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Serdica Math. J. **51** (2025), 443–490

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
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INVARIANTS OF GROUP ACTIONS AND MODULAR FORMS FOR LATTICES OF SU(2, 1)

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Communicated by P. Koshlukov

Dedicated to Acad. Prof. D.Sc. V. Drensky on the occasion of his 75-th birthday

Abstract. Di Cerbo and Stover have classified in five isomorphism classes the quotients of the complex 2-ball by a lattice, which have smooth toroidal compactifications and minimal Haar measure. The present note constructs explicit modular forms of weight three, which embed two of the corresponding Baily-Borel compactifications in a complex projective space of dimension five. The argument is based on multiple applications of the theory of group invariants to complex analytic geometry, as well as on the properties of some Weierstrass functions.

https://doi.org/10.55630/serdica.2025.51.443-490

2020 *Mathematics Subject Classification:* 11F11, 1427.

Key words: Modular forms for lattices of SU(2,1), Abelian functions, invariant under finite biholomorphism groups, projective embeddings of Baily-Borel compactifications of low codimension.

*Research partially supported by the Science Foundation of Sofia University "St. Kliment Ohridski" under Contract 80-10-35/23.05.2025.

1. Introduction. Let $\mathbb{B}/\Gamma$ be a noncompact quotient of the complex 2-ball $\mathbb{B} = \mathrm{SU}(2,1)/\mathrm{S}(\mathrm{U}_2 \times \mathrm{U}_1)$ by a lattice $\Gamma < \mathrm{SU}(2,1)$. Baily and Borel show in [1] that there exist finitely many Γ-orbits κ_j, $1 \leq j \leq h$ of the so called Γ-rational boundary points of $\mathbb{B}$, such that the disjoint union $\widehat{\mathbb{B}/\Gamma} := (\mathbb{B}/\Gamma) \coprod \{\kappa_j \,|\, 1 \leq j \leq h\}$ is a projective variety. This is done by establishing the existence of finitely many Γ-modular forms of sufficiently large weight, which induce a regular embedding of $\widehat{\mathbb{B}/\Gamma}$ in a finite dimensional complex projective space. There arises the problem of construction of explicit Γ-modular forms of comparatively small weight, which provide a regular projective embedding of $\widehat{\mathbb{B}/\Gamma}$ of comparatively small co-dimension. The present note obtains two such embeddings $\Phi_i : \widehat{\mathbb{B}/\Gamma_i} \to \mathbb{P}^5(\mathbb{C})$ by modular forms of weight three, for Baily-Borel compactifications $\widehat{\mathbb{B}/\Gamma_i}$, $1 \leq i \leq 2$, having smooth toroidal compactifications $(\mathbb{B}/\Gamma_i)'$ and minimal Haar measure. The explicit constructions of Φ_i are given in terms of some Weierstrass functions, after several applications of the theory of group invariants of complex analytic and rational functions. By their very definition, the Γ-modular forms can be interpreted as Γ-invariant tensor products of holomorphic differential forms. Namely, a holomorphic function $\phi : \mathbb{B} \to \mathbb{C}$ is a Γ-modular form of weight $n \in \mathbb{N}$ exactly when $\phi(z)\,(dz_1 \wedge dz_2)^{\otimes n}$ is a Γ-invariant holomorphic section of $\left(\Omega^{2,0}_{\mathbb{B}}\right)^{\otimes n} \to \mathbb{B}$, where $\Omega^{2,0}_{\mathbb{B}} \to \mathbb{B}$ is the line bundle of the holomorphic $(2,0)$-forms on the ball $\mathbb{B}$. The Γ-modular forms are very difficult to be constructed explicitly. In order to overcome this disadvantage, Holzapfel's [12] introduces "a transfer" of appropriate rational functions on a birational model Y of $\widehat{\mathbb{B}/\Gamma}$ to Γ-modular forms. On an abelian surface $Y = A$, one can construct explicit quotients of Weierstrass sigma-functions, corresponding to Γ-modular forms of fixed weight. Let A/H be a quotient of an abelian surface A by a finite group H of holomorphic automorphisms, which acts on $\mathbb{B}/\Gamma$ and induces a permutation of the cusps κ_j, $1 \leq j \leq h$. Then there is a latice $\Gamma_H < \mathrm{SU}(2,1)$, containing Γ as a normal subgroup with quotient $\Gamma_H/\Gamma = H$, such that $\mathbb{B}/\Gamma_H$ is birational to A/H. The rational functions on A/H are exactly the H-invariant rational functions on A. This is the second place of the construction, exploiting invariants of group actions. The Baily-Borel compactifications $\widehat{\mathbb{B}/\Gamma_i}$, $1 \leq i \leq 2$, studied by the present article are birational to a bi-elliptic surface $Y_3 = E \times E/\mathbb{Z}_3$, which is a quotient of the Cartesian square $E \times E$ of an elliptic curves E by a cyclic group of order 3. The $\mathbb{Z}_3$-Galois covering $\pi : E \times E \to E \times E/\mathbb{Z}_3 = Y_3$ induces an isomorphism $\pi^* : \mathbb{C}(Y_3) \to \mathbb{C}(E \times E)^{\mathbb{Z}_3}$ of the field $\mathbb{C}(Y_3)$ of the rational functions on Y_3 with the field $\mathbb{C}(E \times E)^{\mathbb{Z}_3}$ of the $\mathbb{Z}_3$-invariant rational functions on $E \times E$. By the means of the Weierstrass sigma-function, we construct such

explicit $e_0, f_{i0}, f_{i1}, f_{i2} \in \mathbb{C}(E\times E)^{\mathbb{Z}_3} = \pi^*(\mathbb{C}(Y_3))$, $1 \le i \le 2$, which correspond to Γ_i-modular forms $\omega_0, \omega_{i0}, \omega_{i1}, \omega_{i2}$ of weight 3. Combining with the transfer $\theta_3(1)$ of the constant $1 \in \mathbb{C}(Y_3)$ and with a standard non-cuspidal modular form $\omega_0^{[3]}$ of weight 3, whose existence is established in [15], the article obtains regular projective embeddings

$$\Phi_i = \left[\theta_3(1) : \omega_0 : \omega_{i0} : \omega_{i1} : \omega_{i2} : \omega_0^{[3]}\right] : \widehat{\mathbb{B}/\Gamma_i} \longrightarrow \mathbb{P}^5(\mathbb{C})$$

of codimension 3 by Γ_i-modular forms of weight 3.

In [12] Holzapfel constructs a ball quotient $\mathbb{B}/\Gamma^{(6)}_{\sqrt{-1}}$ with 6 cusps, which is birational to the abelian surface $A_{\sqrt{-1}} := \left(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\sqrt{-1}\right)\times\left(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\sqrt{-1}\right)$. The same work provides a regular embedding $\widehat{\mathbb{B}/\Gamma^{(6)}_{\sqrt{-1}}} \to \mathbb{P}^{22}(\mathbb{C})$ by explicit $\Gamma^{(6)}_{\sqrt{-1}}$-modular forms of weight 3, subject to explicit relations. Projective embeddings of co-abelian $\widehat{\mathbb{B}/\Gamma}$ by Γ-modular forms of weight 1 or 2 are obtained in the joint article [17] with Kotzev and in [16]. The modular forms of weight 1 for all Galois quotients of a specific Holzapfel's example $\mathbb{B}/\Gamma^{(8)}_{\sqrt{-1}}$ with 8 cusps and minimal model $A_{\sqrt{-1}}$ are described in [14]. For smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ with bi-elliptic minimal model, the study of the Γ-modular forms is initiated by [15] and continues in the present note.

Here is a synopsis of the article. The next, second section provides some preliminaries on Holzapfel's transfer of rational functions on a minimal model Y of a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ to Γ-modular forms. It explains the reduction of the study of a smooth $X = (\mathbb{B}/\Gamma)'$ with a bi-elliptic minimal model $Y_{d,t} = E_1\times E_0/G_{d,t}$ to the description of its $G_{d,t}$-Galois cover $X_0 = (\mathbb{B}/\Gamma_0)'$ with abelian minimal models $E_1\times E_0$, as well as the realization of the Γ-modular forms by $G_{d,t}$-invariant rational functions on the abelian surface $E_1\times E_0$. The preliminaries formulate a sufficient condition from [15], under which Γ-modular forms, corresponding to $G_{d,t}$-invariant abelian functions on $E_1\times E_0$, determine a regular projective embedding of the Baily-Borel compactification. A special attention is paid to Di Cerbo and Stover's classification of the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ with minimal Haar measure, provided by [4]. Let $\mathcal{O}_{\sqrt{-3}} = \mathbb{Z}+\mathbb{Z}\omega$ with $\omega := \mathrm{e}^{\frac{2\pi\sqrt{-1}}{6}}$ be the ring of the Eisenstein integers and $E = (\mathbb{C},+)/(\Lambda,+)$ be the elliptic curve with lattice $\Lambda = \mathcal{O}_{\sqrt{-3}}$. Section 3 constructs explicit G_3-invariant abelian functions $e_0, f_{i0}, f_{i1}, f_{i2}$ on $E\times E$, associated with Γ_i-modular forms $\omega_0, \omega_{i0}, \omega_{i1}, \omega_{i2}$ of weight 3. The fourth section obtains regular embeddings $\Phi_i : \widehat{\mathbb{B}/\Gamma_i} \to \mathbb{P}^5(\mathbb{C})$, $1 \le i \le 2$, after appending to $\omega_0, \omega_{i0}, \omega_{i1}, \omega_{i2}$ some standard modular forms, whose existence is established in [15]. According

to a result of [15], it suffices to show that the generic fibres of the G_3-invariant rational map $f_i = (e_0, f_{i0}, f_{i1}, f_{i2}) : E \times E \to f_i(E \times E) \subset \mathbb{P}^1(\mathbb{C})^4$ consist of a single G_3-orbit, in order to derive the injectiveness of Φ_i. First, we show that the generic fibre of $\Omega_1 = (f_{11}, f_{12}) : E \times E \to \Omega_1(E \times E)$ is of the form $\Omega_1^{-1}(x_3, x_4) = \mathrm{Orb}_{G_3}((u + \Lambda) \times E)$ for $(x_3, x_4) \in \Omega_1(E \times E) \subset \mathbb{P}^1(\mathbb{C})^2$, $u + \Lambda \in E$, while the generic fibre of $\Omega_2 = (f_{21}, f_{22}) : E \times E \to \Omega_2(E \times E)$ is $\Omega_2^{-1}(x_3, x_4) = \mathrm{Orb}_{G_3}(E \times (v + \Lambda))$ for $(x_3, x_4) \in \Omega_1(E \times E) \subset \mathbb{P}^1(\mathbb{C})^2$, $v + \Lambda \in E$. Then the G_3-invariant rational maps $\Upsilon_i = (e_0, f_{i0}) : E \times E \to \Upsilon_i(E \times E) \subseteq \mathbb{P}^1(\mathbb{C})^2$ are proved to have injective restrictions $\Upsilon_1|_{(u+\Lambda)\times E}$, respectively, $\Upsilon_2|_{E\times(v+\Lambda)}$. That suffices for a generic fibre $f_i^{-1}(x_1, \ldots, x_4) = \Upsilon_i^{-1}(x_1, x_2) \cap \Omega_i^{-1}(x_3, x_4) = \mathrm{Orb}_{G_3}(p(x_1, \ldots, x_4))$ of f_i to constitute a single G_3-orbit of a point $p(x_1, \ldots, x_4) \in E \times E$.

2. Preliminaries for modular forms on ball quotients, birational to a bi-elliptic surface. The present section collects some preliminaries for modular forms of a lattice $\Gamma < \mathrm{SU}(2, 1)$, with a special emphasis on the case when $\mathbb{B}/\Gamma$ is birational to a bi-elliptic surface. Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification with minimal model Y. The first subsection recalls Holzapfel's transfer of rational functions on Y to Γ-modular forms, introduced in [12]. Making use of some results from the joint article [3] with Beshkov and Sankaran, the second subsection describes how the study of smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ with bi-elliptic minimal model $Y_{d,t} = E_1 \times E_0/G_{d,t}$ reduces to the study of their unramified $G_{d,t}$-Galois covers $X_o = (\mathbb{B}/\Gamma_o)'$ with abelian minimal model $E_1 \times E_0$, which is a direct product of mutually isogeneous elliptic curves E_1, E_0. As a result, one can construct explicit Γ-modular forms by the means of explicit $G_{d,t}$-invariant rational functions on $E_1 \times E_0$, as explained in the third subsection. The same subsection recalls a result of [15], providing a sufficient condition for $G_{d,t}$-invariant abelian functions on $E_1 \times E_0$ to induce a regular projective embedding of $\widehat{\mathbb{B}/\Gamma}$ by Γ-modular forms of one and a same weight. The final, fourth subsection recalls the main result of Di Cerbo and Stover's [4], classifying the ball quotients $\mathbb{B}/\Gamma$ with smooth toroidal compactification $(\mathbb{B}/\Gamma)'$ and minimal Haar measure.

2.1. Holzapfel's transfer of rational functions to modular forms. Let $\mathbb{B} = \{[t_0 : t_1 : t_2] \in \mathbb{P}^2(\mathbb{C}) \,|\, |t_0|^2 - |t_1|^2 - |t_2|^2 > 0\} = \mathrm{SU}(2, 1)/\mathrm{S}(\mathrm{U}_2 \times \mathrm{U}_1)$ be the complex 2-ball and Γ be a lattice of $\mathrm{SU}(2, 1)$ with noncompact quotient $\mathbb{B}/\Gamma$. A boundary point $p \in \partial\mathbb{B}$ is Γ-rational if the unipotent radical of the stabilizer $\mathrm{Stab}_{\mathrm{SU}(2,1)}(p)$ of p in $\mathrm{SU}(2, 1)$ intersects Γ in its lattice. The set $\partial_\Gamma\mathbb{B}$ of the Γ-rational boundary points is acted by Γ with finitely many orbits, called Γ-cusps.

According to Baily-Borel's [1], the disjoint union $\widehat{\mathbb{B}/\Gamma} := (\mathbb{B}/\Gamma) \coprod (\partial_\Gamma \mathbb{B}/\Gamma)$ of $\mathbb{B}/\Gamma$ with the set $\partial_\Gamma \mathbb{B}/\Gamma = \{\kappa_j \,|\, 1 \leq j \leq h\}$ of the Γ-cusps is a normal projective variety. This is a consequence of the existence of finitely many Γ-modular forms of sufficiently large weight, providing a regular embedding $\widehat{\mathbb{B}/\Gamma} \to \mathbb{P}^N(\mathbb{C})$ for some $N \in \mathbb{N}$. Any $\gamma \in \Gamma < \mathrm{SU}(2,1) < \mathrm{GL}(3,\mathbb{C})$ is a holomorphic map

$$\gamma : \mathbb{B} \longrightarrow \mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 < 1\} \subset \mathbb{C}^2,$$

which can be viewed as an ordered pair $\gamma = (\gamma_1, \gamma_2)$ of holomorphic functions $\gamma_i : \mathbb{B} \to \mathbb{C}$. Let $\mathrm{Jac}(\gamma) \in M_{2\times 2}(\mathbb{C})$ be the Jacobian matrix of $\gamma = (\gamma_1, \gamma_2)$ with respect to $(z_1, z_2) \in \mathbb{B}$. A global holomorphic function $\phi : \mathbb{B} \to \mathbb{C}$ is a Γ-modular form of weight $n \in \mathbb{N}$ if

$$\gamma^*(\phi)(z_1, z_2) := \phi(\gamma(z_1, z_2)) = \frac{\phi(z_1, z_2)}{[\det \mathrm{Jac}(\gamma)]^n}$$

for all $\gamma \in \Gamma$ and all $(z_1, z_2) \in \mathbb{B}$. If $\mathcal{O}_\mathbb{B}$ is the sheaf of the holomorphic functions on $\mathbb{B}$ and $\Omega^{2,0}_\mathbb{B}$ is the sheaf of the holomorphic $(2,0)$-forms then for any $n \in \mathbb{N}$ there is a $\mathbb{C}$-linear embedding

$$j_n : H^0(\mathbb{B}, \mathcal{O}_\mathbb{B}) \longrightarrow H^0\left(\mathbb{B}, \left(\Omega^{2,0}_\mathbb{B}\right)^{\otimes n}\right),$$

$$j_n(\phi(z)) := \phi(z)\,(\mathrm{d}z_1 \wedge \mathrm{d}z_2)^{\otimes n}, \quad \forall \phi \in H^0(\mathbb{B}, \mathcal{O}_\mathbb{B})$$

of the corresponding global sections. A function $\phi \in H^0(\mathbb{B}, \mathcal{O}_\mathbb{B})$ is a Γ-modular form of weight n exactly when $j_n(\phi) \in H^0\left(\mathbb{B}, \left(\Omega^{2,0}_\mathbb{B}\right)^{\otimes n}\right)^\Gamma$ is a Γ-invariant holomorphic section of $\left(\Omega^{2,0}_\mathbb{B}\right)^{\otimes n} \to \mathbb{B}$ and can be viewed as a global holomorphic section $j_n(\phi) \in H^0\left(\mathbb{B}/\Gamma, \left(\Omega^{2,0}_{\mathbb{B}/\Gamma}\right)^{\otimes}\right)$ of $\left(\Omega^{2,0}_{\mathbb{B}/\Gamma}\right)^{\otimes n} \to \mathbb{B}/\Gamma$. The set $\mathcal{M}_n(\Gamma)$ of the Γ-modular forms of weight n is a finite dimensional $\mathbb{C}$-vector space with respect to the pointwise addition and multiplication by a complex number. Identifying $\mathcal{M}_n(\Gamma)$ with $j_n(\mathcal{M}_n(\Gamma))$, one can view the Γ-modular forms as tensor powers of holomorphic differential $(2,0)$-forms on (the possibly singular) ball quotient $\mathbb{B}/\Gamma$. Any $j_n(\phi)$ with $\phi \in \mathcal{M}_n(\Gamma)$ extends over the cusps $\partial_\Gamma \mathbb{B}/\Gamma$ and determines a global holomorphic section $j_n(\phi) : \widehat{\mathbb{B}/\Gamma} \to \left(\Omega^{2,0}_{\mathbb{B}/\Gamma}\right)^{\otimes n}$ over the Baily-Borel compactification. The resolution $\rho : (\mathbb{B}/\Gamma)' \to \widehat{\mathbb{B}/\Gamma}$ of the cuspidal singularities κ_j, $1 \leq j \leq h$ of $\widehat{\mathbb{B}/\Gamma}$ by smooth irreducible elliptic curves D_j results in the toroidal

compactification $(\mathbb{B}/\Gamma)'$. Let $D = \sum_{j=1}^{h} D_j$ be the toroidal compactifying divisor of $\mathbb{B}/\Gamma$. In [6] Hemperly observes that if $X = (\mathbb{B}/\Gamma)'$ is a smooth surface then ρ pulls back isomorphically $j_n(\mathcal{M}_n(\Gamma)) = H^0\left(\widehat{\mathbb{B}/\Gamma}, \left(\Omega^{2,0}_{\mathbb{B}/\Gamma}\right)^{\otimes n}\right)$ to the $\mathbb{C}$-vector space $H^0\left(X, \Omega^{2,0}_X(D)^{\otimes n}\right)$ of the global holomorphic sections of the n-th tensor power of the logarithmic canonical bundle $\Omega^{2,0}_X(D) := \Omega^{2,0}_X \otimes \mathcal{O}_X(D) \to X$. From now on, we assume that the toroidal compactification $X = (\mathbb{B}/\Gamma)'$ is smooth and denote by K_X its canonical divisor. Let

$$\mathcal{L}_X(n(K_X + D)) := \{f \in \mathbb{C}(X)^* \,|\, (f) + n(K_X + D) \geq 0\} \cup \{0\}$$

be the $\mathbb{C}$-vector space of the rational functions $f : X \to \mathbb{P}^1(\mathbb{C})$, whose divisors translate $n(K_X + D)$ to an effective one. An arbitrary global meromorphic section σ of the line bundle $\Omega^{2,0}_X(D) \to X$ provides a $\mathbb{C}$-linear isomorphism

$$\sigma^{\otimes n} : \mathcal{L}_X(n(K_X + D)) \longrightarrow H^0\left(X, \Omega^{2,0}_X(D)^{\otimes n}\right).$$

The blow down $\mathrm{Bl} : X = (\mathbb{B}/\Gamma)' \to Y$ of the smooth irreducible rational (-1)-curves on X yields a minimal surface Y. The map Bl is birational and induces an isomorphism $\mathrm{Bl}^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ of the corresponding (infinite dimensional) $\mathbb{C}$-algebras of rational functions. Let $C = \sum_{j=1}^{h} C_j$ with $C_j := \mathrm{Bl}(D_j)$ be the image of the toroidal compactifying divisor $D = \sum_{j=1}^{h} D_j$ on Y. Any smooth irreducible rational (-1)-curve L_i on $X = (\mathbb{B}/\Gamma)' = (\mathbb{B}/\Gamma) \coprod D$ intersects $D = \coprod_{j=1}^{h} D_j$ in at least 3 points, due to the Kobayashi hyperbolicity of $\mathbb{B}/\Gamma$. That is why, the exceptional divisor $L = \sum_{i=1}^{s} L_i$, $L_i \simeq \mathbb{P}^1(\mathbb{C})$ of Bl maps onto the singular locus $\mathrm{Bl}(L) = C^{\mathrm{sing}}$ of C. An arbitrary point $p \in Y$ is associated with a uniquely determined multiplicity homomorphism

$$m_p : (\mathrm{Div}(Y), +) \longrightarrow (\mathbb{Z}, +)$$

of the divisor group $(\mathrm{Div}(Y), +)$, which takes value $m_p(F) = 1$ on the irreduicble curves $F \subset Y$ thought p and value $m_p(F) = 0$ for $p \notin F$. Bearing in mind that

the canonical divisors of X and Y are related by the equality $K_X = \mathrm{Bl}^*(K_Y) + L$, Holzapfel introduces the subspace

$$\begin{aligned}&\mathcal{L}_Y\left(n(K_Y+C), nC^{\mathrm{sing}}\right)\\ &\quad := \left\{f \in \mathbb{C}(Y)\,|\,(f) + n(K_Y+C) \geq 0,\, m_p(f)+n \geq 0,\, \forall p \in C^{\mathrm{sing}}\right\} \cup \{0\}\end{aligned}$$

of $\mathcal{L}_Y(n(K_Y+C))$ and observes that $\mathrm{Bl} : X \to Y$ induces a $\mathbb{C}$-linear isomorphism

$$\mathrm{Bl}^* : \mathcal{L}_Y\left(n(K_Y+C), nC^{\mathrm{sing}}\right) \longrightarrow \mathcal{L}_X(n(K_X+D)).$$

In [12] Holzapfel refers to the composition

$$\begin{aligned}\theta_n := (\rho^*)^{-1}\,\sigma^{\otimes n}\mathrm{Bl}^* : \mathcal{L}_Y\left(n(K_Y+C), nC^{\mathrm{sing}}\right) \longrightarrow H^0\left(\widehat{\mathbb{B}/\Gamma}, \left(\Omega^{2,0}_{\mathbb{B}/\Gamma}\right)^{\otimes n}\right)\\ = j_n\left(\mathcal{M}_n(\Gamma)\right)\end{aligned}$$

as to the transfer of rational functions on Y to Γ-modular forms of weight n.

