, define the set

$$\widetilde{A}_{12} = \{f_1, \ldots, f_t\}.$$

We shall prove that

$$A^{12} \subseteq \left\langle \widetilde{A}_{12} \right\rangle^{T_{\mathbb{Z}_3}} + I_q.$$

Suppose that there exists $f = f(x_1, x_2, y_1, \ldots, y_n) \in A^{12}$ such that $f \notin \left\langle \widetilde{A}_{12} \right\rangle^{T_{\mathbb{Z}_3}} + I_q$. Define the sets

$$A^{(n)} = \{g \in A^{12} \mid g = g(x_1, x_2, y_1, \ldots, y_n) \notin \left\langle \widetilde{A}_{12} \right\rangle^{T_{\mathbb{Z}_3}} + I_q\},$$

$$M^{(n)} = \{\mu(g) \mid g \in A^{(n)}\}.$$

Note that $M^{(n)}$ is a finite set and $f \in A^{(n)}$. Without loss of generality, we can assume

$$\mu(f) = \min_{<_{dl}} M^{(n)}.$$

Since $\mu(f) \in M^{12}$, there exists $i \in \{1, \dots, t\}$ such that $S_{\mu(f_i)} \preceq S_{\mu(f)}$. By Lemma 6.11, there exists

$$h \in \langle f_i \rangle^{T_{\mathbb{Z}_3}} + I_q \subseteq \left\langle \widetilde{A}_{12} \right\rangle^{T_{\mathbb{Z}_3}} + I_q$$

such that $\mu(h) = \mu(f)$. Without loss of generality, we can suppose the coefficients of $\mu(h)$ and $\mu(f)$ in $h$ and $f$, respectively, equal to 1. If $\widehat{f} = h - f \neq 0$, then $\widehat{f} \in A^{(n)}$ and $\mu(\widehat{f}) <_{dl} \mu(f)$, contradiction. Therefore, $f = h$ and we have again a contradiction. We proved that

$$A^{12} \subseteq \left\langle \widetilde{A}_{12} \right\rangle^{T_{\mathbb{Z}_3}} + I_q$$

as desired. □

**Theorem 6.13.** *Let $\mathbb{F}$ be a finite field with* $\operatorname{char} \mathbb{F} \neq 2$. *If $J$ is a $T_{\mathbb{Z}_3}$-ideal of $\mathcal{L}$ such that $Id(UT_3^{(-)}) \subseteq J$, then $J$ is finitely generated as a $T_{\mathbb{Z}_3}$-ideal.*

Proof. Let $\mathbb{F}$ be a finite field with $q$ elements and let $J$ be a $T_{\mathbb{Z}_3}$-ideal of $\mathcal{L}$ such that $Id(UT_3^{(-)}) = I_q \subseteq J$. By Theorem 5.9, we have

$$J = \langle S \rangle^{T_{\mathbb{Z}_3}} + I_q,$$

where $S \subseteq \operatorname{span} H_q$.

Given $f = f(x_1, \dots, x_r, z_1, \dots, z_s, y_1, \dots, y_n) \in S$, denote by $W$ the set

$$W = \{x_1, \dots, x_r, z_1, \dots, z_s, y_1, \dots, y_n\}.$$

Write

$$f = \sum_{P \subseteq W} f_P$$

where $f_P$ is the sum of all monomials $m$ from $f$ such that:

$$\deg_w m \geq 1 \text{ for every } w \in P, \text{ and } \deg_w m = 0 \text{ for every } w \notin P.$$

Given $w \in W$, since

$$f(0, \dots, 0, w, 0, \dots, 0) = f_{\{w\}}$$

it follows that every $f_{\{w\}} \in J$. In particular,

$$f' = \sum_{P \subseteq W,\ |P| \geq 2} f_P \in J.$$

Given $v, w \in W$, since

$$f'(0, \ldots, 0, v, 0, \ldots, 0, w, 0, \ldots, 0) = f_{\{v,w\}},$$

it follows that every $f_{\{v,w\}}$ lies in $J$. By induction on $|P|$ we can prove that every $f_P$ lies in $J$. Moreover,

$$\langle f(x_1, \ldots, x_r, z_1, \ldots, z_s, y_1, \ldots, y_n)\rangle^{T_{\mathbb{Z}_3}} = \langle f(x_{i_1}, \ldots, x_{i_r}, z_{j_1}, \ldots, z_{j_s}, y_{l_1}, \ldots, y_{l_n})\rangle^{T_{\mathbb{Z}_3}}.$$

