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INTEGRALITY AND SPECIALIZED SYMMETRIC FUNCTIONS

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Abstract. Given $d_1, \dots, d_k$ in the field F, there is a weighted trace function $F^k \to F$ given by $\mathrm{tr}(x_1, \dots, x_k) = \sum d_i x_i$. We prove that F^k satisfies trace identities of the forms $\alpha(d_1, \dots, d_k)x^N =$ a linear combination of terms with lower powers of x; and $\mathrm{tr}(y_1) \cdots \mathrm{tr}(y_n) =$ a linear combination of terms with fewer traces. The approach uses specialized symmetric functions.

1. Introduction. In [3], together with S. Catoiu, we introduced the specialized symmetric functions. These can be regarded from two points of view, either symmetric function theory or trace identities. Given $(d) = (d_1, \dots, d_k)$ we defined the specialized power function $p_n^{(d)}(x_1, \dots, x_k)$ to be $d_1x_1^n + \cdots + d_kx_k^n$. In the special case in which the d_i are all positive integers, $p_n^{(d)}$ can be gotten from the ordinary power symmetric functions by specializing certain groups of variables to be equal, hence the name. We will sometimes suppress the superscript (d) since we have no need to investigate the classical case corresponding to $(d) = (1, \dots, 1)$. There are corresponding analogues of the other classical symmetric functions:

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The elementary symmetric functions, the complete symmetric functions and the Schur functions, and these satisfy most of the relations satisfied by their more classical counterparts, although our focus in the current work will only be on the specialized power symmetric functions. As is standard in symmetric function theory, given a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ we denote the product $p_{\lambda_1} p_{\lambda_2} \cdots$ as p_λ. Our main interest will be in the algebraic relations satisfied by the p_λ.

From the trace identity point of view, let $A = F^k$, be k copies of the characteristic zero field, F, sometimes considered as diagonal $k \times k$ matrices. Since A is commutative, the trace functions on A are simply the linear functionals, and so every non-degenerate trace function on A is of the form

$$\mathrm{tr}(a_1, \ldots, a_k) = d_1 a_1 + \cdots + d_k a_k,$$

for some non-zero $d_1, \ldots, d_k$ in the field. We use the notation $D(d)$ for the corresponding algebra of diagonal matrices with trace. The study of the trace identities of $D(d)$ was initiated by Ioppolo, Koshlukov and La Mattina in [4], in which they found bases for the ideal of identities when $k \leq 2$. If $X = (x_1, \ldots, x_k)$, then the trace of X^n is $d_1 x_1^n + \cdots d_k x_k^n$, which is $p_n^{(d)}(x_1, \ldots, x_k)$. Let $\Lambda^{(d)} \subset F[x_1, \ldots, x_k]$ be the subring generated by the $p_n^{(d)}$. Again, we will suppress the (d) when we don't feel that it is necessary. We will also write Λ_n or $\Lambda_n^{(d)}$ for the algebra generated by $\{p_1, \ldots, p_n\}$. We showed in [3] that if the d_i are all in F then $\Lambda^{(d)}$ has transcendence degree k, so any set of more than k of the p_n will satisfy algebraic relations. In the classical case in which each $d_i = 1$, the Cayley-Hamilton theorem says that X is monic algebraic over Λ of degree n. From the point of view of trace identities, the Cayley-Hamilton theorem is a mixed trace identity (i.e., and identity involving both trace and non-trace terms) in one variable. We would like to know what the trace identities are in this more general case. In the current work we will be focusing mainly on trace identities in one variable. So, pure trace identities in one variable are are relations between the p_λ; and an analogue of Cayley-Hamilton would be a mixed trace identity in one variable.

We define absolute identities to be ones which depend on k, but not otherwise on (d). In the paper [3] we proved that all of the pure, absolute, one variable identities are consequences of the determinantal identities, $|p_{a_i+b_j}| = 0$, for any $a_1, \ldots, a_{k+1}$ and $b_1, \ldots, b_{k+1}$. We also proved that X was algebraic over Λ, but we didn't prove that it was monic. The paper [3] posed six questions or conjectures. In this paper we will address the following two of them:

Conjecture 1 X is integral over Λ.

Conjecture 3 The algebra of $k \times k$ diagonal matrices satisfies an identity of the form $\mathrm{tr}(y_1) \cdots \mathrm{tr}(y_m) =$ a linear combination of trace terms each a product of fewer than m traces, for a certain specified m.