A modular form $\phi \in \mathcal{M}_n(\Gamma)$ is cuspidal if it vanishes at all the cusps $\kappa_j \in \partial_\Gamma \mathbb{B}/\Gamma$, $1 \leq j \leq h$. Denote by $\mathcal{M}^o_n(\Gamma) := \{\phi \in \mathcal{M}_n(\Gamma)\,|\,\phi(\kappa_j) = 0,\, 1 \leq j \leq h\}$ the subspace of the cuspidal Γ-modular forms of weight n and note that $f \in \mathcal{L}_Y\left(n(K_Y+C), nC^{\mathrm{sing}}\right)$ transfers to a cuspidal Γ-modular form $\theta_n(f) = (\rho^*)^{-1}\,\sigma^{\otimes n}\mathrm{Bl}^*(f) \in j_n\left(\mathcal{M}^o_n(\Gamma)\right)$ if and only if the global holomorphic section $\rho^*\theta_n(f) = \sigma^{\otimes n}\mathrm{Bl}^*(f)$ of $\Omega^{2,0}_X(D)^{\otimes n} \to X$ vanishes along $D_j = \rho^{-1}(\kappa_j)$ for all $1 \leq j \leq h$. The global meromorphic section σ of $\Omega^{2,0}_X(D) \to X$ has divisor $(\sigma) = K_X + D$, so that the coefficients of D_j, $1 \leq j \leq h$ in the divisor

$$(\rho^*\theta_n(f)) = \left(\sigma^{\otimes n}\mathrm{Bl}^*(f)\right) = n(\sigma) + (\mathrm{Bl}^*(f)) = n(K_X+D) + (\mathrm{Bl}^*(f)) \subset X$$

of $\rho^*\theta_n(f)$ are positive exactly when the rational function $\mathrm{Bl}^*(f) : X \to \mathbb{P}^1(\mathbb{C})$ has a pole of order $\leq n-1$ along $\sum_{j=1}^{h} D_j = D$. Due to $\mathrm{Bl}(D_j) = C_j$, $1 \leq j \leq h$, this is equivalent to $(f) + nK_Y + (n-1)C \geq 0$ and the subspace $\mathcal{L}_Y\left(nK_Y + (n-1)C, nC^{\mathrm{sing}}\right)$ of $\mathcal{L}_Y\left(n(K_Y+C), nC^{\mathrm{sing}}\right)$ transfers isomorphically onto the cuspidal modular forms,

$$\theta_n\left(\mathcal{L}_Y\left(nK_Y + (n-1)C, nC^{\mathrm{sing}}\right)\right) = j_n\left(\mathcal{M}^o(\Gamma)\right).$$

2.2. Ball quotients, birational to bi-elliptic surfaces and their co-abelian liftings. A minimal smooth projective surface Y of Kodaira dimension $\kappa(Y) = 0$ is bi-elliptic if the second homology group $H_2(Y, \mathbb{Z})$ of Y is of rank $\mathrm{rk}_{\mathbb{Z}} H_2(Y, \mathbb{Z}) = 0$. The term is due to the presence of two fibrations by elliptic curves $Y \to R_1$, $Y \to \mathbb{P}^1(\mathbb{C})$ for such surfaces with bases of genus $g(R_1) = 1$, respectively, $g(\mathbb{P}^1(\mathbb{C})) = 0$. Let $E = (\mathbb{C}, +)/(\Lambda, +)$ with $(\Lambda, +) \simeq (\mathbb{Z}^2, +)$ be an elliptic curve and $u_0 + \Lambda \in E$ be a point of E. Denote by $\mathcal{T}(u_0 + \Lambda) : E \longrightarrow E$, $\mathcal{T}(u_0 + \Lambda)(u + \Lambda) = u_0 + u + \Lambda$, $\forall u + \Lambda \in E$ the translation by $u_0 + \Lambda$ on E. Bagnera and de Franchis's article [2] from 1907 shows that any bi-elliptic surface is a quotient $Y_{d,t} = E_1 \times E_0/G_{d,t}$ of a direct product $E_1 \times E_0$ of elliptic curves $E_j = (\mathbb{C}, +)\,/\,(\Lambda_j, +)$, $(\Lambda_j, +) \simeq \big(\mathbb{Z}^2, +\big)$, $0 \leq j \leq 1$ by a fixed point free group $G_{d,t} = \langle g_d \rangle \times \langle \tau_t \rangle$, generated by

$$g_d = \Big(\mathcal{T}(a_d + \Lambda_1), \mathrm{e}^{\frac{2\pi\sqrt{-1}}{d}}\Big) \quad \text{and} \quad \tau_t = (\mathcal{T}(b_t + \Lambda_1), \mathcal{T}(c_t + \Lambda_0))$$

of order $\mathrm{ord}\mathcal{T}(a_d + \Lambda_1) = d \in \{2, 3, 4, 6\}$, respectively, $\mathrm{ord}\mathcal{T}(b_t + \Lambda_1) = \mathrm{ord}\mathcal{T}(c_t + \Lambda_0) = t \in \{1, 2, 3\}$. Moreover, $\mathrm{e}^{\frac{2\pi\sqrt{-1}}{d}}(c_t + \Lambda_0) = c_t + \Lambda_0$, $\langle a_d + \Lambda_1 \rangle \cap \langle b_t + \Lambda_1 \rangle = \{\Lambda_1\}$ and the group $G_{d,t} \simeq (\mathbb{Z}_d \times \mathbb{Z}_t, +)$ is isomorphic to some of the additive groups

$$\mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_3 \times \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_4 \times \mathbb{Z}_2, \quad \mathbb{Z}_6.$$

Let $\mathrm{Bl} : X = (\mathbb{B}/\Gamma)'$ be the blow down of the smooth irreducible rational (-1)-curves on a smooth toroidal compactification X to a minimal surface Y and $\pi : Y_0 \to Y$ be a finite unramified covering. Corollary 5 from the joint article [3] with Beshkov and Sankaran establishes that the fibered product $X_0 := X \times_Y Y_0$ is a smooth toroidal compactification $X_0 = (\mathbb{B}/\Gamma_0)'$ with minimal model Y_0 and an unramified covering $\zeta : X_0 \to X$ of degree $\deg(\zeta) = \deg(\pi)$. For a bi-elliptic surface $Y_{d,t} = E_1 \times E_0/G_{d,t}$, this result applies to the unramified $G_{d,t}$-Galois covering $\pi : E_1 \times E_0 \longrightarrow Y_{d,t} = E_1 \times E_0/G_{d,t}$ and provides the following

Corollary 1 (Corollary 5 [3])**.** *Let* $\mathrm{Bl} : X = (\mathbb{B}/\Gamma)' \to Y_{d,t}$ *be the blow down of the smooth irreducible rational* (-1)*-curves on a smooth toroidal compactification* $X = (\mathbb{B}/\Gamma)'$ *to a bi-elliptic surface* $Y_{d,t}$*,* $D := X \setminus (\mathbb{B}/\Gamma)$ *be the toroidal compactifying divisor of* $\mathbb{B}/\Gamma$*,* $\pi : E_1 \times E_0 \longrightarrow Y_{d,t} = E_1 \times E_0/G_{d,t}$ *be the canonical unramified* $G_{d,t}$*-Galois covering of* $Y_{d,t}$ *and* $X_0 = X \times_{Y_{d,t}} (E_1 \times E_0) := \{(x, e_1, e_0) \in X \times (E_1 \times E_0) \,|\, \mathrm{Bl}(x) = \pi(e_1, e_0)\}$ *be the fibered product of* X *and*

$E_1 \times E_0$ over $Y_{d,t}$, defined along the commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\mathrm{Pr}_2} & E_1 \times E_0 \\ \downarrow{\scriptstyle \mathrm{Pr}_1} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\mathrm{Bl}} & Y_{d,t} \end{array}$$

for the canonical projections $\mathrm{Pr}_1 : X \times (E_1 \times E_0) \to X$, $\mathrm{Pr}_2 : X \times (E_1 \times E_0) \to E_1 \times E_0$. *Then:*

(*i*) *there is a normal subgroup* Γ_0 *of* Γ *with quotient group* $\Gamma/\Gamma_0 = G_{d,t}$, *such that* $X_0 = (\mathbb{B}/\Gamma_0)'$ *is the smooth toroidal compactification of* $\mathbb{B}/\Gamma_0$;

(*ii*) $\zeta := \mathrm{Pr}_1 : X_0 \to X$ *is an unramified* $G_{d,t}$*-Galois covering, which restricts to unramified* $G_{d,t}$*-Galois coverings* $\zeta : \mathbb{B}/\Gamma_0 \to \mathbb{B}/\Gamma$, $\zeta : T \to D$ *of the ball quotients and of the toroidal compactifying divisors* $T := X_0 \setminus (\mathbb{B}/\Gamma_0)$, *respectively,* $D := X \setminus (\mathbb{B}/\Gamma)$;

(*iii*) *the smooth irreducible rational* (-1)*-curves* $L = \coprod_{i=1}^{s} L_i$ *of* X *pull back to the smooth irreducible rational* (-1)*-curves* $\zeta^{-1}(L) = \coprod_{i=1}^{s} \zeta^{-1}(L_i)$, $\zeta^{-1}(L_i) = \coprod_{j=1}^{dt} L_{ij}$ *on* X_0 *and* $\mathrm{Bl}_0 := \mathrm{Pr}_2 : X_0 = (\mathbb{B}/\Gamma_0)' \to E_1 \times E_0$ *is the blow down of* $\zeta^{-1}(L)$;

(*iv*) $B := \mathrm{Bl}_0(T)$ *and* $C := \mathrm{Bl}(D)$ *are related by an unramified* $G_{d,t}$*-Galois covering* $\pi : B \to C$, *which restricts to an unramified* $G_{d,t}$*-Galois covering* $\pi : B^{\mathrm{sing}} \to C^{\mathrm{sing}}$ *of their singular loci;*

(*v*) *the contractions* $\rho_0 : X_0 = (\mathbb{B}/\Gamma_0)' \to \widehat{\mathbb{B}/\Gamma_0}$, *respectively,* $\rho : X = (\mathbb{B}/\Gamma)' \to \widehat{\mathbb{B}/\Gamma}$ *of the smooth elliptic ireducible components of* T, *respectively,* D *to the cusps provide a commutative diagram*

$$\begin{array}{ccc} \widehat{\mathbb{B}/\Gamma_0} & \xleftarrow{\rho_0} & (\mathbb{B}/\Gamma_0)' \\ \downarrow{\scriptstyle \widehat{\zeta}} & & \downarrow{\scriptstyle \zeta} \\ \widehat{\mathbb{B}/\Gamma} & \xleftarrow{\rho} & (\mathbb{B}/\Gamma)' \end{array} \quad , \tag{1}$$

where $\widehat{\zeta}$ is a $G_{d,t}$-Galois covering, whose branch locus is contained in the Γ-cusps $\partial_\Gamma \mathbb{B}/\Gamma$.

2.3. Realization of co-bi-elliptic modular forms by invariant abelian functions. In the notations from Corollary 1 and Subsection 2.1, Holzapfel's formula (4) from [11] asserts that for any $n \in \mathbb{N}$ there is a $\mathbb{C}$-linear isomorphism

$$\widehat{\zeta}^* : H^0\left(\widehat{\mathbb{B}/\Gamma}, \left(\Omega^{2,0}_{\mathbb{B}/\Gamma}\right)^{\otimes n}\right) \longrightarrow H^0\left(\widehat{\mathbb{B}/\Gamma_0}, \left(\Omega^{2,0}_{\mathbb{B}/\Gamma_0}\right)^{\otimes n}\right)^{G_{d,t}} \tag{2}$$

of the Γ-modular forms of weight n with the $G_{d,t}$-invariant Γ_0-modular forms of the same weight. The article [15] announces briefly that this isomorphism is compatible with the corresponding transfers and induces a $\mathbb{C}$-linear isomorphism

$$\pi^* : \mathcal{L}_{Y_{d,t}}\left(n\left(K_{Y_{d,t}} + C\right), nC^{\mathrm{sing}}\right) \longrightarrow \mathcal{L}_{E_1 \times E_0}\left(nB, nB^{\mathrm{sing}}\right)^{G_{d,t}}.$$

Towards a more detailed explanation, let us observe that (2) and the commutative diagram (1) induce a commutative diagram

$$\begin{array}{ccc} H^0\left(\widehat{\mathbb{B}/\Gamma_0}, \left(\Omega^{2,0}_{\mathbb{B}/\Gamma_0}\right)^{\otimes n}\right)^{G_{d,t}} & \xrightarrow{\rho_0^*} & H^0\left(X_0, \Omega^{2,0}_{X_0}(T)^{\otimes n}\right)^{G_{d,t}} \\ \uparrow{\scriptstyle \widehat{\zeta}^*} & & \uparrow{\scriptstyle \zeta^*} \\ H^0\left(\widehat{\mathbb{B}/\Gamma}, \left(\Omega^{2,0}_{\mathbb{B}/\Gamma}\right)^{\otimes n}\right) & \xrightarrow{\rho^*} & H^0\left(X, \Omega^{2,0}_X(D)^{\otimes n}\right) \end{array} \tag{3}$$

of $\mathbb{C}$-linear isomorphisms. The abelian surface $E_1 \times E_0$ has trivial canonical divisor, so that the canonical divisor $K_{X_0} = \zeta^{-1}(L)$ of $X_0 = (\mathbb{B}/\Gamma_0)'$ coincides with the exceptional divisor $\zeta^{-1}(L)$ of $\mathrm{Bl}_0 : X_0 \to E_1 \times E_0$. An arbitrary meromorphic section σ of the logarithmic canonical bundle $\Omega^{2,0}_X(D) \to X$ pulls back to a $G_{d,t}$-invariant meromorphic section $\zeta^*(\sigma) = \sigma\zeta$ of $\Omega^{2,0}_{X_0}(T) \to X_0$ and there arises a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_{X_0}\left(n\left(K_{X_0} + T\right)\right)^{G_{d,t}} & \xrightarrow{(\sigma\zeta)^{\otimes n}} & H^0\left(X_0, \Omega^{2,0}_{X_0}(T)^{\otimes n}\right)^{G_{d,t}} \\ \uparrow{\scriptstyle \zeta^*} & & \uparrow{\scriptstyle \zeta^*} \\ \mathcal{L}_X(n(K_X + D)) & \xrightarrow{\sigma^{\otimes n}} & H^0\left(X, \Omega^{2,0}_X(D)^{\otimes n}\right) \end{array} \tag{4}$$

of $\mathbb{C}$-linear isomorphisms. Finally, the $\mathbb{C}$-algebra isomorphisms $\mathrm{Bl}^* : \mathbb{C}(Y) \to \mathbb{C}(X)$, $\mathrm{Bl}_0^* : \mathbb{C}(E_1 \times E_0) \to \mathbb{C}(X_0)$ fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_{E_1\times E_0}\left(nB, nB^{\mathrm{sing}}\right)^{G_{d,t}} & \xrightarrow{\mathrm{Bl}_0^*} & \mathcal{L}_{X_0}\left(n\left(K_{X_0}+T\right)\right)^{G_{d,t}} \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \zeta^*} \\ \mathcal{L}_{Y_{d,t}}\left(n\left(K_{Y_{d,t}}+C\right), nC^{\mathrm{sing}}\right) & \xrightarrow{\mathrm{Bl}^*} & \mathcal{L}_X\left(n\left(K_X+D\right)\right) \end{array} \tag{5}$$

of $\mathbb{C}$-linear isomorphisms. The above considerations imply the following

Corollary 2. *Let* $\mathrm{Bl} : X = (\mathbb{B}/\Gamma)' \to Y_{d,t}$ *be the blow down of the smooth irreducible rational* (-1)*-curves on a smooth toroidal compactification* X *with toroidal compactifying divisor* $D := X \setminus (\mathbb{B}/\Gamma)$ *to a bi-elliptic surface* $Y_{d,t} = E_1 \times E_0/G_{d,t}$ *and* $\mathrm{Bl}_0 : X_0 := X \times_{Y_{d,t}} (E_1 \times E_0) = (\mathbb{B}/\Gamma_0)' \to E_1 \times E_0$ *be the blow down of the smooth irreducible rational* (-1)*-curves on the lifting* X_0 *of* X *to a smooth toroidal compactification with toroidal compactifying divisor* $T := X_0 \setminus (\mathbb{B}/\Gamma_0)$ *and abelian minimal model* $E_1 \times E_0$*. Denote by* $B := \mathrm{Bl}_0(T) \subset E_1 \times E_0$*, respectively,* $C := \mathrm{Bl}(D) \subset Y_{d,t}$ *the images of the toroidal compactifying divisors on the corresponding minimal models and put* $\widehat{\zeta} : \widehat{\mathbb{B}/\Gamma_0} \to \widehat{\mathbb{B}/\Gamma}$ *for the* $G_{d,t}$*-Galois covering of the Baily-Borel compactifications, induced by* $\pi : E_1 \times E_0 \to Y_{d,t} = E_1 \times E_0/G_{d,t}$ *then for any* $n \in \mathbb{N}$ *there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{L}_{E_1\times E_0}\left(nB, nB^{\mathrm{sing}}\right)^{G_{d,t}} & \xrightarrow{\widetilde{\theta}_n} & H^0\left(\widehat{\mathbb{B}/\Gamma_0}, \left(\Omega^{2,0}_{\mathbb{B}/\Gamma_0}\right)^{\otimes n}\right)^{G_{d,t}} \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \widehat{\zeta}^*} \\ \mathcal{L}_{Y_{d,t}}\left(n\left(K_{Y_{d,t}}+C\right), nC^{\mathrm{sing}}\right) & \xrightarrow{\theta_n} & H^0\left(\widehat{\mathbb{B}/\Gamma}, \left(\Omega^{2,0}_{\mathbb{B}/\Gamma}\right)^{\otimes n}\right) \end{array} \tag{6}$$

of $\mathbb{C}$*-linear isomorphisms, relating the transfers* θ_n, $\widetilde{\theta}_n$ *to the pull back* π^* *of rational functions and to the pull back* $\widehat{\zeta}^*$ *of modular forms.*

P r o o f. By their very definition, the transfers are the compositions $\theta_n := (\rho^*)^{-1}\sigma^{\otimes n}\mathrm{Bl}^*$, respectively, $\widetilde{\theta}_n := (\rho_0^*)^{-1}(\sigma\zeta)^{\otimes n}\mathrm{Bl}_0^*$. The application of the comutativity of (5) yields $\widetilde{\theta}_n\pi^* = (\rho_0^*)^{-1}(\sigma\zeta)^{\otimes n}\mathrm{Bl}_0^*\pi^* = (\rho_o^*)^{-1}(\sigma\zeta)^{\otimes n}\zeta^*\mathrm{Bl}^*$. Then

the commutativity of (4) implies that $\widetilde{\theta}_n\pi^* = (\rho_0^*)^{-1}\,\zeta^*\sigma^{\otimes n}\mathrm{Bl}^*$. The comutativity of (3) reads as $\rho_0^*\widehat{\zeta}^* = \zeta^*\rho^*$. The left multiplication by $(\rho_0^*)^{-1}$ and the right multiplication by $(\rho^*)^{-1}$ provide the equalities $\widehat{\zeta}^*\,(\rho^*)^{-1} = (\rho_0^*)^{-1}\left(\rho_0^*\widehat{\zeta}^*\right)(\rho^*)^{-1} = (\rho_0^*)^{-1}\,(\zeta^*\rho^*)\,(\rho^*)^{-1} = (\rho_0^*)^{-1}\,\zeta^*$, whereas $\widetilde{\theta}_n\pi^* = \widehat{\zeta}^*\,(\rho^*)^{-1}\,\sigma^{\otimes n}\mathrm{Bl}^* = \widehat{\zeta}^*\theta_n$, which establishes the commutativity of (6). □

In [15] the $G_{d,t}$-action on $\mathcal{L}_{E_1\times E_0}\left(nB, nB^{\mathrm{sing}}\right)$ was mistaken, asserting that it is the product of the usual pull back with the multiplication by the $(-n)$-th power of the linear part of the corresponding element of $G_{d,t}$. In fact, it is the pull back by an entry of $G_{d,t}$. When the weight n is divisible by d, both actions coincide, so that the results of [15] except Theorem 6 are true without any correction. Theorem 6 holds for the rational functions

$$f_{134} := (\pi^*)^{-1}\,(e_{157} + \widetilde{\mathfrak{g}}^*(e_{157}))\,, \qquad f_{234} := (\pi^*)^{-1}\,(e_{467} + \widetilde{\mathfrak{g}}_2^*(e_{467}))\,,$$
$$f_3 := (\pi^*)^{-1}\left(\frac{\sigma(v)\sigma(-v+\lambda_3)}{\sigma(v-\lambda_1)\sigma(-v+\lambda_2)}\right),$$

instead of

$$f_{134} := (\pi^*)^{-1}\,(e_{157} - \widetilde{\mathfrak{g}}_2^*(e_{157}))\,, \qquad f_{234} := (\pi^*)^{-1}\,(e_{467} - \widetilde{\mathfrak{g}}_2^*(e_{467}))\,,$$
$$f_3 := (\pi^*)^{-1}\left(\frac{\sigma(u)\sigma(-u+\lambda_3)}{\sigma(u-\lambda_1)\sigma(-u+\Lambda_2)}\right).$$

In order to introduce the notion of a standard non-cuspidal modular form, we recall the following

Theorem 3 (Theorem 1.1 [15])**.** *Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification with bi-elliptic minimal model $Y_{d,t} = E_1\times E_0/G_{d,t}$, Euler number $e(X) \in \mathbb{N}$ and h cusps $\partial_\Gamma\mathbb{B} = \{\kappa_j \,|\, 1 \le j \le h\}$. Denote by $\mathcal{M}_n^o(\Gamma)$ the spaces of the cuspidal Γ-modular forms of weight $n \in \mathbb{N}$ and put $\mathcal{M}_n(\Gamma)$ for the spaces of all Γ-modular forms of weight n. Then*

$$\dim_{\mathbb{C}}\mathcal{M}_n^o(\Gamma) = 3e(X)\frac{n(n-1)}{2} \quad \textit{and} \quad \dim_{\mathbb{C}}\mathcal{M}_n(\Gamma) = \dim_{\mathbb{C}}\mathcal{M}_n^o(\Gamma) + h$$

for $n \ge 2$, as well as $\mathcal{M}_1^o = \{0\}$, $\dim_{\mathbb{C}}\mathcal{M}_1(\Gamma) \le h$. Moreover, for any $n \ge 2$ and $1 \le j \le h$ there exists $\omega_j^{[n]} \in j_n\mathcal{M}_n(\Gamma) \setminus j_n\mathcal{M}_n^o(\Gamma)$ with

$$\omega_j^{[n]}(\kappa_i) = \delta_{ij} = \begin{cases} 1 & \textit{for } 1 \le i = j \le h, \\ 0 & \textit{for } 1 \le i \ne j \le h, \end{cases} \tag{7}$$

which we call a standard non-cuspidal Γ-modular form of weight n.

Note that $\omega_1^{[n]},\ldots,\omega_h^{[n]} \in j_n\left(\mathcal{M}_n(\Gamma)\right)$ are defined up to summands from $j_n\left(\mathcal{M}_n^o(\Gamma)\right)$.

At the end of Subsection 2.1 we have explained that the transfer $\theta_n(f) \in j_n\left(\mathcal{M}_n(\Gamma)\right)$ of a rational function $f \in \mathcal{L}_{Y_{d,t}}\left(n\left(K_{Y_{d,t}}+C\right), nC^{\mathrm{sing}}\right)$ vanishes at a cusp $\kappa_i \in \partial_\Gamma \mathbb{B}/\Gamma$ if and only if the pole of f along $C_i := \mathrm{Bl}\rho^{-1}(\kappa_i)$ is of order $\leq n-1$. Thus, $\theta_n(f)(\kappa_i) \neq 0$ exactly when f has a pole of order n along C_i. This occurs if and only if the abelian function $\pi^*(f) \in \mathcal{L}_{E_1\times E_0}\left(nB, nB^{\mathrm{sing}}\right)^{G_{d,t}}$ has a pole of order n along $\pi^{-1}(C_i) \subset \pi^{-1}(C) = B$.