Therefore, we can assume that $S$ is formed by polynomials

$$f = f(x_1, \ldots, x_r, z_1, \ldots, z_s, y_1, \ldots, y_n)$$

which are linear combination of monomials $m \in H_q$ such that

$$\deg_{x_1} m, \ldots, \deg_{x_r} m, \deg_{z_1} m, \ldots, \deg_{z_s} m, \deg_{y_1} m, \ldots, \deg_{y_n} m \geq 1.$$

Consider the following sets:

$$S^1 = \bigcup_{n\geq 0} \left\{ \sum_k \alpha_k f^1_{(k_1,\ldots,k_n)} |\ 1 \leq k_1, \ldots, k_n \leq q-1;\ \alpha_k \in \mathbb{F} \right\} \cap J;$$

$$V^1 = \bigcup_{n\geq 0} \left\{ \sum_k \alpha_k g^1_{(k_1,\ldots,k_n)} |\ 1 \leq k_1, \ldots, k_n \leq q-1;\ \alpha_k \in \mathbb{F} \right\} \cap J;$$

$$S^{1,1} = \bigcup_{n\geq 0} \left\{ \sum_{k,l} \alpha_{k,l} f^{11}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}} |\ 0 \leq k_1, \ldots, k_n, l_1, \ldots, l_n \leq q-1;\ \alpha_{k,l} \in \mathbb{F} \right\} \cap J;$$

$$S^{1,2} = \bigcup_{n\geq 0} \left\{ \sum_{k,l} \alpha_{k,l} f^{12}_{\binom{k_1,\ldots,k_n}{l_1,\ldots,l_n}} |\ 0 \leq k_1, \ldots, k_n, l_1, \ldots, l_n \leq q-1;\ \alpha_{k,l} \in \mathbb{F} \right\} \cap J.$$

We also assume $k_i + l_i \geq 1$ in the definition of the sets $S^{1,1}$ e $S^{1,2}$, for every $i = 1, \ldots n$. Note that

$$J = \langle S^1 \cup V^1 \cup S^{1,1} \cup S^{1,2}\rangle^{T_{\mathbb{Z}_3}} + I_q \quad \text{or}$$

$$J = \langle y_1\rangle^{T_{\mathbb{Z}_3}} + \langle S^1 \cup V^1 \cup S^{1,1} \cup S^{1,2}\rangle^{T_{\mathbb{Z}_3}} + I_q.$$

Consider

$$f = \sum_k \alpha_k f^1_{(k_1,\dots,k_n)}(x_1, y_1, \dots, y_n) \in S^1;$$

$$g = \sum_k \alpha_k g^1_{(k_1,\dots,k_n)}(z_1, y_1, \dots, y_n) \in V^1.$$

Then by Lemma 2.2, we have that the polynomials

$$f^1_{(k_1,\dots,k_n)}(x_1, y_1, \dots, y_n),\ g^1_{(k_1,\dots,k_n)}(z_1, y_1, \dots, y_n) \in J,$$

for all $k$. Hence, we can assume that

$$S^1 = \bigcup_{n\geq 0} \left\{ f^1_{(k_1,\dots,k_n)} |\ 1 \leq k_1, \dots, k_n \leq q-1 \right\} \cap J;$$

$$V^1 = \bigcup_{n\geq 0} \left\{ g^1_{(k_1,\dots,k_n)} |\ 1 \leq k_1, \dots, k_n \leq q-1 \right\} \cap J.$$

Now, if $f \in S^{1,1}$ then

$$f(x_1, y_1, \dots, y_n) = \sum_{k,l} \alpha_{k,l} f^{11}_{\binom{k_1,\dots,k_n}{l_1,\dots,l_n}},\ \alpha_{kl} \in \mathbb{F},$$

and

$$f(x_1 + x_2, y_1, \dots, y_n) - f(x_1, y_1, \dots, y_n) - f(x_2, y_1, \dots, y_n) = g \in J.$$