We will prove that Conjecture 1 is true generically and that Conjecture 3 is true. By generically we mean that diagonal matrices satisfy an identity of the form

$$\alpha_n X^n = \sum \alpha_\lambda p_\lambda X^{n-|\lambda|},$$

in which the α are polynomials in $d_1, \ldots, d_k$ and α_n is not identically zero. This will be an identity for any d_i, however; for certain values of the d_i the leading coefficient α_n may become zero. Specifically, if $\sum\{d_i | i \in I\}$ is zero for some $I \subseteq \{1, \ldots, k\}$, then X will not be integral over Λ. We now conjecture that X will be integral in all other cases.

Conjecture 3 has the consequence that any product of m or more traces can be written as a linear combination of terms with at most $m-1$ traces. Conjecture 1 has a somewhat similar application: If X is integral of degree m over Λ_m, then $\Lambda_m = \Lambda$. Because if

$$X^m = \sum a_\lambda p_\lambda X^{m-|\lambda|}$$

where $a_\lambda \in F[d_1, \ldots, d_k]$, then if we multiply by X^i, $i \geq 1$, and take the trace of both sides of the equation, we get

$$p_{m+i} = \sum a_\lambda p_\lambda p_{m-|\lambda|+i}$$

so $p_{m+i} \in \Lambda_{m+i-1}$, and by induction $p_{m+i} \in \Lambda_m$.

There is also a Conjecture 2 in [3] that we have not succeeded in proving, but it seems germane to mention it. Let (d) have t parts with multiplicities $m_1, \ldots, m_t$ and let $m = \prod(m_i + 1)$. Conjecture 2 says that the degree of integrality in Conjecture 1 should be $m-1$ and the number of traces in Conjecture 2 should be m.

We gratefully acknowledge the useful conversations we had with S. Catoiu in the course of this investigation.

2. Non-integrality. The ideal of (pure or mixed) trace identities is homogeneous. Henceforth, whenever we prove the existence or non-existence of identities we will assume homogeneity.

We first show that if there is a set $I \subseteq \{1, \ldots, k\}$ such that $\sum_{i \in I} d_i = 0$, then X cannot be integral over Λ.

Theorem 2.1. *If* $\sum_{i\in I} d_i = 0$, *then* X *cannot be integral over* Λ.

Proof. Assume to the contrary that

$$X^N + \sum a_\lambda p_\lambda X^i = 0,$$

where each $i < N$, and so each λ in the sum has degree $N-i > 0$. If we specialize $x_i = \delta(i \in I)x$, where $\delta(i \in I)$ is 1 or 0, depending on whether $i \in I$ is true, then each $p_j = \sum_m d_m x_m^j$ becomes 0, but X^N does not. □

In the case in which each d_i is either plus or minus 1, the specialized symmetric functions are related to the representations of the general linear Lie algebra. If (d) has k ones and ℓ minus ones then the diagonal matrices $D(d)$ are a trace subalgebra of the Kemer algebra, $M_{k,\ell}$. The trace identities of $M_{k,\ell}$ were found in [2], although Kantor and Trishin had already found the one variable identities in [5]. Translating their results to the language of specialized symmetric functions, given a partition σ of cycle type $(a_1, \ldots, a_k)$ let $T(\sigma)$ be $p_{a_1} \cdots p_{a_k}$.

Theorem 2.2. *Let* λ *be the partition* $(k+1, \ldots, k+1)$ *of* $n = (k+1)(\ell+1)$. *Then the group algebra* FS_n *has a two-sided ideal* I_λ, *and for every* $e \in I_\lambda$, $T(e)$ *is an identity in* Λ, *and all identities for* Λ *are consequences of these.*

3. Initialization step.

Definition 3.1. *We will say that a homogeneous polynomial* $f \in F[d_1, \ldots, d_k; p_1, p_2, \ldots]$ *is* sufficiently monic *if homogeneous in total degree and if it is of the form* $\sum_{\lambda\vdash N} \alpha_\lambda p_\lambda$, *where each* α_λ *is in* $F[d_1, \ldots, d_k]$, $\alpha_{(N)} \neq 0$ *and* $\alpha_{(1^N)} = 1$. *Meaning that* f *is of the form*

$$\alpha_N(d_1, \ldots, d_k)p_N + \cdots + p_1^N.$$

We will say that $\Lambda^{(d)}$ *satisfies such an* f *if* f *vanishes under the substitutions* $p_i \mapsto d_1 x_1 + \ldots + d_k x_k$.