Here is a criterion for $G_{d,t}$-invariant abelian functions on $E_1 \times E_0$ to provide a projective morphism or a projective embedding of $\widehat{\mathbb{B}/\Gamma}$ by Γ-modular forms.

Proposition 4 (Lemma 2 and Proposition 4 (ii) [15])**.** *Let* $\mathrm{Bl} : X = (\mathbb{B}/\Gamma)' \to Y_{d,t}$ *be the blow down of the smooth irreducible rational* (-1)*-curves on a smooth toroidal compactification* $X = (\mathbb{B}/\Gamma)'$ *to a bi-elliptic surface* $Y_{d,t} = E_1 \times E_0/G_{d,t}$*,* $D := X \setminus (\mathbb{B}/\Gamma)$ *be the toroidal compactifying divisor of* $\mathbb{B}/\Gamma$*,* $\partial_\Gamma \mathbb{B}/\Gamma = \{\kappa_j \,|\, 1 \leq j \leq h\}$ *be the set of the* Γ*-cusps,* $\mathrm{Bl}_0 : X_0 := X \times_{Y_{d,t}} (E_1 \times E_0) = (\mathbb{B}/\Gamma_0)' \to E_1 \times E_0$ *be the blow down of the smooth irreducible rational* (-1)*-curves on the* $G_{d,t}$*-Galois cover* $X_0 = (\mathbb{B}/\Gamma_0)'$ *of* X *to its abelian minimal model* $E_1 \times E_0$*,* $T := X_0 \setminus (\mathbb{B}/\Gamma_0)$ *be the toroidal compactifying divisor of* $\mathbb{B}/\Gamma_0$ *and* $n \in d\mathbb{N}$ *be a multiple of* d*, so that* $nK_{Y_{d,t}} = 0$*. Denote* $B := \mathrm{Bl}_0(T)$*,* $C := \mathrm{Bl}(D)$*, put* $i_n := \pi^*|_{\mathcal{L}_{Y_{d,t}}(nC,nC^{\mathrm{sing}})} : \mathcal{L}_{Y_{d,t}}\left(nC, nC^{\mathrm{sing}}\right) \to \mathcal{L}_{E_1\times E_0}\left(nB, nB^{\mathrm{sing}}\right)^{G_{d,t}}$ *and assume that* $f_1,\ldots,f_k \in \mathcal{L}_{E_1\times E_0}\left(nB, nB^{\mathrm{sing}}\right)^{G_{d,t}}$.

(i) If there exist such $1 \leq j_1 < \ldots < j_h \leq k$ *that* $\omega_s^{[n]} := \theta_n i_n^{-1}(f_{j_s}) \in j_n\left(\mathcal{M}_n(\Gamma)\right)$ *with (7) are standard non-cuspidal* Γ*-modular forms of weight* n *then*

$$\Phi = [\theta_n(1) : \theta_n i_n^{-1}(f_1) : \ldots : \theta_n i_n^{-1}(f_k)] : \widehat{\mathbb{B}/\Gamma} \longrightarrow \mathbb{P}^k(\mathbb{C}) \tag{8}$$

is a regular map.

(ii) If there holds (i) and for some $1 \leq m \leq k$ *the generic fibre of the rational map*

$$f = (f_1,\ldots,f_m) : E_1 \times E_0 \longrightarrow \mathbb{P}^1(\mathbb{C})^m$$

consists of a single $G_{d,t}$*-orbit then (8) is a regular projective embedding of the Baily-Borel compactification* $\widehat{\mathbb{B}/\Gamma}$ *by* Γ*-modular forms of weight* n*.*

2.4. Di Cerbo and Stover's classification of the smooth toroidal compactifications with minimal Haar measure. By Theorem 2.5 (3) from

Holzapfel's [13], any abelian minimal model A of a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ is isogeneous to a Cartesian square $E \times E$ of an elliptic curve E. Corollary 10.6.3 from Lange-Birkenhake's monograph [19] specifies that if E has a complex multiplication by an imaginary quadratic number field $\mathbb{Q}(\sqrt{-m})$ for some $m \in \mathbb{N}$ then A is isomorphic to $E \times E$. An arbitrary smooth irreducible elliptic curve $C_0 \subset E \times E$ through the origin is a Lie subgroup $(C_0, +)$ of $(E \times E, +)$. That is why, the identical inclusion of groups $(C_0, +) < (E \times E, +)$ lifts to an injective homomorphism

$$\iota : \mathrm{Lie}(C_0) = \mathbb{C} \longrightarrow \mathbb{C}^2 = \mathrm{Lie}(E \times E)$$

of the coresponding Lie algebras. In particular, ι is a $\mathbb{C}$-linear embedding and its image $\iota(\mathrm{Lie}(C_0)) = \mathrm{Span}_{\mathbb{C}}(\alpha_0, \beta_0)$ is a complex line through the origin, generated by some vector $(\alpha_0, \beta_0) \in \mathbb{C}^2 \setminus \{(0,0)\}$. As a result, if $E = (\mathbb{C}, +)/(\Lambda, +)$ for a latice $(\Lambda, +) \simeq (\mathbb{Z}^2, +)$ of $(\mathbb{C}, +)$ then the elliptic curve C_0 can be represented as the projection

$$C_0 = E(\alpha_0, \beta_0) := \{(\alpha_0 c + \Lambda, \beta_0 c + \Lambda) \in E \times E \,|\, c \in \mathbb{C}\}$$

of the complex line $\mathrm{Span}_{\mathbb{C}}(\alpha_0, \beta_0) \subset \mathbb{C}^2$ by the universal covering

$$\begin{aligned} U_{E\times E} : (\mathbb{C}^2, +) \longrightarrow (E \times E, +) &= (\mathbb{C}^2, +)/(\Lambda \times \Lambda, +) \\ &= [(\mathbb{C}, +)/(\Lambda, +)] \times [(\mathbb{C}, +)/(\Lambda, +)]\,, \end{aligned}$$

$$U_{E\times E}(u, v) = (u, v) + \Lambda \times \Lambda = (u + \Lambda, v + \Lambda), \quad \forall (u, v) \in \mathbb{C}^2.$$

Moreover, if $\alpha_0, \beta_0 \in \mathbb{C}^*$ then

$$E(\alpha_0, \beta_0) = \{(u + \Lambda, v + \Lambda) \in E \times E \,|\, \beta_0 u - \alpha_0 v \in \beta_0 \Lambda + \alpha_0 \Lambda\},$$

as far as the assumption $\beta_0 u - \alpha_0 v = \beta_0 \lambda_2 + \alpha_0 \lambda_1$ for some $\lambda_1, \lambda_2 \in \Lambda$ implies that $\dfrac{u - \lambda_2}{\alpha_0} = \dfrac{v + \lambda_1}{\beta_0}$, whereas

$$(u + \Lambda, v + \Lambda) = \left(\alpha_0 \left(\frac{u - \lambda_2}{\alpha_0}\right) + \Lambda, \beta_0 \left(\frac{v + \lambda_1}{\beta_0}\right) + \Lambda\right) = (\alpha_0 c + \Lambda, \beta_0 c + \lambda)$$

for $c := \dfrac{u - \lambda_2}{\alpha_0}$. An arbitrary smooth ireducible elliptic curve $C_1 \subset E \times E$ is a translate

$$C_1 = E(\alpha_0, \beta_0) + (u_1 + \Lambda, v_1 + \Lambda)$$

$$= \{(\alpha_0 c + u_1 + \Lambda, \beta_0 c + v_1 + \Lambda) \in E \times E \,|\, c \in \mathbb{C}\}$$
$$= \{(u + \Lambda, v + \Lambda) \in E \times E \,|\, \beta_0(u - u_1) - \alpha_0(v - v_1) \in \beta_0\Lambda + \alpha_0\Lambda\}$$

of an elliptic curve $E(\alpha_0, \beta_0) \subset E \times E$ through the origin.

In his article [8] (cf.also [9]) Hirzebruch computes the $\mathrm{SU}(n,1)$-invariant Haar measure $\mathrm{vol}\,(\mathbb{B}^n/\Gamma)$ of a quotient $\mathbb{B}^n/\Gamma$ of the complex n-ball

$$\mathbb{B}^n = \mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}_n \times \mathrm{U}_1)$$

by a lattice $\Gamma < \mathrm{SU}(n,1)$. For $\mathbb{B} = \mathbb{B}^2$ his formula reduces to $\mathrm{vol}\,(\mathbb{B}/\Gamma) = \frac{8\pi^2}{3} e\,(\mathbb{B}/\Gamma)$, where $e\,(\mathbb{B}/\Gamma)$ stands for the Euler number of $\mathbb{B}/\Gamma$. If $\mathbb{B}/\Gamma$ is smooth then $e\,(\mathbb{B}/\Gamma) \in \mathbb{Z}$, whereas $e\,(\mathbb{B}/\Gamma) \in \mathbb{N}$, due to $e\,(\mathbb{B}/\Gamma) = \frac{3}{8\pi^2}\mathrm{vol}\,(\mathbb{B}/\Gamma) \in \mathbb{R}^{>0}$. Hersonsky and Paulin show in [7] that the minimal Haar volume of a compact smooth $\mathbb{B}/\Gamma$ is $8\pi^2$. Parker's [20] establishes that the minimal Haar volume of a noncompact smooth $\mathbb{B}/\Gamma$ is $\frac{8\pi^2}{3}$.

Here is Di Cerbo and Stover's classification of the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ with minimal Haar measure $\mathrm{vol}\,(\mathbb{B}/\Gamma) = \frac{8\pi^2}{3}$.

Theorem 5 (Di Cerbo and Stover, Theorem 1.1 [4])**.** *Let* $E = (\mathbb{C}, +)/(\Lambda, +)$ *be the elliptic curve, whose lattice* $\Lambda = \mathcal{O}_{\sqrt{-3}} = \mathbb{Z} + \mathbb{Z}\omega$, $\omega := \mathrm{e}^{\frac{2\pi\sqrt{-1}}{6}}$ *is the ring of the Eisenstein integers,*

$$g_3 = \left(\mathcal{T}\left(\frac{1+\omega}{3} + \Lambda\right), \omega^2\right), \quad \tau_3 := \left(\mathcal{T}\left(\frac{1}{3} + \Lambda\right), \mathcal{T}\left(\frac{1+\omega}{3} + \Lambda\right)\right),$$

$$G_3 := \langle g_3 \rangle \simeq (\mathbb{Z}_3, +), \quad G_{3,3} := \langle g_3 \rangle \times \langle \tau_3 \rangle \simeq (\mathbb{Z}_3 \times \mathbb{Z}_3, +) \quad \textit{and}$$

$\pi_3 : E \times E \longrightarrow Y_3 = E \times E/G_3$, *respectively,* $\pi_{3,3} : E \times E \longrightarrow Y_{3,3} = E \times E/G_{3,3}$ *be the canonical Galois coverings of the bi-elliptic surfaces* $Y_3, Y_{3,3}$ *by* $E \times E$. *Up to an isomorphism, there exist five smooth toroidal compactifications* $X_i = (\mathbb{B}/\Gamma_i)'$, $0 \leq i \leq 4$ *of minimal Haar measure* $\mathrm{vol}(X_i) = \mathrm{vol}\,(\mathbb{B}/\Gamma_i) = \frac{8\pi^2}{3}$ *or, equivalently, of minimal Euler number* $e(X_i) = e\,(\mathbb{B}/\Gamma_i) = 1$. *If* $\mathrm{Bl}_i : X_i = (\mathbb{B}/\Gamma_i)' \to S_i$ *is the blow down of the smooth irreducible rational* (-1)*-curves on* X_i *to a minimal surface* S_i *and* $D^{(i)} := X_i \setminus (\mathbb{B}/\Gamma_i)$ *is the toroidal compactifying divisor of* $\mathbb{B}/\Gamma_i$ *then* X_0 *is Hirzebruch's example from [10] with abelian minimal model* $S_0 = E \times E$, $\mathrm{Bl}_0\left(D^{(0)}\right) = E(0,1) + E(1,0) + E(1,1) + E(1,\omega)$ *and* $\mathrm{Bl}_0\left(D^{(0)}\right)^{\mathrm{sing}} = \{(\Lambda, \Lambda)\}$,

X_i with $1 \leq i \leq 2$ have bi-elliptic minimal model $S_i = Y_3$, $\mathrm{Bl}_i\left(D^{(i)}\right) = C_0 + C_i$ with $C_0 := \pi_3(E(1,1))$, $C_1 := \pi_3\left(\left(\frac{1}{3}+\Lambda\right)\times E\right)$, $C_2 := \pi_3\left(E\times\left(\frac{1}{3}+\Lambda\right)\right)$ and $\mathrm{Bl}_i\left(D^{(i)}\right)^{\mathrm{sing}} = \left\{\pi_3\left(\frac{1}{3}+\Lambda,\frac{1}{3}+\Lambda\right)\right\}$, while X_i with $3 \leq i \leq 4$ have bi-elliptic minimal model $S_i = Y_{3,3}$, $\mathrm{Bl}_i\left(D^{(i)}\right) = C_0' + C_i'$ with

$$C_0' := \pi_{3,3}(E(1,1+\omega)), \quad C_1' := \pi_{3,3}(\Lambda\times E), \quad C_2' := \pi_{3,3}(E\times\Lambda)$$

and $\mathrm{Bl}_i\left(D^{(i)}\right)^{\mathrm{sing}} = \{\pi_{3,3}(\Lambda,\Lambda)\}$.

From now on, we use throughout the notations from Theorem 5.

3. Explicit invariant abelian functions, associated with co-bi-elliptic modular forms. The present section constructs explicit G_3-invariant abelian functions $e_0, f_{i0}, f_{i1}, f_{i2}$ from

$$\mathcal{L}_{E\times E}\left(3\pi_3^{-1}(C_0+C_i), 3\pi_3^{-1}(C_0+C_i)^{\mathrm{sing}}\right)^{G_3},$$

which induce Γ_i-modular forms of weight 3 and provide regular embeddings $\Phi_i : \widehat{\mathbb{B}/\Gamma_i} \to \mathbb{P}^5(\mathbb{C})$ for the smooth toroidal compactifications $X_i = (\mathbb{B}/\Gamma_i)'$ with $e(X_i) = 1$, $1 \leq i \leq 2$ from Di Cerbo and Stover's Theorem 1.1 from [4], formulated above as Theorem 5. To this end, we use the Weierstrass sigma-function

$$\sigma(z) := z \prod_{\lambda\in\Lambda\setminus\{0\}} \left(1-\frac{z}{\lambda}\right) \mathrm{e}^{\frac{z}{\lambda}+\frac{1}{2}\left(\frac{z}{\lambda}\right)^2}$$

of the lattice $\Lambda = \mathcal{O}_{\sqrt{-3}} = \mathbb{Z}+\mathbb{Z}\omega$, $\omega := \mathrm{e}^{\frac{2\pi\sqrt{-1}}{6}}$. One observes immediately that

$$\sigma(\xi z) = \xi\sigma(z), \quad \forall z \in \mathbb{C}, \quad \forall \xi \in \mathrm{Aut}_{\mathrm{gr}}(E) = \mathcal{O}^*_{\sqrt{-3}} = \langle\omega\rangle, \tag{9}$$

The next lemma formulates the $(\Lambda,+)$-transformation law and Legendre's relation of a Weierstrass sigma-function in the case of $\Lambda = \mathcal{O}_{\sqrt{-3}}$.

Lemma 6 (Section 18.1, Lang [18])**.** *Let $\varepsilon : \Lambda = \mathcal{O}_{\sqrt{-3}} = \mathbb{Z}+\mathbb{Z}\omega \to \{\pm 1\}$ be the map with $\varepsilon(\lambda) = -1$ for $\lambda \in \Lambda \setminus 2\Lambda$ and $\varepsilon(\lambda) = 1$ for $\lambda \in 2\Lambda$. There exists a group homomorphism $\eta : (\Lambda,+) \to (\mathbb{C},+)$, subject to Legendre's relation*

$$\eta(\omega) - \omega\eta(1) = 2\pi\sqrt{-1}, \tag{10}$$

such that

$$\frac{\sigma(z+\lambda)}{\sigma(z)} = \varepsilon(\lambda)\mathrm{e}^{\eta(\lambda)\left(z+\frac{\lambda}{2}\right)}, \quad \forall z \in \mathbb{C}, \quad \forall \lambda \in \Lambda. \tag{11}$$

The construction of one of our G_3-invariant abelian functions on $E \times E$ will provide an explicit formula for η, which we formulate in Corollary 9.

The next lemma formulates some properties of elliptic functions, which will be used several times in the sequel.

Lemma 7. (*i*) (Section 18.1, Lang [18]) *Let* $E = (\mathbb{C},+)/(\Lambda,+)$ *be an elliptic curve with lattice* Λ *and* $\sigma(z)$ *be its associated Weierstrass sigma-function. Then for any complex numbers* $a_i, b_i \in \mathbb{C}$, $1 \le i \le k$ *with* $\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i$ *the* σ*-quotient*

$$f(z) := \prod_{i=1}^{k} \frac{\sigma(z-b_i)}{\sigma(z-a_i)}$$

is $(\Lambda,+)$*-periodic and descends to an elliptic function* $f : E \to \mathbb{P}^1(\mathbb{C})$. *i.e., a rational function on the elliptic curve* E.

(*ii*) (Subsection 1.1, Lang [18]) *If* $f : E = \mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$ *is an elliptic function with poles* $\{a_i + \Lambda \,|\, 1 \le i \le k\}$, *counted with their multiplicities then for any point* $x \in \mathbb{P}^1(\mathbb{C})$ *the fibre* $f^{-1}(x) = \{c_i + \Lambda \,|\, 1 \le i \le k\}$ *consists of* k *points, counted with their multiplicities and having sum* $\sum_{i=1}^{k}(c_i+\Lambda) = \sum_{i=1}^{k}(a_i+\Lambda)$.

Note that the entire function $\sigma : \mathbb{C} \to \mathbb{C}$ is not $(\Lambda,+)$-periodic, but its zero divisor on $\mathbb{C}$ is Λ. That allows to view the origin $\Lambda \in E$ of $E = \mathbb{C}/\Lambda$ as the divisor of σ on E. If we want to construct an abelian function on $E \times E$ (i.e., a rational function on the abelian surface $E \times E$), whose pole divisor is a sum

$$\sum_{j=1}^{k} E(\alpha_j, \beta_j) + (u_j + \Lambda, v_j + \Lambda)$$
$$= \sum_{j=1}^{k} \{(u+\Lambda, v+\Lambda) \in E \times E \,|\, \beta_j u - \alpha_j v + \alpha_j v_j - \beta_j u_j \in \beta_j\Lambda + \alpha_j\Lambda\}$$

of smooth elliptic curves with $\beta_j\Lambda + \alpha_j\Lambda = \Lambda$ then we need to form a σ-quotient, whose denominator is

$$\prod_{j=1}^{k} \sigma(\beta_j u - \alpha_j v + \alpha_j v_j - \beta_j u_j).$$

Throughout, let us denote by $X_i' := \left(\mathbb{B}/\Gamma_i'\right)'$ the smooth toroidal compactifications with abelian minimal model $E \times E$, which admit unramified G_3-Galois coverings $\zeta_i : X_i' \to X_i$ onto Di Cerbo and Stover's examples $X_i = (\mathbb{B}/\Gamma_i)'$, $1 \leq i \leq 2$ from Theorem 5. (For the existence and the properties of X_i' see Corollary 1). Let us consider the blow downs $\mathrm{Bl}_i' : X_i' = \left(\mathbb{B}/\Gamma_i'\right)' \to E \times E$, respectively, $\mathrm{Bl}_i : X_i = (\mathbb{B}/\Gamma_i)' \to Y_3$ of the smooth irreducible rational (-1)-curves and the images $B_i := \mathrm{Bl}_i'\left(X_i' \setminus \left(\mathbb{B}/\Gamma_i'\right)\right)$, respectively, $C_0 + C_i := \mathrm{Bl}_i\left(X_i \setminus (\mathbb{B}/\Gamma_i)\right)$ of the corresponding toroidal compactifying divisors under these blow downs. The unramified G_3-Galois covering $\pi_3 : E \times E \to Y_3 = E \times E/G_3$ restricts to unramified G_3-Galois coverings $\pi_3 : B_i \to C_0 + C_i$ and $\pi_3 : B_i^{\mathrm{sing}} \to (C_0 + C_i)^{\mathrm{sing}}$.

Proposition 8. *Let*

$$\pi_3 : \left(E \times E, B_i, B_i^{\mathrm{sing}}\right) \longrightarrow \left(Y_3, C_0 + C_i, (C_0 + C_i)^{\mathrm{sing}}\right), \quad 1 \leq i \leq 2$$

with $B_i = \mathrm{Orb}_{G_3}(B_{00}) + \mathrm{Orb}_{G_3}(B_{i0})$, $B_{00} := E(1,1)$, $B_{10} := \left(\frac{1}{3} + \Lambda\right) \times E$, $B_{20} := E \times \left(\frac{1}{3} + \Lambda\right)$, $g_3 := \left(\mathcal{T}\left(\frac{1+\omega}{3} + \Lambda\right), \omega^2\right)$ *be the unramified* $G_3 = \langle g_3 \rangle$*-Galois coverings of the minimal models of* $\zeta_i : X_i' = \left(\mathbb{B}/\Gamma_i'\right)' \to X_i = (\mathbb{B}/\Gamma_i)'$. *Then*

$$e_0(u,v) := \frac{e_0^+(u,v)}{e_0^-(u,v)} \quad \textit{with} \tag{12}$$

$$e_0^+(u,v) := \sigma\left(u - v + \frac{2\omega - 4}{3}\right)\sigma\left(u + \omega v + \frac{1 - 5\omega}{3}\right)\sigma\left(u - \omega^2 v\right), \tag{13}$$

$$e_0^-(u,v) := \sigma(u - v)\,\sigma\left(u + \omega v - \left(\frac{1+\omega}{3}\right)\right)\sigma\left(u - \omega^2 v - 2\left(\frac{1+\omega}{3}\right)\right) \tag{14}$$

belongs, both, to $\mathcal{L}_{E\times E}\left(B_1, 3B_1^{\mathrm{sing}}\right)^{G_3}$ *and to* $\mathcal{L}_{E\times E}\left(B_2, 3B_2^{\mathrm{sing}}\right)^{G_3}$.