Note that

$$g + I_q = g(x_1, x_2, y_1, \dots, y_n) + I_q = \sum_{k,l} \beta_{k,l} f^{12}_{\binom{k_1,\dots,k_n}{l_1,\dots,l_n}} + I_q$$

for some $\beta_{k,l} \in \mathbb{F}$. Moreover,

$$(1/2) g(x_1, x_1, y_1, \dots, y_n) = f(x_1, y_1, \dots, y_n).$$

Thus,

$$\langle f \rangle^{T_{\mathbb{Z}_3}} + I_q = \left\langle \sum_{k,l} \beta_{k,l} f^{12}_{\binom{k_1,\dots,k_n}{l_1,\dots,l_n}} \right\rangle^{T_{\mathbb{Z}_3}} + I_q$$

and we can assume $S^{1,1} = \emptyset$.

Therefore

$$J = \langle S^1 \rangle^{T_{\mathbb{Z}_3}} + \langle V^1 \rangle^{T_{\mathbb{Z}_3}} + \langle S^{1,2} \rangle^{T_{\mathbb{Z}_3}} + I_q \quad \text{or}$$

$$J = \langle y_1 \rangle^{T_{\mathbb{Z}_3}} + \langle S^1 \rangle^{T_{\mathbb{Z}_3}} + \langle V^1 \rangle^{T_{\mathbb{Z}_3}} + \langle S^{1,2} \rangle^{T_{\mathbb{Z}_3}} + I_q.$$

Using Lemmas 6.4, 6.5 and Proposition 6.12, there exist finite subsets $\widetilde{S}_1, \widetilde{V}_1, \widetilde{S}_{1,2}$ of $S^1$, $V^1$ and $S^{1,2}$, respectively, such that

$$\langle S^1 \rangle^{T_{\mathbb{Z}_3}} + I_q = \langle \widetilde{S}_1 \rangle^{T_{\mathbb{Z}_3}} + I_q,$$

$$\langle V^1 \rangle^{T_{\mathbb{Z}_3}} + I_q = \langle \widetilde{V}_1 \rangle^{T_{\mathbb{Z}_3}} + I_q,$$

$$\langle S^{1,2} \rangle^{T_{\mathbb{Z}_3}} + I_q = \langle \widetilde{S}_{1,2} \rangle^{T_{\mathbb{Z}_3}} + I_q.$$

It follows that

$$J = \langle \widetilde{S}_1 \rangle^{T_{\mathbb{Z}_3}} + \langle \widetilde{V}_1 \rangle^{T_{\mathbb{Z}_3}} + \langle \widetilde{S}_{1,2} \rangle^{T_{\mathbb{Z}_3}} + I_q \quad \text{or}$$

$$J = \langle y_1 \rangle^{T_{\mathbb{Z}_3}} + \langle \widetilde{S}_1 \rangle^{T_{\mathbb{Z}_3}} + \langle \widetilde{V}_1 \rangle^{T_{\mathbb{Z}_3}} + \langle \widetilde{S}_{1,2} \rangle^{T_{\mathbb{Z}_3}} + I_q.$$

Since $I_q$ is finitely generated as a $T_{\mathbb{Z}_3}$-ideal, we conclude that $J$ is also finitely generated as a $T_{\mathbb{Z}_3}$-ideal. □

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*Daniela Martinez Correa*
*Universidade de São Paulo (USP)*
*Instituto de Matemática, Estatística e Ciência da Computação*
*R. do Matão, 1010 – Butantã, 05508-090, São Paulo, SP, Brazil*
*e-mail:* danielam.correa@ime.usp.br

*Dimas José Gonçalves*
*Universidade Federal de São Carlos (UFSCar)*
*Departamento de Matemática*
*13565-905, São Carlos, SP, Brazil*
*e-mail:* dimasjg@ufscar.br

*Evandro Riva*
*Universidade Tecnológica Federal do Paraná (UTFPR)*
*Departamento de Matemática*
*Av. Sete de Setembro, 3165, 80230-901, Curitiba, PR, Brazil*
*e-mail:* evandroriva@utfpr.edu.br

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