Definition 3.2. *Likewise, we will say that a homogeneous polynomial* f *in* $F[d_1, \ldots, d_k; p_1, p_2, \ldots, t]$ *is* sufficiently monic *if it is homogeneous in total degree and if it is of the form* $\sum \alpha_\lambda p_\lambda t^{N-|\lambda|}$, *where each* α_λ *is in* $F[d_1, \ldots, d_k]$, $\alpha_\emptyset \neq 0$ *and* $\alpha_{(1^N)} = 1$. *This means that* f *is of the form*

$$\alpha_\emptyset(d_1, \ldots, d_k)t^N + \cdots + p_1^N.$$

We will say that x satisfies such an f if $\Lambda^{(d)}$ satisfies $f|_{t=x}$. We will also say that x is sufficiently monic over $\Lambda^{(d)}$ in this case.

The main step in the remaining proofs is to prove that each of $\Lambda^{(d)}$ and X satisfies sufficiently monic polynomials, which we will do by induction on k, the length of (d). It turns out that either of these two conditions implies the other, but we will only need that a monic polynomial satisfied by X over Λ implies a relation amongst the p_λ. The obvious method would be to take traces, but there is a technical problem. Consider the following identity satisfied by $D(a,b)$:

$$ab(a+b)X^3 - 3abp_1X^2 - (a^2-ab+b^2)p_2X + (a+b)p_1^2X \\ - (abp_3 + (a+b)p_2p_1 - p_1^3)I = 0$$

If we take the trace of both sides, using $\mathrm{tr}(X^i) = p_i$ and $\mathrm{tr}(I) = a+b$, we simply get $0 = 0$. The problem disappears if we multiply by X before taking trace, and this works in general.

Lemma 3.3. *Let $f = \sum a_\lambda p_\lambda X^{N-|\lambda|}$ be a sufficiently monic element of $\Lambda[X]$. Let $g = \mathrm{tr}(X \cdot f)$, namely $g = \sum a_\lambda p_\lambda p_{N+1-|\lambda|}$. Then the coefficient of p_{N+1} in g is $a_\emptyset \neq 0$ and the coefficient of $p_{1^{N+1}}$ in g is $a_{1^N} = 1$, and so $\mathrm{tr}(X \cdot f)$ is sufficiently monic.*

We now turn to an induction proof of the existence of sufficiently monic polynomial identities. For the initialization step, we jump over the trivial case of $k = 1$ to do $k = 2$. This actually was done by Ioppolo, Koshlukov and La Mattina in [4], but we choose a different proof (leading to a higher degree!) because it illustrates the process we use in the induction step. We start with this easy observation.

Lemma 3.4. *The polynomial $\sum a_\lambda p_\lambda X^i$ is an identity for $D(d)$ if and only if $\sum a_\lambda p_\lambda x_j^i = 0$ for all $j = 1, \ldots, k$.*

P r o o f. $\sum a_\lambda p_\lambda X^i$ is a diagonal matrix and the j^{th} entry is $\sum a_\lambda p_\lambda x_j^i$. □

In the case of $k = 2$, we write a and b in place of d_1 and d_2, and x and y in place of x_1 and x_2. So

$$p_1 = ax + by \text{ and } p_2 = ax^2 + by^2.$$

Hence, $ax = p_1 - by$ and $ax^2 = p_2 - by^2$, and since $(ax)^2 = a(ax^2)$ we have the identity $(p_1 - by)^2 = a(p_2 - by^2)$, which simplifies to

$$b(a+b)y^2 - 2bp_1y + (p_1^2 - ap_2) = 0.$$

By a similar argument,

$$a(a+b)x^2 - 2ap_1x + (p_1^2 - bp_2) = 0.$$

Setting $x = y = t$ in each and multiplying these two equations gives an equation over Λ which vanishes if $t = x$ or y, hence, by Lemma 3.4 it is a trace identity for $D(a,b)$. We observe that the coefficient of X^4 is $ab(a+b)^2$ will be non-zero unless a, b or $a+b$ is zero. We also observe the occurrence of p_1^4. In sum:

Lemma 3.5. *When $(d) = (a,b)$, X satisfies a sufficiently monic identity of the form*

$$ab(a+b)^2X^4 + \cdots + p_1^4$$

Multiplying by X and taking traces as in Lemma 3.3, gives

Corollary 3.6. *When $(d) = (a,b)$, $\Lambda^{(d)}$ satisfies a sufficiently monic identity of the form*

$$ab(a+b)^2p_5 + \cdots + p_1^5.$$

We chose to multiply by X before taking traces because, had we not, the last term would have been multiplied by $a+b$, the trace of the identity matrix, in which case the p_1^4 would have vanished in the $a+b=0$ case. As it is, it is valid for all a and b.