P r o o f. Towards the $(\Lambda, +)$-periodicity of $e_0(u,v)$ with respect to u, one makes use of (11) and $\omega^2 - \omega + 1 = 0$, in order to compute that

$$\begin{aligned}\frac{e_0^+(u+\lambda, v)}{e_0^+(u,v)} &= \frac{\sigma\left(u - v + \frac{2\omega-4}{3} + \lambda\right)}{\sigma\left(u - v + \frac{2\omega-4}{3}\right)} \frac{\sigma\left(u + \omega v + \frac{1-5\omega}{3} + \lambda\right)}{\sigma\left(u + \omega v + \frac{1-5\omega}{3}\right)} \frac{\sigma\left(u - \omega^2 v + \lambda\right)}{\sigma(u - \omega^2 v)} \\ &= \varepsilon(\lambda)^3 \mathrm{e}^{\eta(\lambda)\left[u - v + \frac{2\omega-4}{3} + \frac{\lambda}{2}\right]} \mathrm{e}^{\eta(\lambda)\left[u + \omega v + \frac{1-5\omega}{3} + \frac{\lambda}{2}\right]} \mathrm{e}^{\eta(\lambda)\left[u - \omega^2 v + \frac{\lambda}{2}\right]}\end{aligned}$$

$$= \varepsilon(\lambda)\mathrm{e}^{\eta(\lambda)\left(3u+\frac{3\lambda}{3}-1-\omega\right)}$$

Similarly,

$$\frac{e_0^-(u+\lambda,v)}{e_0^-(u,v)} = \frac{\sigma(u-v+\lambda)}{\sigma(u-v)}\frac{\sigma\left(u+\omega v-\left(\frac{1+\omega}{3}\right)+\lambda\right)}{\sigma\left(u+\omega v-\left(\frac{1+\omega}{3}\right)\right)}\frac{\sigma\left(u-\omega^2 v-2\left(\frac{1+\omega}{3}\right)+\lambda\right)}{\sigma\left(u-\omega^2 v-2\left(\frac{1+\omega}{3}\right)\right)}$$
$$= \varepsilon(\lambda)^3\mathrm{e}^{\eta(\lambda)\left[u-v+\frac{\lambda}{2}\right]}\mathrm{e}^{\eta(\lambda)\left[u+\omega v-\left(\frac{1+\omega}{3}\right)+\frac{\lambda}{2}\right]}\mathrm{e}^{\eta(\lambda)\left[u-\omega^2 v-2\left(\frac{1+\omega}{3}\right)+\frac{\lambda}{2}\right]},$$

$$\frac{e_0^-(u+\lambda,v)}{e_0^-(u,v)} = \varepsilon(\lambda)\mathrm{e}^{\eta(\lambda)\left(3u+\frac{3\lambda}{3}-1-\omega\right)}, \quad \forall\lambda\in\Lambda, \quad \forall(u,v)\in\mathbb{C}^2 \tag{15}$$

by (11) and $\omega^2-\omega+1=0$. As a result,

$$\frac{e_0(u+\lambda,v)}{e_0(u,v)} = \frac{e_0^+(u+\lambda,v)}{e_0^+(u,v)}\frac{e_0^-(u,v)}{e_0^-(u+\lambda,v)} = 1, \quad \forall\lambda\in\Lambda, \quad \forall(u,v)\in\mathbb{C}^2.$$

This establishes the $(\Lambda,+)$-periodicity of $e_0(u,v)$ with respect to u. Towards the $(\Lambda,+)$-periodicity of $e_0(u,v)$ with respect to v, note that $u-\omega^{2i}v+c_i = -\omega^{2i}(-\omega^{4i}u+v-\omega^{4i}c_i)$ for $1\le i\le 2$, $c_i\in\mathbb{C}$, $(u,v)\in\mathbb{C}^2$ and apply (9) to represent

$$e_0^+(u,v) = \sigma\left(u-v+\frac{2\omega-4}{3}\right)\sigma\left(-\omega^2 u+v-\left(\frac{\omega+4}{3}\right)\right)\sigma\left(\omega u+v\right),$$

$$e_0^-(u,v) = \sigma(u-v)\sigma\left(-\omega^2 u+v+\frac{\omega-2}{3}\right)\sigma\left(\omega u+v+\frac{2-4\omega}{3}\right).$$

Then for any $\lambda\in\Lambda$ and $(u,v)\in\mathbb{C}^2$ one has

$$\frac{e_0^+(u,v+\lambda)}{e_0^+(u,v)} = \frac{\sigma\left(u-v+\frac{2\omega-4}{3}-\lambda\right)}{\sigma\left(u-v+\frac{2\omega-4}{3}\right)}\frac{\sigma\left(-\omega^2 u+v-\left(\frac{\omega+4}{3}\right)+\lambda\right)}{\sigma\left(-\omega^2 u+v-\left(\frac{\omega+4}{3}\right)\right)}\frac{\sigma(\omega u+v+\lambda)}{\sigma(\omega u+v)}$$
$$= \varepsilon(\lambda)^3\mathrm{e}^{-\eta(\lambda)\left(u-v+\frac{2\omega-4}{3}-\frac{\lambda}{2}\right)}\mathrm{e}^{\eta(\lambda)\left[-\omega^2 u+v-\left(\frac{\omega+4}{3}\right)+\frac{\lambda}{2}\right]}\mathrm{e}^{\eta(\lambda)\left(\omega u+v+\frac{\lambda}{2}\right)}$$
$$= \varepsilon(\lambda)\mathrm{e}^{\eta(\lambda)\left(3v+\frac{3\lambda}{2}-\omega\right)},$$

$$\frac{e_0^-(u,v+\lambda)}{e_0^-(u,v)} = \frac{\sigma(u-v-\lambda)}{\sigma(u-v)}\frac{\sigma\left(-\omega^2 u+v+\frac{\omega-2}{3}+\lambda\right)}{\sigma\left(-\omega^2 u+v+\frac{\omega-2}{3}\right)}\frac{\sigma\left(\omega u+v+\frac{2-4\omega}{3}+\lambda\right)}{\sigma\left(\omega u+v+\frac{2-4\omega}{3}\right)}$$
$$= \varepsilon(\lambda)^3\mathrm{e}^{-\eta(\lambda)\left(u-v-\frac{\lambda}{2}\right)}\mathrm{e}^{\eta(\lambda)\left(-\omega^2 u+v+\frac{\omega-2}{3}+\frac{\lambda}{2}\right)}\mathrm{e}^{\eta(\lambda)\left(\omega u+v+\frac{2-4\omega}{3}+\frac{\lambda}{2}\right)},$$

$$\frac{e_0^-(u,v+\lambda)}{e_0^-(u,v)} = \varepsilon(\lambda)\mathrm{e}^{\eta(\lambda)\left(3v+\frac{3\lambda}{2}-\omega\right)}, \quad \forall\lambda\in\Lambda, \quad \forall(u,v)\in\mathbb{C}^2, \tag{16}$$

according to (11). Altogether,

$$\frac{e_0(u,v+\lambda)}{e_0(u,v)} = \frac{e_0^+(u,v+\lambda)}{e_0^+(u,v)}\frac{e^-(u,v)}{e_0^-(u,v+\lambda)} = 1, \quad \forall\lambda\in\Lambda, \quad (u,v)\in\mathbb{C}^2$$

and $e_0(u,v)$ is $(\Lambda,+)$-periodic with respect to v. The above considerations establish that $e_0(u,v)$ is $(\Lambda\times\Lambda,+)$-periodic and can be viewed as an abelian function on $E\times E$.

We claim that the pole divisor of $e_0(u,v)$ on $E\times E$, which coincides with the zero divisor of $e_0^-(u,v)$ is the G_3-orbit $\sum_{j=0}^{2} g_3^j(B_{00})$ of B_{00}, in order to conclude that e_0 belongs to $\mathcal{L}_{E\times E}(B_1) := \{f\in\mathbb{C}(E\times E)\,|\,(f)+B_1\geq 0\}\cup\{0\}$ and to $\mathcal{L}_{E\times E}(B_2)$. Indeed,

$$B_{00} := E(1,1) = \{(u+\Lambda,v+\Lambda)\in E\times E\,|\,u-v\in\Lambda\}, \tag{17}$$

$$\begin{aligned} B_{01} :=& g_3(B_{00}) = E(1,\omega^2)+\left(\frac{1+\omega}{3}+\Lambda,\Lambda\right)\\ =&\left\{(u+\Lambda,v+\Lambda)\in E\times E\,\middle|\,\omega^2\left[u-\left(\frac{1+\omega}{3}\right)\right]-v\in\omega^2\Lambda+\Lambda=\Lambda\right\}\\ =&\left\{(u+\Lambda,v+\Lambda)\in E\times E\,|\,u+\omega v-\left(\frac{1+\omega}{3}\right)\in\Lambda\right\} \end{aligned} \tag{18}$$

$$\begin{aligned} B_{02} :=& g_3^2(B_{00}) = E(1,-\omega)+\left(\frac{2+2\omega}{3}+\Lambda,\Lambda\right)\\ =&\left\{(u+\Lambda,v+\Lambda)\in E\times E\,\middle|\,-\omega\left[u-\left(\frac{2+2\omega}{3}\right)\right]-v\in-\omega\Lambda+\Lambda=\Lambda\right\}\\ =&\left\{(u+\Lambda,v+\Lambda)\in E\times E\,|\,u-\omega^2v-2\left(\frac{1+\omega}{3}\right)\in\Lambda\right\}, \end{aligned} \tag{19}$$

after multiplying the equation of B_{01} by $(-\omega)\in\mathrm{Aut}_{\mathrm{gr}}(E)=\mathcal{O}^*_{\sqrt{-3}}$ and the equation of B_{02} by $\omega^2\in\mathcal{O}^*_{\sqrt{-3}}$. Bearing in mind that $z\in\mathbb{C}$ is a root of the equation $\sigma(z)=0$ if and only if $z\in\Lambda$, one concludes that $(e_0)_\infty = \sum_{j=0}^{2} B_{0j} =$

$\sum_{j=0}^{2} g_3^j(B_{00})$. Due to $\sum_{j=0}^{2} g_3^j(B_{00}) < B_i$ for $1 \le i \le 2$, there follow $e_0 \in \mathcal{L}_{E\times E}(B_1)$ and $e_0 \in \mathcal{L}_{E\times E}(B_2)$.

Towards the G_3-invariance of e_0, let us note that the pull backs of $e_0^+(u,v)$ and $e_0^-(u,v)$ by $g_3 = \left(\mathcal{T}\left(\frac{1+\omega}{3}+\Lambda\right), \omega^2\right)$ are not correctly defined and depend on the choice of a lifting of g_3 to the affine linear group $\mathrm{Aff}(\mathbb{C}^2) = \mathcal{T}(\mathbb{C}^2) \rtimes \mathrm{GL}(2,\mathbb{C})$ of $\mathbb{C}^2$. However, any two liftings of g_3 transform their $(\Lambda\times\Lambda,+)$-periodic quotient $e_0 = \dfrac{e_0^+}{e_0^-}$ into one and a same abelian function. Throughout, we fix the lifting

$$\widetilde{g}_3 := \left(\mathcal{T}\left(\frac{1+\omega}{3}\right), \omega^2\right) \in \mathrm{Aff}(\mathbb{C}^2)$$

of g_3. Then

$$g_3^*(e_0) = \widetilde{g}_3^*\left(\frac{e_0^+}{e_0^-}\right) = \frac{\widetilde{g}_3^*(e_0^+)}{\widetilde{g}_3{}^*(e_0^-)} = \frac{e_0^+\left(u+\frac{1+\omega}{3}, \omega^2 v\right)}{e_0^-\left(u+\frac{1+\omega}{3}, \omega^2 v\right)} \quad \text{with}$$

$$\widetilde{g}_3^*(e_0^+)(u,v) = \sigma\left(u-v+\frac{2-4\omega}{3}\right)\sigma\left(u+\omega v+\frac{1+\omega}{3}\right)\sigma(u-\omega^2 v+\omega-1),$$

$$\widetilde{g}_3{}^*(e_0^-)(u,v) = \sigma(u-v)\sigma\left(u+\omega v-\left(\frac{1+\omega}{3}\right)\right)\sigma\left(u-\omega^2 v+\frac{1+\omega}{3}\right),$$

whereas

$$\begin{aligned}\frac{\widetilde{g}_3{}^*(e_0^-)}{e_0^-} &= \frac{\sigma\left(u-\omega^2 v+\frac{1+\omega}{3}\right)}{\sigma\left(u-\omega^2 v-2\left(\frac{1+\omega}{3}\right)\right)} \\ &= \frac{\sigma\left(u-\omega^2 v-2\left(\frac{1+\omega}{3}\right)+1+\omega\right)}{\sigma\left(u-\omega^2 v-2\left(\frac{1+\omega}{3}\right)\right)} = -\mathrm{e}^{\eta(1+\omega)\left[u-\omega^2 v-2\left(\frac{1+\omega}{3}\right)+\frac{1+\omega}{2}\right]},\end{aligned}$$

$$\frac{\widetilde{g}_3^*(e_0^-)}{e_0^-} = -\mathrm{e}^{\eta(1+\omega)\left[u-\omega^2 v-\left(\frac{1+\omega}{6}\right)\right]}, \tag{20}$$

respectively,

$$\frac{\widetilde{g}_3^*(e_0^+)}{e_0^+} = \frac{\sigma\left(u-v+\frac{2-4\omega}{3}\right)}{\sigma\left(u-v+\frac{2\omega-4}{3}\right)}\frac{\sigma\left(u+\omega v+\frac{1+\omega}{3}\right)}{\sigma\left(u+\omega v+\frac{1-5\omega}{3}\right)}\frac{\sigma\left(u-\omega^2 v+\omega-1\right)}{\sigma(u-\omega^2 v)}$$

$$
\begin{aligned}
&= \frac{\sigma\big(u-v+\frac{2\omega-4}{3}+2-2\omega\big)}{\sigma\big(u-v+\frac{2\omega-4}{3}\big)}\,\frac{\sigma\big(u+\omega v+\frac{1-5\omega}{3}+2\omega\big)}{\sigma\big(u+\omega v+\frac{1-5\omega}{3}\big)}\,\frac{\sigma(u-\omega^2 v+\omega-1)}{\sigma(u-\omega^2 v)}\\
&= -\mathrm{e}^{\eta(2-2\omega)\left(u-v+\frac{2\omega-4}{3}+1-\omega\right)}\mathrm{e}^{\eta(2\omega)\left(u+\omega v+\frac{1-5\omega}{3}+\omega\right)}\mathrm{e}^{\eta(\omega-1)\left(u-\omega^2 v+\frac{\omega-1}{2}\right)}\\
&= -\mathrm{e}^{\eta(2-2\omega)\left[u-v-\left(\frac{1+\omega}{3}\right)\right]}\mathrm{e}^{\eta(2\omega)\left(u+\omega v+\frac{1-2\omega}{3}\right)}\mathrm{e}^{\eta(\omega-1)\left(u-\omega^2 v+\frac{\omega-1}{2}\right)}\\
&= -\mathrm{e}^{M_0^+ u+N_0^+ v+P_0^+}
\end{aligned}
$$

with

$$
\begin{aligned}
M_0^+ &:= \eta(2-2\omega)+\eta(2\omega)+\eta(\omega-1)=\eta(2-2\omega+2\omega+\omega-1)=\eta(1+\omega)\in\mathbb{C},\\
N_0^+ &:= -\eta(2-2\omega)+\omega\eta(2\omega)-\omega^2\eta(\omega-1)\in\mathbb{C},\\
P_0^+ &:= -\left(\frac{1+\omega}{3}\right)\eta(2-2\omega)+\left(\frac{1-2\omega}{3}\right)\eta(2\omega)+\left(\frac{\omega-1}{2}\right)\eta(\omega-1)\in\mathbb{C}.
\end{aligned}
$$

Thus,

$$
\frac{g_3^*(e_0)}{e_0}=\frac{\widetilde{g}_3^*(e_0)}{e_0}=\frac{\widetilde{g}_3^*(e_0^+)}{e_0^+}\,\frac{e_0^-}{\widetilde{g}_3^*(e_0^-)}=\frac{\mathrm{e}^{M_0^+ u+N_0^+ v+P_0^+}}{\mathrm{e}^{\eta(1+\omega)\left[u-\omega^2 v-\left(\frac{1+\omega}{6}\right)\right]}}=\mathrm{e}^{Nv+P}\quad\text{with}
$$

$$
N:=N_0^+ +\omega^2\eta(1+\omega)\in\mathbb{C}\quad\text{and}\quad P:=P_0^+ +\left(\frac{1+\omega}{6}\right)\eta(1+\omega)\in\mathbb{C}.
$$

Bearing in mind that $\eta:(\Lambda,+)\to(\mathbb{C},+)$ is a homomorphism of the $\mathbb{Z}$-module $\Lambda=\mathbb{Z}+\mathbb{Z}\omega$, one expresses

$$
\begin{aligned}
N &= -2\eta(1)+2\eta(\omega)+2\omega\eta(\omega)-\omega^2\eta(\omega)+\omega^2\eta(1)+\omega^2\eta(1)+\omega^2\eta(\omega)\\
&= 2(1+\omega)\eta(\omega)+2(\omega^2-1)\eta(1)=2(1+\omega)[\eta(\omega)+\omega^2\eta(1)],
\end{aligned}
$$

$$
\begin{aligned}
P = &-2\left(\frac{1+\omega}{3}\right)\eta(1)+2\left(\frac{1+\omega}{3}\right)\eta(\omega)+\left(\frac{2-4\omega}{3}\right)\eta(\omega)+\left(\frac{\omega-1}{2}\right)\eta(\omega)\\
&+\left(\frac{1-\omega}{2}\right)\eta(1)+\left(\frac{1+\omega}{6}\right)\eta(1)+\left(\frac{1+\omega}{6}\right)\eta(\omega)=\eta(\omega)-\omega\eta(1).
\end{aligned}
$$

The application of Legendre's relation (10) to N and P yields

$$
N=2(1+\omega)[(2\omega-1)\eta(1)+2\pi\sqrt{-1}],\quad P=2\pi\sqrt{-1},\quad\text{whereas}
$$

$$
\frac{g_3^*(e_0)}{e_0}=\mathrm{e}^{Nv}=\mathrm{e}^{2(1+\omega)\left[(2\omega-1)\eta(1)+2\pi\sqrt{-1}\right]v}.
$$

One can view $\dfrac{g_3^*(e_0)}{e_0} : E \to \mathbb{P}^1(\mathbb{C})$ as an elliptic function on $v + \Lambda \in E$ without zeroes and poles, in order to conclude that $\dfrac{g_3^*(e_0)}{e_0}$ is a constant. That holds exactly when $N = 0$. In such a way, we obtain the explicit value

$$\eta(1) = \frac{2\pi\sqrt{-1}}{1-2\omega} = \frac{2\pi\sqrt{-1}(1+2\omega^2)}{(1-2\omega)(1+2\omega^2)} = \left(\frac{2\omega-1}{3}\right)2\pi\sqrt{-1} \tag{21}$$

of Weierstrass eta-function of $\Lambda = \mathcal{O}_{\sqrt{-3}} = \mathbb{Z} + \mathbb{Z}\omega$ at $1 \in \Lambda$ and derive that $g_3^*(e_0) = e_0$, whereas $e_0 \in \mathcal{L}_{E\times E}(B_1)^{G_3}$, $e_0 \in \mathcal{L}_{E\times E}(B_2)^{G_3}$.

There remains to be checked that the divisor (e_0) of e_0 is of multiplicity $m_p(e_0) \geq -3$ with respect to any singular point p of B_1 or B_2. To this end, note that by Corollary 1 (iv), the singular loci $B_i^{\mathrm{sing}} = \pi_3^{-1}\,(C_0 + C_i)^{\mathrm{sing}}$ of B_i, $1 \leq i \leq 2$ coincide with the fibre $B_i^{\mathrm{sing}} = \pi_3^{-1}(p_0)$ of the unramified G_3-Galois covering $\pi_3 : E \times E \to Y_3 = E \times E/G_3$ over the single singular point $p_0 = \pi_3\left(\dfrac{1}{3} + \Lambda, \dfrac{1}{3} + \Lambda\right)$ of $C_0 + C_1$ and $C_0 + C_2$. Namely, if $p_{00} := \left(\dfrac{1}{3} + \Lambda, \dfrac{1}{3} + \Lambda\right)$. then

$$\begin{aligned} B_1^{\mathrm{sing}} = B_2^{\mathrm{sing}} &= \mathrm{Orb}_{G_3}(p_{00}) \\ &= \left\{ p_{0i} := g_3^i(p_{00}) = \left(\frac{\omega^{2i}}{3} + \Lambda, \frac{\omega^{2i}}{3} + \Lambda\right) \;\middle|\; 0 \leq i \leq 2 \right\}. \end{aligned}$$

Bearing in mind that $(e_0)_\infty = \sum\limits_{j\in\mathbb{Z}_3} g_3^j(B_{00})$, note that $g_3^i(p_{00}) \in g_3^j(B_{00})$ is equivalent to $p_{00} \in g_3^{j-i}(B_{00})$, whereas

$$\begin{aligned} m_{g_3^i(p_{00})}(e_0)_\infty &= m_{g_3^i(p_{00})}\left(\sum_{j\in\mathbb{Z}_3} g_3^j(B_{00})\right) = \sum_{j\in\mathbb{Z}_3} m_{g_3^i(p_{00})}\left(g_3^j(B_{00})\right) \\ &= \sum_{j\in\mathbb{Z}_3} m_{p_{00}}\left(g_3^{j-i}(B_{00})\right) = \sum_{k\in\mathbb{Z}_3} m_{p_{00}}\left(g_3^k(B_{00})\right) \\ &= m_{p_{00}}\left(\sum_{k\in\mathbb{Z}_3} g_3^k(B_{00})\right) = m_{p_{00}}(e_0)_\infty. \end{aligned}$$

Straightforward verification establishes that p_{00} belongs to (17), (18), (19), so that

$$m_{g_3^i(p_{00})}(e_0) = m_{g_3^i(p_{00})}(e_0)_0 - m_{g_3^i(p_{00})}(e_0)_\infty \geq -m_{g_3^i(p_{00})}(e_0)_\infty = -m_{p_{00}}(e_0)_\infty$$

$$= -m_{p_{00}}\left(\sum_{j\in\mathbb{Z}_3} g^j(B_{00})\right) = -\left(\sum_{j\in\mathbb{Z}_3} m_{p_{00}}\left(g_3^j(B_{00})\right)\right) = -3.$$

That justifies $e_0 \in \mathcal{L}_{E\times E}\left(B_1, 3B_1^{\mathrm{sing}}\right)^{G_3}$, $e_0 \in \mathcal{L}_{E\times E}\left(B_2, 3B_2^{\mathrm{sing}}\right)^{G_3}$ and concludes the proof of the proposition. □

The explicit value (21) of the Weierstrass eta-function $\eta : (\Lambda, +) \to (\mathbb{C}, +)$ of $\Lambda = \mathcal{O}_{\sqrt{-3}} = \mathbb{Z}+\mathbb{Z}\omega$ at 1 and Legendre's relation (10) allow to derive an explicit formula for η at all the points of $\mathcal{O}_{\sqrt{-3}}$.

Corollary 9. *Weierstrass eta-function $\eta : (\Lambda, +) \to (\mathbb{C}, +)$ of the ring $\Lambda = \mathcal{O}_{\sqrt{-3}} = \mathbb{Z} + \mathbb{Z}\omega$ of the Eisenstein integers acts by the rule*

$$\eta(\lambda) = \overline{\lambda}\eta(1) = \overline{\lambda}\left(\frac{2\omega-1}{3}\right)2\pi\sqrt{-1}, \quad \forall\lambda\in\Lambda. \tag{22}$$

Proof. Recall from (21) that $\eta(1) = \left(\frac{2\omega-1}{3}\right)2\pi\sqrt{-1}$. Combining this equality with Legendre's relation (10), one concludes that

$$\eta(\omega) = \omega\left[\left(\frac{2\omega-1}{3}\right)2\pi\sqrt{-1}\right] + 2\pi\sqrt{-1} = \left(\frac{1+\omega}{3}\right)2\pi\sqrt{-1}.$$

An arbitrary $\lambda \in \Lambda = \mathcal{O}_{\sqrt{-3}} = \mathbb{Z}+\mathbb{Z}\omega$ is of the form $\lambda = p+q\omega$ for some $p, q \in \mathbb{Z}$. Therefore

$$\begin{aligned}\eta(\lambda) &= \eta(p+q\omega) = p\eta(1) + q\eta(\omega)\\ &= \left[p\left(\frac{2\omega-1}{3}\right) + q\left(\frac{1+\omega}{3}\right)\right]2\pi\sqrt{-1} = \left[\frac{(q-p)+(2p+q)\omega}{3}\right]2\pi\sqrt{-1}.\end{aligned}$$

On the other hand,

$$\overline{\lambda}\eta(1) = (p-q\omega^2)\left(\frac{2\omega-1}{3}\right)2\pi\sqrt{-1} = \left[\frac{(q-p)+(2p+q)\omega}{3}\right]2\pi\sqrt{-1},$$

whereas $\eta(\lambda) = \overline{\lambda}\eta(1)$ for all $\lambda \in \Lambda = \mathcal{O}_{\sqrt{-3}}$. □

Proposition 10. *In the notations from Proposition* 8 *let*

$$f_0^+(u,v) := \sigma\left(u+v-\left(\frac{1+2\omega}{3}\right)\right)\sigma\left(u-\omega v-\frac{2}{3}\right)\sigma\left(u+\omega^2 v-\frac{\omega}{3}\right). \tag{23}$$

$$e_1^+(u) := \sigma\left(u+\frac{1}{3}\right)\sigma\left(u+\frac{\omega^2}{3}\right)\sigma\left(u-\frac{\omega}{3}\right), \tag{24}$$

$$e_1^-(u) := \sigma\left(u-\frac{1}{3}\right)\sigma\left(u-\frac{\omega^2}{3}\right)\sigma\left(u+\frac{\omega}{3}\right), \tag{25}$$

$$e_2^+(v) := \sigma(v)\sigma\left(v+\frac{\omega-2}{3}\right)\sigma\left(v+\frac{2-\omega}{3}\right) \tag{26}$$

$$e_2^-(v) := \sigma\left(v-\frac{1}{3}\right)\sigma\left(v-\frac{\omega^2}{3}\right)\sigma\left(v+\frac{\omega}{3}\right). \tag{27}$$

Then:

(*i*) $f_0(u,v) := \dfrac{f_0^+(u,v)}{e_0^-(u,v)}$ *with* $e_0^-(u,v)$ *from* (14) *belongs to* $\mathcal{L}_{E\times E}\left(\mathrm{Orb}_{G_3}(B_{00}), 2B_1^{\mathrm{sing}}\right)$, $\mathcal{L}_{E\times E}\left(\mathrm{Orb}_{G_3}(B_{00}), 2B_2^{\mathrm{sing}}\right)$ *and* $g_3^*(f_0)=\omega^2 f_0$;

(*ii*) $e_1(u) := \dfrac{e_1^+(u)}{e_1^-(u)}$ *belongs to* $\mathcal{L}_{E\times E}\left(\mathrm{Orb}_{G_3}(B_{10}), B_1^{\mathrm{sing}}\right)$ *and* $g_3^*(e_1)=-\omega e_1$;

(*iii*) $e_2(v) := \dfrac{e_2^+(v)}{e_2^-(v)}$ *belongs to* $\mathcal{L}_{E\times E}\left(\mathrm{Orb}_{G_3}(B_{20}), B_2^{\mathrm{sing}}\right)$ *and* $g_3^*(e_2)=-\omega e_2$;

(*iv*) *the product* $f_{10}(u,v) := f_0(u,v)e_1(u)$ *belongs to* $\mathcal{L}_{E\times E}\left(B_1, 3B_1^{\mathrm{sing}}\right)^{G_3}$, *while* $f_{20}(u,v) := f_0(u,v)e_2(v)$ *is from* $\mathcal{L}_{E\times E}\left(B_2, 3B_2^{\mathrm{sing}}\right)^{G_3}$.

P r o o f. (*i*) Towards the $(\Lambda,+)$-periodicity of $f_0(u,v)$ with respect to u, one makes use of (11), in order to compute that

$$\begin{aligned}\frac{f_0^+(u+\lambda,v)}{f_0^+(u,v)} &= \frac{\sigma\left(u+v-\left(\frac{1+2\omega}{3}\right)+\lambda\right)}{\sigma\left(u+v-\left(\frac{1+2\omega}{3}\right)\right)}\frac{\sigma\left(u-\omega v-\frac{2}{3}+\lambda\right)}{\sigma\left(u-\omega v-\frac{2}{3}\right)}\frac{\sigma\left(u+\omega^2 v-\frac{\omega}{3}+\lambda\right)}{\sigma\left(u+\omega^2 v-\frac{\omega}{3}\right)}\\ &= \varepsilon(\lambda)^3 \mathrm{e}^{\eta(\lambda)\left[u+v-\left(\frac{1+2\omega}{3}\right)+\frac{\lambda}{2}\right]}\mathrm{e}^{\eta(\lambda)\left(u-\omega v-\frac{2}{3}+\frac{\lambda}{2}\right)}\mathrm{e}^{\eta(\lambda)\left(u+\omega^2 v-\frac{\omega}{3}+\frac{\lambda}{2}\right)}\\ &= \varepsilon(\lambda)\mathrm{e}^{\eta(\lambda)\left(3u+\frac{3\lambda}{2}-1-\omega\right)}.\end{aligned}$$

Combining with (15), one concludes that

$$\frac{f_0(u+\lambda,v)}{f_0(u,v)} = \frac{f_0^+(u+\lambda,v)}{f_0^+(u,v)}\frac{e_0^-(u,v)}{e_0^-(u+\lambda,v)} = 1, \quad \forall \lambda\in\Lambda, \quad \forall (u,v)\in\mathbb{C}^2.$$

In order to check the $(\Lambda,+)$-periodicity of $f_0(u,v)$ with respect to v, one observes that $u+\omega^{2i}v+c_i = \omega^{2i}\left(\omega^{4i}u+v+\omega^{4i}c_i\right)$ for $1\le i\le 2$, $c_i\in\mathbb{C}$, $(u,v)\in\mathbb{C}^2$ and

exploits (9), in order to express

$$f_0^+(u,v)=\sigma\left(u+v-\left(\frac{1+2\omega}{3}\right)\right)\sigma\left(\omega^2u+v-\frac{2\omega^2}{3}\right)\sigma\left(-\omega u+v+\frac{\omega^2}{3}\right).$$

Then

$$\begin{aligned}&\frac{f_0^+(u,v+\lambda)}{f_0^+(u,v)}\\&=\frac{\sigma\left(u+v-\left(\frac{1+2\omega}{3}\right)+\lambda\right)}{\sigma\left(u+v-\left(\frac{1+2\omega}{3}\right)\right)}\frac{\sigma\left(\omega^2u+v-\frac{2\omega^2}{3}+\lambda\right)}{\sigma\left(\omega^2u+v-\frac{2\omega^2}{3}\right)}\frac{\sigma\left(-\omega u+v+\frac{\omega^2}{3}+\lambda\right)}{\sigma\left(-\omega u+v+\frac{\omega^2}{3}\right)}\\&=\varepsilon(\lambda)^3\mathrm{e}^{\eta(\lambda)\left[u+v-\left(\frac{1+2\omega}{3}\right)+\frac{\lambda}{2}\right]}\mathrm{e}^{\eta(\lambda)\left(\omega^2u+v-\frac{2\omega^2}{3}+\frac{\lambda}{2}\right)}\mathrm{e}^{\eta(\lambda)\left(-\omega u+v+\frac{\omega^2}{3}+\frac{\lambda}{2}\right)}\\&=\varepsilon(\lambda)\mathrm{e}^{\eta(\lambda)\left(3v+\frac{3\lambda}{2}-\omega\right)}\end{aligned}$$

and (16) imply

$$\frac{f_0(u,v+\lambda)}{f_0(u,v)}=\frac{f_0^+(u,v+\lambda)}{f_0^+(u,v)}\frac{e_0^-(u,v)}{e_0^-(u,v+\lambda)}=1,\quad\forall\lambda\in\Lambda,\quad\forall(u,v)\in\mathbb{C}^2$$

and conclude the verification of the $(\Lambda\times\Lambda,+)$-periodicity of $f_0(u,v)$.

The pole divisor of $f_0(u,v)$ coincides with the zero divisor $(e_0^-)_0$ of the entire function $e^-(u,v)$. In the proof of Proposition 8 we have established that $\left(e_0^-\right)_0=\sum_{j=0}^{2}g_3^j(B_{00})$ and for any singular point $p=g_3^k(p_{00})\in B_1^{\mathrm{sing}}=B_2^{\mathrm{sing}}=\mathrm{Orb}_{G_3}(p_{00})$, $p_{00}=\left(\frac{1}{3}+\Lambda,\frac{1}{3}+\Lambda\right)$, the multiplicity $m_p(f_0)_\infty=m_p(e_0^-)_0=3$. One needs to show that the zero divisor $(f_0)_0=(f_0^+)_0$ of f_0 is of multiplicity $m_{g_3^k(p_{00})}(f_0)_0\geq 1$ with respect to any singular point $g_3^k(p_{00})\in B_1^{\mathrm{sing}}=B_2^{\mathrm{sing}}$, in order to conclude that $m_{g_3^k(p_{00})}(f_0)=m_{g_3^k(p_{00})}(f_0)_0-m_{g_3^k(p_{00})}(f_0)_\infty\geq 1-3=-2$ and to derive that

$$f_0\in\mathcal{L}_{E\times E}\left(\mathrm{Orb}_{G_3}(B_{00}),2B_i^{\mathrm{sing}}\right)\quad\text{for}\quad 1\leq i\leq 2.$$

Making use of $\sigma(z)=0$ if and only if $z\in\Lambda$, one observes that $(f_0)_0=(f_0^+)_0=\sum_{i=0}^{2}B'_{0i}$ for

$$B'_{00}:=\left\{(u+\Lambda,v+\Lambda)\in E\times E\,\middle|\,u+v-\left(\frac{1+2\omega}{3}\right)\in\Lambda\right\},$$

$$B'_{01} := \left\{ (u+\Lambda, v+\Lambda) \in E \times E \,\middle|\, u - \omega v - \frac{2}{3} \in \Lambda \right\},$$

$$B'_{02} := \left\{ (u+\Lambda, v+\Lambda) \in E \times E \,\middle|\, u + \omega^2 v - \frac{\omega}{3} \in \Lambda \right\}.$$

Straightforwardly, $p_{00} = \left(\frac{1}{3}+\Lambda, \frac{1}{3}+\Lambda\right) \in B'_{02}$, $g_3(p_{00}) = \left(\frac{\omega^2}{3}+\Lambda, \frac{\omega^2}{3}+\Lambda\right) \in B'_{00}$, $g_3^2(p_{00}) = \left(-\frac{\omega}{3}+\Lambda, -\frac{\omega}{3}+\Lambda\right) \in B'_{01}$, whereas

$$m_{g_3^k(p_{00})}(f_0)_0 = m_{g_3^k(p_{00})}\left(\sum_{i=0}^{2} B'_{0i}\right) \geq 1, \forall 0 \leq k \leq 2$$

and $f_0 \in \mathcal{L}_{E\times E}\left(\mathrm{Orb}_{G_3}(B_{00}), 2B_i^{\mathrm{sing}}\right)$, $\forall 1 \leq i \leq 2$.

Towards $g_3^*(f_0) = \omega^2 f_0$, one makes use of (22), in order to express $\eta(1+\omega) = (1-\omega^2)\eta(1)$ and to reduce (20) to the form

$$\frac{\widetilde{g_3^*}(e_0^-)}{e_0^-} = -\mathrm{e}^{\eta(1)(1-\omega^2)\left[u-\omega^2 v-\left(\frac{1+\omega}{6}\right)\right]} = -\mathrm{e}^{\eta(1)\left[(2-\omega)u+(1-2\omega)v-\frac{1}{2}\right]}.$$

On the other hand, (11) allows to compute that

$$\begin{aligned}
\frac{\widetilde{g_3^*}(f_0^+)}{f_0^+}(u,v) &= \frac{f_0^+\left(u+\frac{1+\omega}{3}, \omega^2 v\right)}{f_0^+(u,v)} = \frac{\sigma\left(u+v+\frac{\omega-1}{3}\right)}{\sigma\left(u+v-\left(\frac{1+2\omega}{3}\right)\right)} \frac{\sigma\left(u-\omega v+\frac{1}{3}\right)}{\sigma\left(u-\omega v-\frac{2}{3}\right)} \\
&= \frac{\sigma\left(u+v-\left(\frac{1+2\omega}{3}\right)+\omega\right)}{\sigma\left(u+v-\left(\frac{1+2\omega}{3}\right)\right)} \frac{\sigma\left(u-\omega v-\frac{2}{3}+1\right)}{\sigma\left(u-\omega v-\frac{2}{3}\right)} \\
&= \mathrm{e}^{\eta(\omega)\left[u+v-\left(\frac{1+2\omega}{3}\right)+\frac{\omega}{2}\right]} \mathrm{e}^{\eta(1)\left(u-\omega v-\frac{2}{3}+\frac{1}{2}\right)}.
\end{aligned}$$

Due to $\eta(\omega) = -\omega^2\eta(1)$ by (22), there follow

$$\begin{aligned}
\frac{\widetilde{g_3}^*(f_0^+)}{f_0^+}(u,v) &= \mathrm{e}^{\eta(1)\left\{(-\omega^2)\left[u+v-\left(\frac{\omega+2}{6}\right)\right]+u-\omega v-\frac{1}{6}\right\}} \\
&= \mathrm{e}^{\eta(1)\left[(2-\omega)u+(1-2\omega)v-\frac{1}{2}+\frac{2\omega-1}{6}\right]} = \mathrm{e}^{\eta(1)\left[(2-\omega)u+(1-2\omega)v-\frac{1}{2}\right]} \mathrm{e}^{\frac{(2\omega-1)^2}{18}2\pi\sqrt{-1}} \\
&= \mathrm{e}^{\eta(1)\left[(2-\omega)u+(1-2\omega)v-\frac{1}{2}\right]} \mathrm{e}^{-\frac{2\pi\sqrt{-1}}{6}} = \omega^{-1}\mathrm{e}^{\eta(1)\left[(2-\omega)u+(1-2\omega)v-\frac{1}{2}\right]} \quad \text{and}
\end{aligned}$$

$$\frac{g_3^*(f_0)}{f_0} = \frac{\widetilde{g_3^*}(f_0)}{f_0} = \frac{\widetilde{g_3^*}(f_0^+)}{f_0^+} \frac{e_0^-}{\widetilde{g_3^*}(e_0^-)} = -\omega^{-1} = -\frac{1}{\omega} = \omega^2.$$

That concludes the proof of (i).

(ii) and (iii) According to Lemma (7) (i), the quotients $e_1(u) := \dfrac{e_1^+(u)}{e_1^-(u)}$ and $e_2(v) := \dfrac{e_2^+(v)}{e_2^-(v)}$ are elliptic functions on E and can be viewed as abelian functions on $E \times E$. Bearing in mind that $\sigma(z) = 0$ if and only if $z \in \Lambda$, one concludes that the pole divisors of e_1 and e_2 are

$$\begin{aligned}(e_1)_\infty = (e_1^-)_0 &= \left[\left(\frac{1}{3}+\Lambda\right)\times E\right] + \left[\left(\frac{\omega^2}{3}+\Lambda\right)\times E\right] + \left[\left(-\frac{\omega}{3}+\Lambda\right)\times E\right] \\ &= \sum_{i=0}^{2}\left[\mathcal{T}\left(i\left(\frac{1+\omega}{3}+\Lambda\right)\right)\left(\frac{1}{3}+\Lambda\right)\right]\times E = \sum_{i=0}^{2} g_3^i\left(\left(\frac{1}{3}+\Lambda\right)\times E\right) \\ &= \mathrm{Orb}_{G_3}(B_{10}),\end{aligned}$$

respectively,

$$\begin{aligned}(e_2)_\infty = (e_2^-)_0 &= \left[E\times\left(\frac{1}{3}+\Lambda\right)\right] + \left[E\times\left(\frac{\omega^2}{3}+\Lambda\right)\right] + \left[E\times\left(-\frac{\omega}{3}+\Lambda\right)\right] \\ &= \sum_{i=0}^{2} E\times\omega^{2i}\left(\frac{1}{3}+\Lambda\right) = \sum_{i=0}^{2} g_3^i\left(E\times\left(\frac{1}{3}+\Lambda\right)\right) = \mathrm{Orb}_{G_3}(B_{20}).\end{aligned}$$

Since $p_{00} = \left(\dfrac{1}{3}+\Lambda, \dfrac{1}{3}+\Lambda\right) \in B_{i0}$ for $1 \le i \le 2$ and $g_3^j(p_{00}) \notin B_{i0}$ for $1 \le j \le 2$, $1 \le i \le 2$, the multiplicities

$$m_{g_3^j(p_{00})}\left(g_3^k(B_{i0})\right) = m_{g_3^{j-k}(p_{00})}(B_{i0}) = \delta_{jk} = \begin{cases} 1 & \text{for } 0 \le j = k \le 2, \\ 0 & \text{for } 0 \le j \ne k \le 2. \end{cases}$$

As a result, $m_{g_3^j(p_{00})}(\mathrm{Orb}_{G_3}(B_{i0})) = 1$ for all $0 \le j \le 2$, which implies that the multiplicities

$$m_{g_3^j(p_{00})}(e_i) = m_{g_3^j(p_{00})}(e_i)_0 - m_{g_3^j(p_{00})}(e_i)_\infty \ge -m_{g_3^j(p_{00})}(\mathrm{Orb}_{G_3}(B_{i0})) = -1$$

and e_i belong to $\mathcal{L}_{E\times E}\left(\mathrm{Orb}_{G_3}(B_{i0}), B_i^{\mathrm{sing}}\right)$ for $1 \le i \le 2$.

Towards the verification of $g_3^*(e_i) = -\omega e_i$, $1 \le i \le 2$, let us note that

$$\widetilde{g_3^*}(e_1^+)(u) = e_1^+\left(u+\frac{1+\omega}{3}\right) = \sigma\left(u+\frac{2+\omega}{3}\right)\sigma\left(u+\frac{2\omega}{3}\right)\sigma\left(u+\frac{1}{3}\right),$$

$$\widetilde{g}_3^*(e_1^-)(u) = e_1^-\left(u+\frac{1+\omega}{3}\right) = \sigma\left(u+\frac{\omega}{3}\right)\sigma\left(u+\frac{2}{3}\right)\sigma\left(u+\frac{1+2\omega}{3}\right)$$

and apply (11), (22), in order to observe that $\eta(\omega) = -\omega^2\eta(1)$ and to obtain that

$$\frac{\widetilde{g}_3^*(e_1^+)}{e_1^+}(u) = \frac{\sigma\left(u+\frac{2+\omega}{3}\right)}{\sigma\left(u+\frac{\omega-1}{3}\right)}\frac{\sigma\left(u+\frac{2\omega}{3}\right)}{\sigma\left(u-\frac{\omega}{3}\right)} = \frac{\sigma\left(u+\frac{\omega-1}{3}+1\right)}{\sigma\left(u+\frac{\omega-1}{3}\right)}\frac{\sigma\left(u-\frac{\omega}{3}+\omega\right)}{\sigma\left(u-\frac{\omega}{3}\right)}$$

$$= \mathrm{e}^{\eta(1)\left(u+\frac{\omega-1}{3}+\frac{1}{2}\right)}\mathrm{e}^{\eta(\omega)\left(u-\frac{\omega}{3}+\frac{\omega}{2}\right)} = \mathrm{e}^{\eta(1)\left[u+\frac{1+2\omega}{6}-\omega^2\left(u+\frac{\omega}{6}\right)\right]} = \mathrm{e}^{\eta(1)\left[(2-\omega)u+\frac{1+\omega}{3}\right]},$$

$$\frac{\widetilde{g}_3^*(e_1^-)}{e_1^-}(u) = \frac{\sigma\left(u+\frac{2}{3}\right)}{\sigma\left(u-\frac{1}{3}\right)}\frac{\sigma\left(u+\frac{1+2\omega}{3}\right)}{\sigma\left(u+\frac{1-\omega}{3}\right)} = \frac{\sigma\left(u-\frac{1}{3}+1\right)}{\sigma\left(u-\frac{1}{3}\right)}\frac{\sigma\left(u+\frac{1-\omega}{3}+\omega\right)}{\sigma\left(u+\frac{1-\omega}{3}\right)}$$

$$= \mathrm{e}^{\eta(1)\left(u-\frac{1}{3}+\frac{1}{2}\right)}\mathrm{e}^{\eta(\omega)\left(u+\frac{1-\omega}{3}+\frac{\omega}{2}\right)} = \mathrm{e}^{\eta(1)\left[u+\frac{1}{6}-\omega^2\left(u+\frac{\omega+2}{6}\right)\right]} = \mathrm{e}^{\eta(1)\left[(2-\omega)u+\frac{2-\omega}{3}\right]}.$$

As a result,

$$\frac{g_3^*(e_1)}{e_1} = \frac{\widetilde{g}_3^*(e_1)}{e_1} = \frac{\widetilde{g}_3^*(e_1^+)}{e_1^+}\frac{e_1^-}{\widetilde{g}_3^*(e_1^-)} = \mathrm{e}^{\eta(1)\left(\frac{2\omega-1}{3}\right)}$$

$$= \mathrm{e}^{\frac{(2\omega-1)^2}{9}2\pi\sqrt{-1}} = \mathrm{e}^{-\frac{2\pi\sqrt{-1}}{3}} = \frac{1}{\omega^2} = -\omega,$$

which concludes the proof of (*ii*).

Similarly,

$$\omega^2 v + \frac{\omega-2}{3} = \omega^2\left(v+\frac{1+\omega}{3}\right), \quad \omega^2 v + \frac{2-\omega}{3} = \omega^2\left[v-\left(\frac{1+\omega}{3}\right)\right]$$

and (9) imply

$$\widetilde{g}_3^*(e_2^+)(v) = e_2^+(\omega^2 v) = \sigma(v)\sigma\left(v+\frac{1+\omega}{3}\right)\sigma\left(v-\left(\frac{1+\omega}{3}\right)\right),$$

while

$$\omega^2 v - \frac{1}{3} = \omega^2\left(v+\frac{\omega}{3}\right),\ \omega^2 v - \frac{\omega^2}{3} = \omega^2\left(v-\frac{1}{3}\right),\ \omega^2 v + \frac{\omega}{3} = \omega^2\left(v-\frac{\omega^2}{3}\right)$$

and (9) suffice for

$$\widetilde{g}_3^*(e_2^-)(v) = e_2^-(\omega^2 v) = \sigma\left(v+\frac{\omega}{3}\right)\sigma\left(v-\frac{1}{3}\right)\sigma\left(v-\frac{\omega^2}{3}\right) = e_2^-(v).$$

Making use of (11) and (22), one derives that

$$\begin{aligned}\frac{g_3^*(e_2)}{e_2} &= \frac{\widetilde{g_3}^*(e_2)}{e_2} = \frac{\widetilde{g}_3^*(e_2^+)}{e_2^+}\frac{e_2^-}{\widetilde{g}_3^*(e_2^-)} = \frac{\widetilde{g}_3^*(e_2^+)}{e_2^+} = \frac{\sigma\left(v+\frac{1+\omega}{3}\right)}{\sigma\left(v+\frac{\omega-2}{3}\right)}\frac{\sigma\left(v-\left(\frac{1+\omega}{3}\right)\right)}{\sigma\left(v+\frac{2-\omega}{3}\right)}\\ &= \frac{\sigma\left(v+\frac{\omega-2}{3}+1\right)}{\sigma\left(v+\frac{\omega-2}{3}\right)}\frac{\sigma\left(v+\frac{2-\omega}{3}-1\right)}{\sigma\left(v+\frac{2-\omega}{3}\right)} = \mathrm{e}^{\eta(1)\left(v+\frac{\omega-2}{3}+\frac{1}{2}\right)}\mathrm{e}^{-\eta(1)\left(v+\frac{2-\omega}{3}-\frac{1}{2}\right)}\\ &= \mathrm{e}^{\eta(1)\left(\frac{2\omega-1}{3}\right)} = -\omega\end{aligned}$$

and verifies (iii).