4. The induction step. In this section we assume that for every (d) with k parts, the diagonal matrix X is sufficiently monic over $\Lambda^{(d)}$ and that $\Lambda^{(d)}$ satisfies a sufficiently monic polynomial. We let $(d^+) = (d_1, \ldots, d_{k+1})$ have $k+1$ parts and for each $1 \le i \le k+1$ we let $(\hat{d_i})$ be (d^+) with d_i removed. The strategy is to construct sufficiently monic polynomials $g_i(t) \in \Lambda^{(d^+)}[t]$ such that $g_i(x_i) = 0$. Then their product will be an identity for X by Lemma 3.4. To save notation, we will write p_λ for $p_\lambda^{(d)}$ and q_λ for $p_\lambda^{(d^+)}$. Given $f \in F[d_1, \ldots, d_k, p_1, p_2, \ldots]$, for each $1 \le i \le k+1$ we define $g_i \in F[d_1, \ldots, d_{k+1}, q_1, q_2, \ldots, t]$ via

$$g_i(t) = f(\hat{d_i}, q_1 - d_it, q_2 - d_it^2, \ldots].$$

Lemma 4.1. *If f is an identity for $\Lambda^{(d)}$, then each $g_i(x_i)$ is an identity for $\Lambda^{(d^+)}$.*

Proof. If $t = x_i$, then each $q_j - d_it^i$ becomes $\sum_{\alpha \ne i} d_\alpha x_\alpha$ which equals $p_j^{(\hat{d_i})}$, so $g_i(x_i) = f(\hat{d_i}, p_i^{(\hat{d_i})}) = 0$. □

At this point it is useful to see specifically how g_i is related to f.

Lemma 4.2. *If $f = p_\lambda = p_{\lambda_1} \cdots p_{\lambda_a}$, then g_i equals*

$$(q_{\lambda_1} - d_i t^i) \cdots (q_{\lambda_a} - d_i t^{\lambda_a}).$$

In particular, the highest degree term in t is $(-d_i)^a t^{|\lambda|}$ and the lowest degree term is q_λ.

The number of non-zero parts to a partition λ is called the length of λ and is denoted $\ell(\lambda)$.

Corollary 4.3. *If $f = \sum_{\lambda \vdash N} \alpha_\lambda p_\lambda$, then the term in $g_i(t)$ of highest degree in t is $\sum_{\lambda \vdash N} \alpha_\lambda (-d)^{\ell(\lambda)}$.*

In the case of sufficiently monic polynomials we have

Lemma 4.4. *If $f = \sum_{\lambda \vdash N} \alpha_\lambda p_\lambda$ is sufficiently monic, then so is each g_i.*

P r o o f. By Lemma 4.2, the terms of g_i not involving t will be $g_i(\hat{d}_i, q_1, q_2, \ldots)$, and so if f has a summand of p_1^N, then g_i will have a summand of q_1^N. And, if $f = \alpha_N(d) p_N + \cdots$, then the highest degree term of g_i in t will be t^N times $(-d_i)\alpha_N(\hat{d}_i)+$ higher degree terms in d_i. □

Theorem 4.5. *Let f be a sufficiently monic identity for X over $\Lambda^{(d)}$. Then the product $g_1(X) \cdots g_{k+1}(X)$ will be a sufficiently monic identity for X over $\Lambda^{(d^+)}$.*

This proves Conjecture 1. Conjecture 2 follows from multilinearization:

P r o o f o f C o n j e c t u r e 2. Theorem 4.5 implies that the generic diagonal polynomial X will satisfy a identity of the form

$$\mathrm{tr}(X)^N + \text{terms involving fewer traces} = 0.$$

We can now multilinearize: Let $X = c_1 y_1 + \cdots + c_N y_n$ and take the coefficient of $c_1 \cdots c_N$, getting

$$\mathrm{tr}(y_1) \cdots \mathrm{tr}(y_N) + \text{terms involving fewer traces} = 0. \qquad \square$$

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