(iv) is an immediate consequence of (i), (ii) and (iii). Namely, the products $f_{i0} = f_0 e_i$, $1 \le i \le 2$ of the $(\Lambda\times\Lambda,+)$-periodic f_0 and e_i are $(\Lambda\times\Lambda,+)$-periodic. The pole divisors of f_{i0} are $(f_{i0})_\infty = (f_0e_i)_\infty = (f_0)_\infty + (e_i)_\infty = \mathrm{Orb}_{G_3}(B_{00}) + \mathrm{Orb}_{G_3}(B_{i0}) = B_i$. The multiplicities of the divisors of f_{i0} with respect to a singular point

$$g_3^k(p_{00}) \in B_1^{\mathrm{sing}} = B_2^{\mathrm{sing}} = \mathrm{Orb}_{G_3}(p_{00}), \quad p_{00} = \left(\frac{1}{3}+\Lambda, \frac{1}{3}+\Lambda\right)$$

are $m_{g_3^k(p_{00})}(f_{i0}) = m_{g_3^k(p_{00})}(f_0e_i) = m_{g_3^i(p_{00})}(f_0) + m_{g_3^k(p_{00})}(e_i) \ge -2-1 = -3$. Finally, f_{i0} are G_3-invariant, according to $g_3^*(f_{i0}) = g_3^*(f_0e_i) = g_3^*(f_0)g_3^*(e_i) = (\omega^2 f_0)(-\omega e_i) = f_0e_i = f_{i0}$. That is why, the elliptic functions f_{i0} belongs to $\mathcal{L}_{E\times E}\left(B_i, 3B_i^{\mathrm{sing}}\right)^{G_3}$ for $1 \le i \le 2$. □

Proposition 11. *In the notations from Proposition* 8 *let*

(28)
$$f'_{11}(u) := \frac{\sigma(u)\sigma\left(u-\frac{2}{3}\right)}{\sigma\left(u-\frac{1}{3}\right)^2}, \quad f'_{12}(u) := \frac{\sigma(u)^2\sigma(u-1)}{\sigma\left(u-\frac{1}{3}\right)^3}, \quad f_{1i}(u) := \prod_{i=0}^{2}(g_3^*)^i(f'_{1i})(u),$$

for $1 \le i \le 2$,

(29)
$$f_{21}(v) := \frac{\sigma(v)^3}{\sigma\left(v-\frac{1}{3}\right)\sigma\left(v-\frac{\omega^2}{3}\right)\sigma\left(v+\frac{\omega}{3}\right)},$$

(30)
$$f'_{22}(v) := \frac{\sigma(v)^2\sigma(v-1)}{\sigma\left(v-\frac{1}{3}\right)^3}, \quad f_{22}(v) := \prod_{i=0}^{2}(g_3^*)^i(f'_{22})(v).$$

Then f_{11} *belongs to* $\mathcal{L}_{E\times E}\left(2B_1, 2B_1^{\mathrm{sing}}\right)^{G_3}$, f_{12} *belongs to* $\mathcal{L}_{E\times E}\left(3B_1, 3B_1^{\mathrm{sing}}\right)^{G_3}$, f_{21} *belongs to* $\mathcal{L}_{E\times E}\left(B_2, B_2^{\mathrm{sing}}\right)^{G_3}$ *and* f_{22} *belongs to* $\mathcal{L}_{E\times E}\left(3B_2, 3B_2^{\mathrm{sing}}\right)^{G_3}$.

P r o o f. According to Lemma 7 (i), the functions f'_{11}, f'_{12}, f_{21}, f'_{22} of a single complex variable are $(\Lambda,+)$-periodic and descend to elliptic functions on E. Thus, for any $0 \le k \le 2$, $1 \le j \le 2$, $\lambda \in \Lambda$, $u \in \mathbb{C}$ there holds $(g_3^*)^k (f'_{1j})(u+\lambda) = f'_{1j}\left(u+\lambda+k\left(\frac{1+\omega}{3}\right)\right) = f'_{1j}\left(u+k\left(\frac{1+\omega}{3}\right)\right) = (g_3^*)^k (f'_{1j})(u)$, whereas $f_{1j}(u+\lambda) = \prod\limits_{k=0}^{2} (g_3^*)^k (f'_{1j})(u+\lambda) = \prod\limits_{k=0}^{2} (g_3^*)^k (f'_{1j})(u) = f_{1j}(u)$. Bearing in mind that $\omega^2\Lambda = \omega^2\mathcal{O}_{\sqrt{-3}} = \mathcal{O}_{\sqrt{-3}} = \Lambda$, one observes that $(g_3^*)^k (f'_{22})(v+\lambda) = f'_{22}\left(\omega^{2k}(v+\lambda)\right) = f'_{22}\left(\omega^{2k}v\right) = (g_3^*)^k (f'_{22})(v)$ for all $0 \le k \le 2$, $\lambda \in \Lambda$, $v \in \mathbb{C}$ and derives that $f_{22}(v+\lambda) = \prod\limits_{k=0}^{2} (g_3^*)^k (f'_{22})(v+\lambda) = \prod\limits_{k=0}^{2} (g_3^*)^k (f'_{22})(v) = f_{22}(v)$. In such a way, we have shown that $f_{ij} : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$, $\forall (i,j) \ne (2,1)$ are $(\Lambda,+)$-periodic and can be viewed as abelian functions on $E \times E$.

By its very definition, f_{21} has divisor $(f_{21}) = 3(E \times \Lambda) - \mathrm{Orb}_{G_3}(B_{20})$ and belongs to $\mathcal{L}_{E\times E}(B_2)$. For any singular point $g_3^k(p_{00}) \in B_2^{\mathrm{sing}} = \mathrm{Orb}_{G_3}(p_{00})$, $p_{00} = \left(\frac{1}{3}+\Lambda, \frac{1}{3}+\Lambda\right)$, $0 \le k \le 2$, note that $m_{g_3^k(p_{00})}(E \times \Lambda) = 0$, as well as $m_{g_3^k(p_{00})}\left(g_3^l(B_{20})\right) = \delta_{kl} = 1$ for $0 \le k = l \le 2$ and $m_{g_3^k(p_{00})}\left(g_3^l(B_{20})\right) = \delta_{kl} = 0$ for $0 \le k \ne l \le 2$. Thus, $m_{g_3^k(p_{00})}\left(\mathrm{Orb}_{G_3}(B_{20})\right) = 1$, whereas $m_{g_3^k(p_{00})}(f_{21}) = -1$ and $f_{21} \in \mathcal{L}_{E\times E}\left(B_2, B_2^{\mathrm{sing}}\right)$.

Observe that for arbitrary $\nu \in \{0,\infty\}$, $0 \le k \le 2$, $(i,j) \ne (2,1)$, the equality $\nu = (g_3^*)^k (f'_{ij})(u+\Lambda, v+\Lambda) = f'_{ij}\left(g_3^k(u+\Lambda, v+\Lambda)\right)$ of points of $\mathbb{P}^1(\mathbb{C})$ is equivalent to the equality $g_3^k\left((g_3^*)^k (f'_{ij})\right)_\nu = (f'_{ij})_\nu$ of the corresponding zero or pole divisors. Therefore $(f_{ij})_\nu = \left(\prod\limits_{k=0}^{2} (g_3^*)^k (f'_{ij})\right)_\nu = \sum\limits_{k=0}^{2} \left((g_3^*)^k (f'_{ij})\right)_\nu = \sum\limits_{k=0}^{2} g_3^{-k}\left((f'_{ij})_\nu\right) = \mathrm{Orb}_{G_3}(f'_{ij})_\nu$ and the complete divisors of f_{ij} are $(f_{ij}) = (f_{ij})_0 - (f_{ij})_\infty = \mathrm{Orb}_{G_3}(f'_{ij})_0 - \mathrm{Orb}_{G_3}(f'_{ij})_\infty = \mathrm{Orb}_{G_3}\left((f'_{ij})_0 - (f'_{ij})_\infty\right) = \mathrm{Orb}_{G_3}(f'_{ij})$. By their very definitions, the divisors of f'_{ij} with $(i,j) \ne (2,1)$ are

$$(f'_{11}) = (\Lambda \times E) + \left[\left(\frac{2}{3} + \Lambda\right) \times E\right] - 2B_{10},$$
$$(f'_{12}) = 3(\Lambda \times E) - 3B_{10}, \quad (f'_{22}) = 3(E \times \Lambda) - 3B_{20}.$$

Therefore $(f_{11}) = \mathrm{Orb}_{G_3}(\Lambda \times E) + \mathrm{Orb}_{G_3}\left(\left(\frac{2}{3} + \Lambda\right) \times E\right) - 2\mathrm{Orb}_{G_3}(B_{10})$, $(f_{12}) = 3\mathrm{Orb}_{G_3}(\Lambda \times E) - 3\mathrm{Orb}_{G_3}(B_{10})$, $(f_{22}) = 3\mathrm{Orb}_{G_3}(E \times \Lambda) - 3\mathrm{Orb}_{G_3}(B_{20})$. The above considerations establish that $f_{11} \in \mathcal{L}_{E\times E}(2B_1)$ and $f_{i2} \in \mathcal{L}_{E\times E}(3B_i)$ for $1 \leq i \leq 2$.

The multiplicities of the complete divisors of f_{ij} with respect to the singular points $g_3^k(p_{00}) \in B_1^{\mathrm{sing}} = B_2^{\mathrm{sing}} = \mathrm{Orb}_{G_3}(p_{00})$, $p_{00} = \left(\frac{1}{3} + \Lambda, \frac{1}{3} + \Lambda\right)$, $0 \leq k \leq 2$ are

$$m_{g_3^k(p_{00})}(f_{ij}) = m_{g_3^k(p_{00})}\left(\mathrm{Orb}_{G_3}(f'_{ij})\right) = m_{g_3^k(p_{00})}\left(\sum_{l\in\mathbb{Z}_3} g_3^l(f'_{ij})\right)$$
$$= \sum_{l\in\mathbb{Z}_3} m_{g_3^k(p_{00})}\left(g_3^l(f'_{ij})\right) = \sum_{l\in\mathbb{Z}_3} m_{g_3^{k-l}(p_{00})}(f'_{ij}) = \sum_{l\in\mathbb{Z}_3} m_{g_3^l(p_{00})}(f'_{ij}).$$

The elliptic curves $\Lambda \times E$, $\left(\frac{2}{3} + \Lambda\right) \times E$ and $E \times \Lambda$ are disjoint from the singular loci $B_1^{\mathrm{sing}} = B_2^{\mathrm{sing}} = \mathrm{Orb}_{G_3}(p_{00})$. Moreover, $m_{p_{00}}(B_{i0}) = 1$ for $1 \leq i \leq 2$ and $m_{g_3^l(p_{00})}(B_{i0}) = 0$ for $1 \leq l \leq 2$, $1 \leq i \leq 2$, so that $m_{g_3^k(p_{00})}(f_{ij}) = m_{p_{00}}(f'_{ij})$ for all $(i,j) \neq (2,1)$. In particular, $m_{g_3^k(p_{00})}(f_{11}) = m_{p_{00}}(f'_{11}) = -2m_{p_{00}}(B_{10}) = -2$ and $m_{g_3^k(p_{00})}(f_{i2}) = m_{p_{00}}(f'_{i2}) = -3m_{p_{00}}(B_{i0}) = -3$ for $1 \leq i \leq 2$. In such a way, we have justified that $f_{11} \in \mathcal{L}_{E\times E}\left(2B_1, 2B_1^{\mathrm{sing}}\right)$ and $f_{i2} \in \mathcal{L}_{E\times E}\left(3B_i, 3B_i^{\mathrm{sing}}\right)$ for $1 \leq i \leq 2$.

The G_3-invariance of f_{ij} with $(i,j) \in \{(1,1),(1,2),(2,2)\}$ follows from their very definitions and from $g_3^3 = \mathrm{Id}_{E\times E}$. According to $\omega^2 v - \frac{1}{3} = \omega^2\left(v + \frac{\omega}{3}\right)$, $\omega^2 v - \frac{\omega^2}{3} = \omega^2\left(v - \frac{1}{3}\right)$, $\omega^2 v + \frac{\omega}{3} = \omega^2\left(v - \frac{\omega^2}{3}\right)$ and (9), there holds

$$g_3^*(f_{21}(v)) = f_{21}(\omega^2 v) = \frac{\sigma(v)^3}{\sigma\left(v + \frac{\omega}{3}\right)\sigma\left(v - \frac{1}{3}\right)\sigma\left(v - \frac{\omega^2}{3}\right)} = f_{21}(v).$$

That proves the G_3-invariance of all f_{ij} and completes the proof of the proposition. □

4. Explicit projective embeddings of Baily-Borel compactifications by modular forms of weight 3. The present section establishes the main results of the article – Theorem 14 and Theorem 15. They provide regular embeddings $\Phi_i : \widehat{\mathbb{B}/\Gamma_i} \to \mathbb{P}^5(\mathbb{C})$ by modular forms of weight 3 for the Baily-Borel compactifications of the ball quotients $\mathbb{B}/\Gamma_i$, $1 \leq i \leq 2$ with smooth toroidal compactifications $(\mathbb{B}/\Gamma_i)'$ and minimal Haar measure $\mathrm{vol}\,(\mathbb{B}/\Gamma_i) = \dfrac{8\pi^2}{3}$, described by Di Cerbo and Stover in Theorem 1.1 from [4] - cf. also Theorem 5 from the present article. In both cases, the regularity and the injectiveness of Φ_i are derived by the criterion from Proposition 4. That is why, we proceed by counting the G_3-orbits of the generic fibres of the rational maps

$$f_i = (e_0, f_{i0}, f_{i1}, f_{i2}) : E \times E \longrightarrow f_i(E \times E) \subset \mathbb{P}^1(\mathbb{C})^4,$$

given by the G_3-invariant abelian functions from Proposition 8, Proposition 10 and Proposition 11.

Proposition 12. *In the notations from Proposition 8 and (28) from Proposition 11, the generic fibres of the* $\left\langle \mathcal{T}\left(\dfrac{1+\omega}{3} + \Lambda\right)\right\rangle$*-invariant rational map*

$$f_1 = (f_{11}, f_{12}) : E \longrightarrow f_1(E) \subset \mathbb{P}^1(\mathbb{C})^2 \tag{31}$$

consist of a single $\left\langle \mathcal{T}\left(\dfrac{1+\omega}{3} + \Lambda\right)\right\rangle$*-orbit.*

P r o o f. Note that the $\left\langle \mathcal{T}\left(\dfrac{1+\omega}{3} + \Lambda\right)\right\rangle$-invariance of $f_{11}(u), f_{12}(u) : E \to \mathbb{P}^1(\mathbb{C})$ implies the $\left\langle \mathcal{T}\left(\dfrac{1+\omega}{3} + \Lambda\right)\right\rangle$-invariance of all the fibres of (31). Let us assume that the generic fibre $f_1^{-1}(x_1, x_2) = f_{11}^{-1}(x_1) \cap f_{12}^{-1}(x_2)$ of f_1 over $(x_1, x_2) \in f_1(E)$, $x_1, x_2 \in \mathbb{P}^1(\mathbb{C})$ consists of $k \geq 2$ orbits of $\left\langle \mathcal{T}\left(\dfrac{1+\omega}{3} + \Lambda\right)\right\rangle$. Since all the fibres of the $\left\langle \mathcal{T}\left(\dfrac{1+\omega}{3} + \Lambda\right)\right\rangle$-invariant elliptic function $f_{11} : E \to \mathbb{P}^1(\mathbb{C})$ of order 6 consist of 2 orbits of $\left\langle \mathcal{T}\left(\dfrac{1+\omega}{3} + \Lambda\right)\right\rangle \simeq (\mathbb{Z}_3, +)$, there follows $f_{11}^{-1}(x_1) = f_1^{-1}(x_1, x_2) = f_{11}^{-1}(x_1) \cap f_{12}^{-1}(x_2) \subset f_{12}^{-1}(x_2)$. Let $f_{11}^{-1}(x_1) =$

$\mathrm{Orb}_{\langle\mathcal{T}(\frac{1+\omega}{3}+\Lambda)\rangle}(u(x_1)+\Lambda)\coprod\mathrm{Orb}_{\langle\mathcal{T}(\frac{1+\omega}{3}+\Lambda)\rangle}(u'(x_1)+\Lambda)$ and note that

$$\begin{aligned}f_{11}(u) &:= \prod_{i=0}^{2}\left(\mathcal{T}\left(\frac{1+\omega}{3}+\Lambda\right)^{*}\right)^{i}\left(\frac{\sigma(u)\sigma\left(u-\frac{2}{3}\right)}{\sigma\left(u-\frac{1}{3}\right)^{2}}\right)\\ &= \prod_{i=0}^{2}\frac{\sigma\left(u+i\left(\frac{1+\omega}{3}\right)\right)\sigma\left(u-\frac{2}{3}+i\left(\frac{1+\omega}{3}\right)\right)}{\sigma\left(u-\frac{1}{3}+i\left(\frac{1+\omega}{3}\right)\right)^{2}}\\ &= \frac{\sigma(u)\sigma\left(u-\frac{2}{3}\right)\sigma\left(u+\frac{1+\omega}{3}\right)\sigma\left(u+\frac{\omega-1}{3}\right)\sigma\left(u+\frac{2+2\omega}{3}\right)\sigma\left(u+\frac{2\omega}{3}\right)}{\sigma\left(u-\frac{1}{3}\right)^{2}\sigma\left(u+\frac{\omega}{3}\right)^{2}\sigma\left(u+\frac{2\omega+1}{3}\right)^{2}}\end{aligned}$$

has double poles at $\frac{1}{3}+\Lambda$, $-\frac{\omega}{3}+\Lambda$, $\frac{-2\omega-1}{3}+\Lambda=\frac{\omega^2}{3}+\Lambda$. According to Lemma 7 (ii), for any $x_1\in\mathbb{P}^1(\mathbb{C})$ the fibre $f_{11}^{-1}(x_1)$ consists of six points with sum

$$\begin{aligned}\Lambda &= 2\left[\left(\frac{1}{3}+\Lambda\right)+\left(-\frac{\omega}{3}+\Lambda\right)+\left(\frac{\omega^2}{3}+\Lambda\right)\right]\\ &= \left[\sum_{i=0}^{2}u(x_1)+i\left(\frac{1+\omega}{3}\right)+\Lambda\right]+\left[\sum_{i=0}^{2}u'(x_1)+i\left(\frac{1+\omega}{3}\right)+\Lambda\right]\\ &= 3u(x_1)+3u'(x_1)+\Lambda,\end{aligned}$$

as far as $\sum_{i=0}^{2}i\left(\frac{1+\omega}{3}\right)+\Lambda=1+\omega+\Lambda=\Lambda$. Thus, for a generic $x_1\in\mathbb{P}^1(\mathbb{C})$ there exists $\lambda(x_1)\in\Lambda$ with $3u(x_1)+3u'(x_1)=\lambda(x_1)$. One can view $u(x_1)$ and $u'(x_1)$ as local holomorphic functions of $x_1\in\mathbb{P}^1(\mathbb{C})$, in order to interpret $\lambda(x_1):\mathbb{P}^1(\mathbb{C})\to\Lambda\subset\mathbb{C}$ as a local holomorphic function. The continuity of $\lambda(x_1)$ and the discreteness of Λ requires $\lambda(x_1)\equiv\lambda\in\Lambda$ to be constant and $u'(x_1)=\frac{\lambda}{3}-u(x_1)$ for generic and, therefore, any $x_1\in\mathbb{P}^1(\mathbb{C})$. As a result, the inclusion

$$\mathrm{Orb}_{\langle\mathcal{T}(\frac{1+\omega}{3}+\Lambda)\rangle}(u(x_1)+\Lambda)\coprod\mathrm{Orb}_{\langle\mathcal{T}(\frac{1+\omega}{3}+\Lambda)\rangle}(u'(x_1)+\Lambda)=f_{11}^{-1}(x_1)\subset f_{12}^{-1}(x_2)$$

for a generic point $(x_1,x_2)\in f_1(E)\subset\mathbb{P}^1(\mathbb{C})^2$ implies that

$$f_{12}(u(x_1)+\Lambda)=f_{12}(u'(x_1)+\Lambda)=f_{12}\left(\frac{\lambda}{3}-u(x_1)+\Lambda\right).$$

Note that the image $f_1(E)$ of E under $f_1=(f_{11},f_{12})$ projects onto $f_{11}(E)=\mathbb{P}^1(\mathbb{C})$ and for any $x_1\in\mathbb{P}^1(\mathbb{C})$ there exists $x_2\in\mathbb{P}^1(\mathbb{C})$ with $(x_1,x_2)\in f_1(E)$.

That is why, the elliptic function

$$\begin{aligned} f_{12}(u) &:= \prod_{i=0}^{2} \left(\mathcal{T} \left(\frac{1+\omega}{3} + \Lambda \right)^{*} \right)^{i} \left(\frac{\sigma(u)^2 \sigma(u-1)}{\sigma\left(u-\frac{1}{3}\right)^3} \right) \\ &= \prod_{i=0}^{2} \frac{\sigma\left(u + i\left(\frac{1+\omega}{3}\right)\right)^2 \sigma\left(u-1+i\left(\frac{1+\omega}{3}\right)\right)}{\sigma\left(u-\frac{1}{3}+i\left(\frac{1+\omega}{3}\right)\right)^3} \\ &= \frac{\sigma(u)^2\sigma(u-1)\sigma\left(u+\frac{1+\omega}{3}\right)^2 \sigma\left(u+\frac{\omega-2}{3}\right) \sigma\left(u+\frac{2+2\omega}{3}\right)^2 \sigma\left(u+\frac{2\omega-1}{3}\right)}{\sigma\left(u-\frac{1}{3}\right)^3 \sigma\left(u+\frac{\omega}{3}\right)^3 \sigma\left(u+\frac{2\omega+1}{3}\right)^3} \end{aligned}$$

is subject to the functional equation

$$f_{12}(u) = f_{12}\left(\frac{\lambda}{3} - u\right). \tag{32}$$

The set of the triple poles of $f_{12}(u)$ on E is $\Delta_\infty = \left\{ \frac{1}{3} + \Lambda, \frac{\omega^2}{3} + \Lambda, -\frac{\omega}{3} + \Lambda \right\}$, while the set of the double zeroes is $\Delta = \left\{ \Lambda, -\left(\frac{1+\omega}{3}\right) + \Lambda, \frac{1+\omega}{3} + \Lambda \right\}$. The elliptic function

$$\begin{aligned} f_{12}\left(\frac{\lambda}{3} - u\right) = \frac{\sigma\left(\frac{\lambda}{3}-u\right)^2 \sigma\left(\frac{\lambda}{3}-u-1\right)}{\sigma\left(\frac{\lambda}{3}-u-\frac{1}{3}\right)^3} \\ \frac{\sigma\left(\frac{\lambda}{3}-u+\frac{1+\omega}{3}\right)^2 \sigma\left(\frac{\lambda}{3}-u+\frac{\omega-2}{3}\right)}{\sigma\left(\frac{\lambda}{3}-u+\frac{\omega}{3}\right)^3} \frac{\sigma\left(\frac{\lambda}{3}-u+\frac{2+2\omega}{3}\right)^2 \sigma\left(\frac{\lambda}{3}-u+\frac{2\omega-1}{3}\right)}{\sigma\left(\frac{\lambda}{3}-u+\frac{2\omega+1}{3}\right)^3}, \end{aligned}$$

has triple poles at $\Delta'_\infty = \left\{ \frac{\lambda-1}{3} + \Lambda, \frac{\lambda+\omega}{3} + \Lambda, \frac{\lambda+2\omega+1}{3} + \Lambda \right\}$ and double zeroes at $\Delta' = \left\{ \frac{\lambda}{3} + \Lambda, \frac{\lambda+1+\omega}{3} + \Lambda, \frac{\lambda+2+2\omega}{3} + \Lambda \right\}$. The functional equation (32) requires the coincidence $\Delta_\infty = \Delta'_\infty$ of the sets of the triple poles, as well as the coincidence $\Delta = \Delta'$ of the sets of the double zeroes of $f_{12}(u)$ and $f_{12}\left(\frac{\lambda}{3} - u\right)$. In particular, $\frac{\lambda-1}{3} + \Lambda \in \Delta'_\infty = \Delta_\infty$, which holds in one of the following three cases:

$$\text{Case 1: } \frac{\lambda-1}{3} + \Lambda = \frac{1}{3} + \Lambda, \tag{33}$$

$$\text{Case 2:}\quad \frac{\lambda-1}{3}+\Lambda=\frac{\omega^2}{3}+\Lambda \text{ or} \tag{34}$$

$$\text{Case 3:}\quad \frac{\lambda-1}{3}+\Lambda=-\frac{\omega}{3}+\Lambda. \tag{35}$$

If (33) then $\frac{2}{3}+\Lambda=\frac{\lambda}{3}+\Lambda\in\Delta'=\Delta$ is an absurd. Similarly, (34) implies $\frac{\omega}{3}+\Lambda=\frac{\lambda}{3}+\Lambda\in\Delta'=\Delta$ and leads to a contradiction. Finally, (35) forces $-\frac{\omega^2}{3}+\Lambda=\frac{\lambda}{3}+\Lambda\in\Delta'=\Delta$, which is not the case under consideration. In such a way, we have shown that the functional equation (32) does not hold and the generic fibre of (31) consists of a single orbit of $\left\langle \mathcal{T}\left(\frac{1+\omega}{3}+\Lambda\right)\right\rangle$. □

The next proposition plays a crucial role in the proofs of Theorems 14 and 15.

Proposition 13. *Let*

$$\psi_1(z,t)=\prod_{i=0}^{2}\frac{\sigma(z-b_i(t))}{\sigma(z-a_i(t))},\quad \psi_2(z,t)=\prod_{i=0}^{2}\frac{\sigma(z-c_i(t))}{\sigma(z-a_i(t))}\quad \textit{with}$$

$$a_i(t):=\omega^{2i}t+\frac{\lambda_i}{3},\quad b_i(t):=\omega^{2i}t+\frac{\mu_i}{3},\quad c_i(t):=-\omega^{2i}t+\frac{\nu_i}{3},$$

$$\sum_{i=0}^{2}\lambda_i=\sum_{i=0}^{2}\mu_i=\sum_{i=0}^{2}\nu_i\in 3\Lambda,$$

$\lambda_i,\mu_i,\nu_i\in\Lambda$ *be elliptic functions on* $E=(\mathbb{C},+)/(\Lambda,+)$, $\Lambda=\mathcal{O}_{\sqrt{-3}}$, *depending on a parameter* $t\in\mathbb{C}$. *Suppose that* $\frac{\lambda_0}{3}+\Lambda\neq\frac{\mu_0}{3}+\Lambda$ *and the set*

$$\Sigma:=\left\{\frac{\omega^{2i}(\nu_0-\lambda_i)}{3}+\Lambda\,\middle|\,0\le i\le 2\right\}$$

is not contained in

$$\Sigma_1:=\left\{\frac{\omega^{2i}(\nu_0-\mu_i)}{3}+\Lambda\,\middle|\,0\le i\le 2\right\}.$$

Then for any $\epsilon=\pm1$ *the rational map*

$$\Psi_\epsilon=(\psi_1,\epsilon\psi_2):E\to\Psi_\epsilon(E)\subset\mathbb{P}^1(\mathbb{C})^2$$

is injective.

P r o o f. The cardinalities of the exceptional fibres of Ψ_ϵ are bounded by the ones of the generic $\Psi_\epsilon^{-1}(x_1, x_2)$, so that it suffices to show that $\left|\Psi_\epsilon^{-1}(x_1, x_2)\right| = 1$ for a generic $(x_1, x_2) \in \Psi_\epsilon(E) \subset \mathbb{P}^1(\mathbb{C})^2$. Let us assume that $\Psi_\epsilon^{-1}(x_1, x_2) = \psi_1^{-1}(x_1) \cap (\epsilon\psi_2)^{-1}(x_2)$ contains at least two points for a generic $(x_1, x_2) \in \Psi_\epsilon(E) \subset \mathbb{P}^1(\mathbb{C})^2$. According to Lemma 7 (ii), any fibre of the elliptic functions $\epsilon\psi_i$, $1 \leq i \leq 2$ of order 3 with sum of the poles $\sum_{i=0}^{2}(a_i(t) + \Lambda) = \frac{1}{3}\left(\sum_{i=0}^{2}\lambda_i\right) + \Lambda = \Lambda$ is of the form

$$(\epsilon\psi_i)^{-1}(x_i) = \{u_i(x_i) + \Lambda, u_i'(x_i) + \Lambda, -u_i(x_i) - u_i'(x_i) + \Lambda\}, \quad x_i \in \mathbb{P}^1(\mathbb{C}).$$

The points of $\psi_1^{-1}(x_1)$ can be labeled in such a way that $u_1(x_1)+\Lambda$ and $u_1'(x_1)+\Lambda$ belong to $\Psi_\epsilon^{-1}(x_1, x_2) = \psi_1^{-1}(x_1) \cap (\epsilon\psi_2)^{-1}(x_2) \subseteq (\epsilon\psi_2)^{-1}(x_2)$. Therefore, the third point $-u_1(x_1) - u_1'(x_1) + \Lambda$ of $(\epsilon\psi_2)^{-1}(x_2)$ is also common to $\psi_1^{-1}(x_1)$ and $(\epsilon\psi_2)^{-1}(x_2)$. Thus, $\Psi_\epsilon^{-1}(x_1, x_2) = \psi_1^{-1}(x_1) = (\epsilon\psi_2)^{-1}(x_2)$ for generic $(x_1, x_2) \in \Psi_\epsilon(E) \subset \mathbb{P}^1(\mathbb{C})^2$ and $\Psi_\epsilon : E \to \Psi_\epsilon(E)$ is (a possibly ramified) covering of degree 3. The canonical projections $\mathrm{Pr}_1 : \mathbb{P}^1(\mathbb{C})^2 \to \mathbb{P}^1(\mathbb{C})$, $\mathrm{Pr}_1(x_1, x_2) = x_1$, $\forall (x_1, x_2) \in \mathbb{P}^1(\mathbb{C})^2$, $\mathrm{Pr}_2 : \mathbb{P}^1(\mathbb{C})^2 \to \mathbb{P}^1(\mathbb{C})$, $\mathrm{Pr}_2(x_1, x_2) = x_2$, $\forall (x_1, x_2) \in \mathbb{P}^1(\mathbb{C})^2$ on the factors satisfy the commutative diagrams

$$\begin{array}{ccc} E & \xrightarrow{\Psi_\epsilon} & \Psi_\epsilon(E) \\ {\scriptstyle\psi_1}\downarrow & \swarrow{\scriptstyle \mathrm{Pr}_1} & \\ \mathbb{P}^1(\mathbb{C}) & & \end{array} \quad \text{and} \quad \begin{array}{ccc} E & \xrightarrow{\Psi_\epsilon} & \Psi_\epsilon(E) \\ {\scriptstyle\epsilon\psi_2}\downarrow & \swarrow{\scriptstyle \mathrm{Pr}_2} & \\ \mathbb{P}^1(\mathbb{C}) & & \end{array}$$

of surjective morphisms. Since Ψ_ϵ, ψ_1 and $\epsilon\psi_2$ are of one and a same degree $\deg(\Psi_\epsilon) = \deg(\psi_1) = \deg(\epsilon\psi_2) = 3$, the restrictions $\mathrm{Pr}_1|_{\Psi_\epsilon(E)} : \Psi_\epsilon(E) \to \mathbb{P}^1(\mathbb{C})$ and $\mathrm{Pr}_2|_{\Psi_\epsilon(E)} : \Psi_\epsilon(E) \to \mathbb{P}^1(\mathbb{C})$ are of degree $\deg\left(\mathrm{Pr}_1|_{\Psi_\epsilon(E)}\right) = \deg\left(\mathrm{Pr}_2|_{\Psi_\epsilon(E)}\right) = 1$, i.e., bijective morphisms of $\Psi_\epsilon(E)$ onto $\mathbb{P}^1(\mathbb{C})$. Denoting

$$h_1 := \mathrm{Pr}_2\left(\mathrm{Pr}_1|_{\Psi_\epsilon(E)}\right)^{-1} : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}),$$

$$h_2 := \mathrm{Pr}_1\left(\mathrm{Pr}_2|_{\Psi_\epsilon(E)}\right)^{-1} : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}),$$

one observes that $\epsilon\psi_2 = h_1\psi_1$, $\psi_1 = h_2(\epsilon\psi_2)$, whereas $\psi_1 = h_2h_1\psi_1$, $\epsilon\psi_2 = h_1h_2(\epsilon\psi_2)$. Due to the surjectiveness of $\psi_1 : E \to \mathbb{P}^1(\mathbb{C})$, $\epsilon\psi_2 : E \to \mathbb{P}^1(\mathbb{C})$, that suffices for $h_2h_1 = \mathrm{Id}_{\mathbb{P}^1(\mathbb{C})}$, $h_1h_2 = \mathrm{Id}_{\mathbb{P}^1(\mathbb{C})}$. In such a way, we have established

that $h_1 : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a holomorphic automorphism with inverse $h_1^{-1} = h_2$ and $h_1 \in \mathrm{PGL}(2, \mathbb{C})$. If h_1 lifts to a matrix

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}) \tag{36}$$

then $h_1\psi_1 = \epsilon\psi_2$ is equivalent to

$$\frac{p\psi_1 + q}{r\psi_1 + s} = \epsilon\psi_2. \tag{37}$$

If $\psi^-(z,t) := \prod_{i=0}^{2} \sigma(z - a_i(t))$, $\psi_1^+(z,t) := \prod_{i=0}^{2} \sigma(z - b_i(t))$, $\psi_2^+(z,t) := \prod_{i=0}^{2} \sigma(z - c_i(t))$ then $\psi_1(z,t) = \dfrac{\psi_1^+(z,t)}{\psi^-(z,t)}$, $\psi_2(z,t) = \dfrac{\psi_2^+(z,t)}{\psi^-(z,t)}$ and (37) is equivalent to

$$p\psi_1^+(z,t) + q\psi^-(z,t) = r\epsilon\frac{\psi_1^+(z,t)\psi_2^+(z,t)}{\psi^-(z,t)} + s\epsilon\psi_2^+(z,t).$$

Let us assume that $r \neq 0$. Then

$$\psi_3(z,t) := \frac{\psi_1^+(z,t)\psi_2^+(z,t)}{\psi^-(z,t)} = \frac{p}{r\epsilon}\psi_1^+(z,t) + \frac{q}{r\epsilon}\psi^-(z,t) - \frac{s}{r}\psi_2^+(z,t)$$

is an entire function on $\mathbb{C}^2$. By the very definition of $\psi^-(z,t)$, there holds $\psi^-(a_0(t),t) = \psi^-\left(t + \dfrac{\lambda_0}{3}, t\right) = 0$ for all $t \in \mathbb{C}$. Bearing in mind that $\sigma(u) = 0$ if and only if $u \in \Lambda$, one observes that

$$\psi_1^+\left(t + \frac{\lambda_0}{3}, t\right) = \sigma\left(\frac{\lambda_0 - \mu_0}{3}\right)\sigma\left((2-\omega)t + \frac{\lambda_0 - \mu_1}{3}\right)\sigma\left((1+\omega)t + \frac{\lambda_0 - \mu_2}{3}\right) \not\equiv 0,$$

$$\epsilon\psi_2\left(t + \frac{\lambda_0}{3}, t\right) = \epsilon\sigma\left(2t + \frac{\lambda_0 - \nu_0}{3}\right)\sigma\left(\omega t + \frac{\lambda_0 - \nu_1}{3}\right)\sigma\left(-\omega^2 t + \frac{\lambda_0 - \nu_2}{3}\right) \not\equiv 0$$

do not vanish identically for $\dfrac{\lambda_0-\mu_0}{3}\notin\Lambda$. Thus, $\psi_3\left(t+\dfrac{\lambda_0}{3},t\right)\equiv\infty$, which is a contradiction, establishing $r=0$ and reducing (37) to

$$p\psi_1(z,t)+q=s\epsilon\psi_2(z,t)\quad\text{with}\quad p,s\in\mathbb{C}^*, \tag{38}$$

due to (36). By the very definition of $\psi_2(z,t)$, one has

$$\psi_2^+(c_0(t),t)=\psi_2^+\left(-t+\frac{\nu_0}{3},t\right)\equiv 0,$$

$$\psi^-\left(-t+\frac{\nu_0}{3},t\right)=\sigma\left(-2t+\frac{\nu_0-\lambda_0}{3}\right)\sigma\left(-\omega t+\frac{\nu_0-\lambda_1}{3}\right)\sigma\left(\omega^2 t+\frac{\nu_0-\lambda_2}{3}\right)\not\equiv 0,$$

whereas $\psi_2\left(-t+\dfrac{\nu_0}{3},t\right)=\dfrac{\psi_2^+\left(-t+\frac{\nu_0}{3},t\right)}{\psi^-\left(-t+\frac{\nu_0}{3},t\right)}\equiv 0$. In such a way, (38) specializes to

$$p\psi_1^+\left(-t+\frac{\nu_0}{3},t\right)+q\psi^-\left(-t+\frac{\nu_0}{3},t\right)\equiv 0. \tag{39}$$

Bearing in mind that $\sigma(u)=0$ if and only if $u\in\Lambda$, one observes that the entire functions

$$\psi_1^+\left(-t+\frac{\nu_0}{3},t\right)=\sigma\left(-2t+\frac{\nu_0-\mu_0}{3}\right)\sigma\left(-\omega t+\frac{\nu_0-\mu_1}{3}\right)\sigma\left(\omega^2 t+\frac{\nu_0-\mu_2}{3}\right)$$

and

$$\psi^-\left(-t+\frac{\nu_0}{3},t\right)=\sigma\left(-2t+\frac{\nu_0-\lambda_0}{3}\right)\sigma\left(-\omega t+\frac{\nu_0-\lambda_1}{3}\right)\sigma\left(\omega^2 t+\frac{\nu_0-\lambda_2}{3}\right)$$

are not $(\Lambda,+)$-periodic, but their zero divisors

$$\widetilde{S}_1^+:=\left\{t_0\in\mathbb{C}\,\middle|\,\psi_1^+\left(-t_0+\frac{\nu_0}{3},t_0\right)=0\right\},\ \widetilde{S}^-:=\left\{t_0\in\mathbb{C}\,\middle|\,\psi^-\left(-t_0+\frac{\nu_0}{3},t_0\right)=0\right\}$$

are $(\Lambda,+)$-invariant, due to $-2,-\omega,\omega^2\in\mathrm{End}(E)=\{\xi\in\mathbb{C}\,|\,\xi\Lambda\subseteq\Lambda\}$. That allows to consider the $(\Lambda,+)$-orbit spaces $S_1^+:=\widetilde{S}_1^+/\Lambda$ $S^-:=\widetilde{S}^-/\Lambda$ as subsets of the elliptic curve $E=(\mathbb{C},+)/(\Lambda,+)$. More precisely,

$$S_1^+=\left[\left(\frac{\nu_0-\mu_0}{6}+\Lambda\right)+E^{2-\mathrm{tor}}\right]\cup\left\{-\frac{\omega^2(\nu_0-\mu_1)}{3}+\Lambda,\ \frac{\omega(\nu_0-\mu_2)}{3}+\Lambda\right\},$$

$$S^-=\left[\left(\frac{\nu_0-\lambda_0}{6}+\Lambda\right)+E^{2-\mathrm{tor}}\right]\cup\left\{-\frac{\omega^2(\nu_0-\lambda_1)}{3}+\Lambda,\ \frac{\omega(\nu_0-\lambda_2)}{3}+\Lambda\right\}.$$

It suffices to show the existence of a point $t_0+\Lambda\in S^-\setminus S_1^+$, because

$$\psi^-\left(-t_0+\frac{\nu}{3},t_0\right)=0\ \text{ and }\ p\psi_1^+\left(-t_0+\frac{\nu_0}{3},t_0\right)\in\mathbb{C}^*$$

contradict (39) and imply that $\deg(\Psi_\epsilon)=1$. If $S^-\setminus S_1^+=\emptyset$ then $S^-\subseteq S_1^+$, whereas $2S^-\subseteq 2S_1^+$ for

$$2S^-:=\left\{2(t_0+\Lambda)\,|\,t_0+\Lambda\in S^-\right\}=\left\{\frac{\omega^{2i}(\nu_0-\lambda_i)}{3}+\Lambda\,\Big|\,0\le i\le 2\right\}=\Sigma,$$

$$2S_1^+:=\left\{2(t_0+\Lambda)\,|\,t_0+\Lambda\in S_1^+\right\}=\left\{\frac{\omega^{2i}(\nu_0-\mu_i)}{3}+\Lambda\,\Big|\,0\le i\le 2\right\}=\Sigma_1,$$

due to $-\dfrac{2\omega^{2i}(\nu_0-\lambda_i)}{3}+\Lambda=\dfrac{\omega^{2i}(\nu_0-\lambda_i)}{3}+\Lambda$ for $1\le i\le 2$. In such a way, the inclusion $S^-\subseteq S_1^+$ leads to $\Sigma=2S^-\subseteq 2S_1^+=\Sigma_1$, which contradicts the assumption $\Sigma\nsubseteqq\Sigma_1$ and concludes the proof of the proposition. □

In order to state and to prove the main results of the article, let us note that the divisors $B_i=\mathrm{Orb}_{G_3}(B_{00})+\mathrm{Orb}_{G_3}(B_{i0})$, $1\le i\le 2$ are effective, so that

$$\mathcal{L}_{Y_3}\left(m_1B_i,n_1B_i^{\mathrm{sing}}\right)\subseteq\mathcal{L}_{Y_3}\left(m_2B_i,n_2B_i^{\mathrm{sing}}\right)$$

for arbitrary natural numbers $m_i,n_i\in\mathbb{N}$, subject to the inequalities $m_1\le m_2$, $n_1\le n_2$.

Theorem 14. *Let* $\mathrm{Bl}_1:\left(X_1=(\mathbb{B}/\Gamma_1)',D^{(1)}\right)\to(Y_3,C_0+C_1)$ *be the blow down of the smooth irreducible rational* (-1)*-curves on Di Cerbo and Stover's smooth toroidal compactification of minimal Haar measure* $\mathrm{vol}\,(\mathbb{B}/\Gamma_1)=\dfrac{8\pi^2}{3}$*, described in Theorem 5,* $\pi_3:E\times E\to Y_3=E\times E/G_3$ *be the unramified* G_3*-Galois covering of its bi-elliptic minimal model* Y_3*,* $G_3=\langle g_3\rangle$*,* $g_3=\left(\mathcal{T}\left(\dfrac{1+\omega}{3}+\Lambda\right),\omega^2\right)$*,* $C_0:=\pi_3(E(1,1))$*,* $C_1=\pi_3\left(\left(\dfrac{1}{3}+\Lambda\right)\times E\right)$ *and* $\rho_1:X_1=(\mathbb{B}/\Gamma_1)'\to\widehat{\mathbb{B}/\Gamma_1}$ *be the contraction of the toroidal compactifying divisor* $D^{(1)}:=X_1\setminus(\mathbb{B}/\Gamma_1)=D_0+D_1$ *to the cusps* $\kappa_i=\rho_1(D_i)$*,* $0\le i\le 1$*. Consider the rational functions*

$$\varphi_0\in\mathcal{L}_{Y_3}\left(C_0+C_1,3(C_0+C_1)^{\mathrm{sing}}\right),\quad\varphi_{10}\in\mathcal{L}_{Y_3}\left(C_0+C_1,3(C_0+C_1)^{\mathrm{sing}}\right),$$

$$\varphi_{11}\in\mathcal{L}_{Y_3}\left(2(C_0+C_1),2(C_0+C_1)^{\mathrm{sing}}\right),\ \varphi_{12}\in\mathcal{L}_{Y_3}\left(3(C_0+C_1),3(C_0+C_1)^{\mathrm{sing}}\right),$$

which pull back to the G_3*-invariant abelian functions*

$$\pi_3^*(\varphi_0)(u,v) = e_0(u,v) = \frac{e_0^+(u,v)}{e_0^-(u,v)} \in \mathcal{L}_{E\times E}\left(B_1, 3B_1^{\mathrm{sing}}\right)^{G_3},$$

$$\pi_3^*(\varphi_{10})(u,v) = f_{10}(u,v) = \frac{f_0^+(u,v)}{e_0^-(u,v)}\,\frac{e_1^+(u)}{e_1^-(u)} \in \mathcal{L}_{E\times E}\left(B_1, 3B_1^{\mathrm{sing}}\right)^{G_3},$$

$$\pi_3^*(\varphi_{11})(u) = f_{11}(u) = \prod_{i=0}^{2}(g_3^*)^i\left(\frac{\sigma(u)\sigma\left(u-\frac{2}{3}\right)}{\sigma\left(u-\frac{1}{3}\right)^2}\right) \in \mathcal{L}_{E\times E}\left(2B_1, 2B_1^{\mathrm{sing}}\right)^{G_3},$$

$$\pi_3^*(\varphi_{12})(u) = f_{12}(u) = \prod_{i=0}^{2}(g_3^*)^i\left(\frac{\sigma(u)^2\sigma(u-1)}{\sigma\left(u-\frac{1}{3}\right)^3}\right) \in \mathcal{L}_{E\times E}\left(3B_1, 3B_1^{\mathrm{sing}}\right)^{G_3}$$

with $e_0^+(u,v)$ *from* (13), $e_0^-(u,v)$ *from* (14), $f_0^+(u,v)$ *from* (23), $e_1^+(u)$ *from* (24), $e_1^-(u)$ *from* (25), *the transfer* $\theta_3 : \mathcal{L}_{Y_3}(3(C_0+C_1), 3(C_0+C_1)^{\mathrm{sing}}) \to j_3\left(\mathcal{M}_3(\Gamma_1)\right)$ *of rational functions on* Y_3 *to* Γ_1*-modular forms of weight* 3 *and a standard non-cuspidal* Γ_1*-modular form* $\omega_0^{[3]} \in j_3\left(\mathcal{M}_3(\Gamma_1)\right) \setminus j_3\left(\mathcal{M}_3^o(\Gamma_1)\right)$ *with* $\omega_0^{[3]}(\kappa_0) = 1$, $\omega_0^{[3]}(\kappa_1) = 0$. *Then*

$$\Phi_1 = \left[\theta_3(1) : \theta_3(\varphi_0) : \theta_3(\varphi_{10}) : \theta_3(\varphi_{11}) : \theta_3(\varphi_{12}) : \omega_0^{[3]}\right] : \widehat{\mathbb{B}/\Gamma_1} \longrightarrow \mathbb{P}^5(\mathbb{C}) \tag{40}$$

is a regular embedding of the Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma_1}$ *of* $\mathbb{B}/\Gamma_1$.

P r o o f. Note that $f_{12} = \pi_3^*(\varphi_{12})$ has pole divisor $(f_{12})_\infty = 3\mathrm{Orb}_{G_3}(B_{10})$, so that φ_{12} has a pole of order 3 along $C_1 = \pi_3(B_{10})$. By the remark before Proposition 4, the transfer $\theta_3(\varphi_{12}) \in j_3\left(\mathcal{M}_3(\Gamma_1)\right) \setminus j_3\left(\mathcal{M}_3^o(\Gamma_1)\right)$ has values $\theta_3(\varphi_{12})(\kappa_0) = 0$, $\theta_3(\varphi_{12})(\kappa_1) \in \mathbb{C}^*$ at the Γ_1-cusps and gives rise to a standard non-cuspidal modular form $\omega_1^{[3]} := \dfrac{\theta_3(\varphi_{12})}{\theta_3(\varphi_{12})(\kappa_1)}$. Once $\theta_3(1)$, $\theta_3(\varphi_{12})(\kappa_1)\omega_1^{[3]}$, $\omega_0^{[3]}$ are homogeneous components of Φ_1, Proposition 4 (i) applies to provide the regularity of (40). According to Proposition 4 (ii), it suffices to establish that the generic fibre of the G_3-invariant rational map

$$f_1 = (e_0, f_{10}, f_{11}, f_{12}) : E\times E \longrightarrow f_1(E\times E) \subset \mathbb{P}^1(\mathbb{C})^4 \tag{41}$$

consists of a single G_3-orbit, in order to conclude that (40) is a regular embedding. Let us introduce the rational maps $\Upsilon_1 = (e_0, f_{10}) : E\times E \to \Upsilon_1(E\times E) \subseteq \mathbb{P}^1(\mathbb{C})^2$, $\Omega_1 = (f_{11}, f_{12}) : E\times E \to \Omega_1(E\times E) \subset \mathbb{P}^1(\mathbb{C})^2$, in order to represent the fibre of $f_1 = (\Upsilon_1, \Omega_1)$ over $x = (x_1, \ldots, x_4) \in f_1(E\times E) \subset \mathbb{P}^1(\mathbb{C})^4$, $x_i \in \mathbb{P}^1(\mathbb{C})$

as an intersection $f_1^{-1}(x) = \Upsilon_1^{-1}(x_1, x_2) \cap \Omega_1^{-1}(x_3, x_4)$. By Proposition 12, the $\left\langle \mathcal{T}\left(\frac{1+\omega}{3} + \Lambda\right)\right\rangle$-invariant rational map $\Delta_1 = (f_{11}, f_{12}) : E \longrightarrow \Delta_1(E) \subset \mathbb{P}^1(\mathbb{C})^2$ has generic fibre $\Delta_1^{-1}(x_3, x_4) = \mathrm{Orb}_{\langle \mathcal{T}\left(\frac{1+\omega}{3}+\Lambda\right)\rangle}\,(u(x_3, x_4) + \Lambda)$, which consists of a single $\left\langle \mathcal{T}\left(\frac{1+\omega}{3} + \Lambda\right)\right\rangle$-orbit. Thus,

$$\Omega_1^{-1}(x_3, x_4) := \{(u + \Lambda, v + \Lambda) \in E \times E \,|\, f_{11}(u + \Lambda) = x_3,\ f_{12}(u + \Lambda) = x_4\} =$$

$$= \Delta_1^{-1}(x_3, x_4) \times E = \mathrm{Orb}_{\langle \mathcal{T}\left(\frac{1+\omega}{3}+\Lambda\right)\rangle}\,(u(x_3, x_4) + \Lambda) \times E = \coprod_{i=0}^{2} p_i(x_3, x_4) \times E$$

with $p_i(x_3, x_4) := u(x_3, x_4) + i\left(\frac{1+\omega}{3}\right) + \Lambda$ is a disjoint union of three vertical elliptic curves. As a result, one represents

$$f_1^{-1}(x) = \Upsilon_1^{-1}(x_1, x_2) \cap \Omega_1^{-1}(x_3, x_4) = \coprod_{i=0}^{2} \left[(p_i(x_3, x_4) \times E) \cap \Upsilon_1^{-1}(x_1, x_2)\right].$$

If $\Xi_{p_i(x_3,x_4)} = (e_0\,(p_i(x_3, x_4),\ ),\, f_{10}\,(p_i(x_3, x_4),\ )) : E \to \Xi_{p_i(x_3,x_4)}(E) \subset \mathbb{P}^1(\mathbb{C})^2$ then $f_1^{-1}(x) = \coprod_{i=0}^{2} p_i(x_3, x_4) \times \Xi^{-1}_{p_i(x_3,x_4)}(x_1, x_2)$. It suffices to show the injectiveness of $\Xi_{p_i(x_3,x_4)} : E \to \Xi_{p_i(x_3,x_4)}(E) \subset \mathbb{P}^1(\mathbb{C})^2$ for all $p_i(x_3, x_4) \in E$, in order to conclude that the fibres of the G_3-invariant map (41) are generically of cardinality 3 and constitute a single G_3-orbit. Note that for a fixed generic $t + \Lambda \in E$ the degree of

$$\Xi_{t+\Lambda} = \left(\frac{e_0^+(t + \Lambda,\ )}{e_0^-(t + \Lambda,\ )},\, \frac{f_0^+(t + \Lambda,\ )}{e_0^-(t + \Lambda,\ )}\,\frac{e_1^+(t + \Lambda)}{e_1^-(t + \Lambda)}\right) : E \longrightarrow \Xi_{t+\Lambda}(E) \subset \mathbb{P}^1(\mathbb{C})^2$$

coincides with the degree of

$$\Psi_{t+\Lambda} = \left(\frac{e_0^+(t + \Lambda,\ )}{e_0^-(t + \Lambda,\ )},\, \frac{f_0^+(t + \Lambda,\ )}{e_0^-(t + \Lambda,\ )}\right)$$
$$= (e_0, f_0)\,(t + \Lambda,\ ) : E \longrightarrow \Psi_{t+\Lambda}(E) \subset \mathbb{P}^1(\mathbb{C})^2.$$

Combining $t - \omega^{2i} z + c_i = -\omega^{2i}\left(z - \omega^{4i} t - \omega^{4i} c_i\right)$ for $0 \leq i \leq 2$, $c_i \in \mathbb{C}$, $(t, z) \in \mathbb{C}^2$ with with (9), one represents

$$e_0^+(t, z) = -\sigma\left(z - t + \frac{4 - 2\omega}{3}\right)\sigma\left(z - \omega^2 t - \left(\frac{\omega + 4}{3}\right)\right)\sigma(z + \omega t),$$

$$e_0^-(t,z) = -\sigma(z-t)\sigma\left(z-\omega^2 t+\frac{\omega-2}{3}\right)\sigma\left(z+\omega t+\frac{2-4\omega}{3}\right).$$

Similarly, $t+\omega^{2i}z+c_i = \omega^{2i}\left(z+\omega^{4i}t+\omega^{4i}c_i\right)$ with $0\le i\le 2$, $c_i\in\mathbb{C}$, $(t,z)\in\mathbb{C}^2$ and (9) provide

$$f_0^+(t,z) = \sigma\left(z+t-\left(\frac{1+2\omega}{3}\right)\right)\sigma\left(z+\omega^2 t-\frac{2\omega^2}{3}\right)\sigma\left(z-\omega t+\frac{\omega^2}{3}\right).$$

In order to apply Proposition 13 to $\Psi_{t+\Lambda}$, let us introduce $a_i(t) := \omega^{2i}t+\dfrac{\lambda_i}{3}$, $b_i(t) := \omega^{2i}t+\dfrac{\mu_i}{3}$, $c_i(t) := -\omega^{2i}t+\dfrac{\nu_i}{3}$ with $\lambda_0=0$, $\lambda_1=2-\omega$, $\lambda_2=4\omega-2$, $\mu_0=2\omega-4$, $\mu_1=\omega+4$, $\mu_2=0$, $\nu_0=1+2\omega$, $\nu_1=2\omega^2$, $\nu_2=-\omega^2$ and express $e_0(t,z)=\dfrac{e_0^+(t,z)}{e_0^-(t,z)}=\prod\limits_{i=0}^{2}\dfrac{\sigma(z-b_i(t))}{\sigma(z-a_i(t))}$, $f_0(t,z)=\dfrac{f_0^+(t,z)}{e_0^-(t,z)}=-\prod\limits_{i=0}^{2}\dfrac{\sigma(z-c_i(t))}{\sigma(z-a_i(t))}$. As far as $\sum\limits_{i=0}^{2}\lambda_i=\sum\limits_{i=0}^{2}\mu_i=\sum\limits_{i=0}^{2}\nu_i=3\omega\in 3\Lambda=3\mathcal{O}_{\sqrt{-3}}$, $\dfrac{\lambda_0}{3}+\Lambda=\Lambda\neq\dfrac{-1-\omega}{3}+\Lambda=\dfrac{\mu_0}{3}+\Lambda$ and the set $\Sigma:=\left\{\dfrac{\omega^{2i}(\nu_0-\lambda_i)}{3}+\Lambda\,\middle|\,0\le i\le 2\right\}=\left\{-\dfrac{\omega^2}{3}+\Lambda\right\}$ is not contained in the set $\Sigma_1:=\left\{\dfrac{\omega^{2i}(\nu_0-\mu_i)}{3}+\Lambda\,\middle|\,0\le i\le 2\right\}=\left\{-\dfrac{1}{3}+\Lambda\right\}$, the assumptions of Proposition 13 hold for the rational map $\Psi_{t+\Lambda}$ and imply its injectiveness for all $t+\Lambda\in E$. □

Theorem 15. *Let* $\mathrm{Bl}_2:\left(X_2=(\mathbb{B}/\Gamma_2)',D^{(2)}\right)\to(Y_3,C_0+C_2)$ *be the blow down of the smooth irreducible rational* (-1)*-curves on Di Cerbo and Stover's smooth toroidal compactification of minimal Haar measure* $\mathrm{vol}\,(\mathbb{B}/\Gamma_2)=\dfrac{8\pi^2}{3}$*, described in Theorem* 5*,* $\pi_3:E\times E\to Y_3=E\times E/G_3$ *be the unramified* G_3*-Galois covering of its bi-elliptic minimal model* Y_3*,* $G_3=\langle g_3\rangle$*,* $g_3=\left(\mathcal{T}\left(\dfrac{1+\omega}{3}+\Lambda\right),\omega^2\right)$*,* $C_0:=\pi_3(E(1,1))$*,* $C_2=\pi_3\left(E\times\left(\dfrac{1}{3}+\Lambda\right)\right)$ *and* $\rho_2:X_1=(\mathbb{B}/\Gamma_2)'\to\widehat{\mathbb{B}/\Gamma_2}$ *be the contraction of the toroidal compactifying divisor* $D^{(2)}:=X_2\setminus(\mathbb{B}/\Gamma_2)=D_0+D_2$ *to the cusps* $\kappa_i=\rho_1(D_i)$*,* $i\in\{0,2\}$*. Consider the rational functions*

$$\varphi_0\in\mathcal{L}_{Y_3}\left(C_0+C_2,3(C_0+C_2)^{\mathrm{sing}}\right),\quad \varphi_{20}\in\mathcal{L}_{Y_3}\left(C_0+C_2,3(C_0+C_2)^{\mathrm{sing}}\right),$$

$$\varphi_{21}\in\mathcal{L}_{Y_3}\left(C_0+C_2,(C_0+C_2)^{\mathrm{sing}}\right),\quad \varphi_{22}\in\mathcal{L}_{Y_3}\left(3(C_0+C_2),3(C_0+C_2)^{\mathrm{sing}}\right),$$

which pull back to the G_3-invariant abelian functions

$$\pi_3^*(\varphi_0)(u,v) = e_0(u,v) = \frac{e_0^+(u,v)}{e_0^-(u,v)} \in \mathcal{L}_{E\times E}\left(B_2, 3B_2^{\mathrm{sing}}\right)^{G_3},$$

$$\pi_3^*(\varphi_{20})(u,v) = f_{20}(u,v) = \frac{f_0^+(u,v)}{e_0^-(u,v)}\,\frac{e_2^+(v)}{e_2^-(v)} \in \mathcal{L}_{E\times E}\left(B_2, 3B_2^{\mathrm{sing}}\right)^{G_3},$$

$$\pi_3^*(\varphi_{21})(v) = f_{21}(v) := \frac{\sigma(v)^3}{\sigma\left(v-\frac{1}{3}\right)\sigma\left(v-\frac{\omega^2}{3}\right)\sigma\left(v+\frac{\omega}{3}\right)} \in \mathcal{L}_{E\times E}\left(B_2, B_2^{\mathrm{sing}}\right)^{G_3},$$

$$\pi_3^*(\varphi_{22})(v) = f_{22}(v) = \prod_{i=0}^{2}(g_3^*)^i\left(\frac{\sigma(v)^2\sigma(v-1)}{\sigma\left(v-\frac{1}{3}\right)^3}\right) \in \mathcal{L}_{E\times E}\left(3B_2, 3B_2^{\mathrm{sing}}\right)^{G_3}$$

with $e_0^+(u,v)$ from (13), *$e_0^-(u,v)$ from* (14), *$f_0^+(u,v)$ from* (23), *$e_2^+(v)$ from* (26), *$e_2^-(v)$ from* (27), *the transfer $\theta_3 : \mathcal{L}_{Y_3}(3(C_0+C_2), 3(C_0+C_2)^{\mathrm{sing}}) \to j_3\left(\mathcal{M}_3(\Gamma_2)\right)$ of rational functions on Y_3 to Γ_2-modular forms of weight 3 and a standard non-cuspidal Γ_2-modular form $\omega_0^{[3]} \in j_3\left(\mathcal{M}_3(\Gamma_2)\right) \setminus j_3\left(\mathcal{M}_3^o(\Gamma_2)\right)$ with $\omega_0^{[3]}(\kappa_0) = 1$, $\omega_0^{[3]}(\kappa_2) = 0$. Then*

$$\Phi_2 = \left[\theta_3(1) : \theta_3(\varphi_0) : \theta_3(\varphi_{20}) : \theta_3(\varphi_{21}) : \theta_3(\varphi_{22}) : \omega_0^{[3]}\right] : \widehat{\mathbb{B}/\Gamma_2} \longrightarrow \mathbb{P}^5(\mathbb{C}) \tag{42}$$

is a regular embedding of the Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma_2}$ of $\mathbb{B}/\Gamma_2$.

P r o o f. Note that the modular form $\theta_3(\varphi_{22}) \in j_3\mathcal{M}_3(\Gamma_2)$ has values $\theta_3(\varphi_{22})(\kappa_0) = 0$, $\theta_3(\varphi_{22})(\kappa_2) \in \mathbb{C}^*$ at the cusps, as far as $f_{22} = \pi_3^*(\varphi_{22})$ has a pole of order 3 along the divisor $\mathrm{Orb}_{G_3}(B_{20}) \subset E\times E$, whereas φ_{22} has a pole of order 3 along $C_2 = \pi_3(B_{20})$. The presence of components $\theta_3(1)$, $\theta_3(\varphi_{22})$, $\omega_0^{[3]}$ of Φ_2 guarantees the regularity of Φ_2, according to Proposition 4 (i). We claim that the generic fibre of the G_3-invariant rational map

$$f_2 = (e_0, f_{20}, f_{21}, f_{22}) : E\times E \longrightarrow f_2(E\times E) \subset \mathbb{P}^1(\mathbb{C})^4 \tag{43}$$

consists of a single G_3-orbit. To this end, let us consider the rational maps $\Upsilon_2 = (e_0, f_{20}) : E\times E \to \Upsilon_2(E\times E) \subseteq \mathbb{P}^1(\mathbb{C})^2$, $\Omega_2 = (f_{21}, f_{22}) : E\times E \to \Omega_2(E\times E) \subset \mathbb{P}^1(\mathbb{C})^2$ and express $f_2 = (\Upsilon_2, \Omega_2) : E\times E \to f_2(E\times E) \subset \mathbb{P}^1(\mathbb{C})^4$. All the fibres $f_{21}^{-1}(x_3)$, $x_3 \in \mathbb{P}^1(\mathbb{C})$ of the $\langle\omega^2\rangle$-invariant elliptic function $f_{21} : E \to \mathbb{P}^1(\mathbb{C})$ of order 3 consist of a single $\langle\omega^2\rangle$-orbit, i.e., $f_{21}^{-1}(x_3) = \mathrm{Orb}_{\langle\omega^2\rangle}\left(v(x_3)+\Lambda\right)$ for some $v(x_3)+\Lambda \in E$. The fibres of the $\langle\omega^2\rangle$-invariant rational map $\Delta_2 = (f_{21}, f_{22}) : E \to \Delta_2(E) \subset \mathbb{P}^1(\mathbb{C})^2$ are contained and, therefore, coincide with the fibres of

f_{21}, i.e. $\Delta_2^{-1}(x_3,x_4) = f_{21}^{-1}(x_3) = \mathrm{Orb}_{\langle\omega^2\rangle}(v(x_3)+\Lambda)$ for all $(x_3,x_4) \in \Delta_2(E) \subset \mathbb{P}^1(\mathbb{C})^2$. As a result, the fibres of Ω_2 are disjoint unions of horizontal elliptic curves $\Omega_2^{-1}(x_3,x_4) = E \times \Delta_2^{-1}(x_3,x_4) = E \times \mathrm{Orb}_{\langle\omega^2\rangle}(v(x_3)+\Lambda)$. One can assume that $v(x_3)+\Lambda$ does not belong to the fixed point group

$$\begin{aligned}\mathrm{Fix}_E(\omega^2) :=& \{v+\Lambda \,|\, \omega^2(v+\Lambda) = v+\Lambda\} = \left\{ i\left(\frac{1+\omega}{3}+\Lambda\right) \,\middle|\, 0 \le i \le 2 \right\} \\ =& \left\langle \frac{1+\omega}{3}+\Lambda \right\rangle \simeq (\mathbb{Z}_3,+)\end{aligned}$$

of ω^2, so that $\Omega_2^{-1}(x_3,x_4) = \coprod_{i=0}^{2} E \times p_i(x_3)$ with $p_i(x_3) := \omega^{2i} v(x_3) + \Lambda$ has three irreducible components. As a result,

$$f_2^{-1}(x) = \Upsilon_2^{-1}(x_1,x_2) \cap \Omega_2^{-1}(x_3,x_4) = \coprod_{i=0}^{2} \left[(E \times p_i(x_3)) \cap \Upsilon_2^{-1}(x_1,x_2) \right].$$

Let us denote $\Xi_{p_i(x_3)} = (e_0(\ ,p_i(x_3)),\ f_{20}(\ ,p_i(x_3)) : E \to \Xi_{p_i(x_3)}(E) \subset \mathbb{P}^1(\mathbb{C})^2$ and observe that

$$f_2^{-1}(x) = \coprod_{i=0}^{2} \left[(E \times p_i(x_3)) \cap \Upsilon_2^{-1}(x_1,x_2) \right] = \coprod_{i=0}^{2} \Xi_{p_i(x_3)}^{-1}(x_1,x_2) \times p_i(x_3).$$

We claim that $\Xi_{t+\Lambda} = (e_0(\ ,t+\Lambda), f_{20}(\ ,t+\Lambda)) : E \to \Xi_{t+\Lambda}(E) \subset \mathbb{P}^1(\mathbb{C})^2$ is a rational map of degree 1 for all $t+\Lambda \in E$, in order to conclude that the generic fibre of the G_3-invariant rational map (43) consists of a single G_3-orbit. To this end, let us observe that the degree of

$$\Xi_{t+\Lambda} = \left(\frac{e_0^+(\ ,t+\Lambda)}{e_0^-(\ ,t+\Lambda)}, \frac{f_0^+(\ ,t+\Lambda)}{e_0^-(\ ,t+\Lambda)} \frac{e_2^+(t+\Lambda)}{e_2^-(t+\Lambda)} \right) : E \longrightarrow \Xi_{t+\Lambda}(E) \subset \mathbb{P}^1(\mathbb{C})^2$$

coincides with the degree of

$$\begin{aligned}\Psi_{t+\Lambda} &= \left(\frac{e_0^+(\ ,t+\Lambda)}{e_0^-(\ ,t+\Lambda)}, \frac{f_0^+(\ ,t+\Lambda)}{e_0^-(\ ,t+\Lambda)} \right) \\ &= (e_0,f_0)(\ ,t+\Lambda) : E \longrightarrow \Psi_{t+\Lambda}(E) \subset \mathbb{P}^1(\mathbb{C})^2.\end{aligned}$$

After representing $e_0(z,t) = \prod_{i=0}^{2} \frac{\sigma(z-b_i(t))}{\sigma(z-a_i(t))}$, $f_0(z,t) = \prod_{i=0}^{2} \frac{\sigma(z-c_i(t))}{\sigma(z-a_i(t))}$ by affine linear functions $a_i(t) := \omega^{2i}t + \frac{\lambda_i}{3}$, $b_i(t) := \omega^{2i}t + \frac{\mu_i}{3}$, $c_i(t) := -\omega^{2i}t + \frac{\nu_i}{3}$ with

$\lambda_0 = 0$, $\lambda_1 = 2+2\omega$, $\lambda_2 = 1+\omega$, $\mu_0 = 4-2\omega$, $\mu_1 = 0$, $\mu_2 = 5\omega - 1$, $\nu_0 = 1+2\omega$, $\nu_1 = \omega$, $\nu_2 = 2$, one observes that $\sum_{i=0}^{2}\lambda_i = \sum_{i=0}^{2}\mu_i = \sum_{i=0}^{2}\nu_i = 3+3\omega \in 3\Lambda = 3\mathcal{O}_{\sqrt{-3}}$, $\frac{\lambda_0}{3}+\Lambda = \Lambda \neq \frac{1+\omega}{3}+\Lambda = \frac{\mu_0}{3}+\Lambda$ and the set $\Sigma := \left\{ \frac{\omega^{2i}(\nu_0-\lambda_i)}{3}+\Lambda \,\middle|\, 0\leq i\leq 2 \right\} = \left\{ -\frac{\omega^2}{3}+\Lambda \right\}$ is not contained in $\Sigma_1 := \left\{ \frac{\omega^{2i}(\nu_0-\mu_i)}{3}+\Lambda \,\middle|\, 0\leq i\leq 2 \right\} = \left\{ \frac{\omega}{3}+\Lambda \right\}$. Thus, Proposition 13 applies to provide the injectiveness of $\Psi_{t+\Lambda} = (e_0(\ ,t+\Lambda), f_0(\ ,t+\Lambda)) : E \to \Psi_{t+\Lambda}(E) \subset \mathbb{P}^1(\mathbb{C})^2$ for all $t+\Lambda \in E$ and to concludes the proof of the theorem. □

5. Conclusion. By Di Cerbo and Stover's Theorem 1.1 [4], formulated above as Theorem 5, the isomorphism classes of the smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ with minimal Haar measure $\mathrm{vol}(X) = \mathrm{vol}(\mathbb{B}/\Gamma) = \frac{8\pi^2}{3}$ have five representatives $X_i = (\mathbb{B}/\Gamma_i)'$, $0 \leq i \leq 4$. Proposition 17 from [16] provides a regular embedding $\widehat{\mathbb{B}/\Gamma_0} \to \mathbb{P}^5(\mathbb{C})$ by Γ_0-modular forms of weight 2 for Hirzebruch's example $X_0 = (\mathbb{B}/\Gamma_0)'$ with abelian minimal model $E \times E$. The present article constructs regular embeddings $\widehat{\mathbb{B}/\Gamma_i} \to \mathbb{P}^5(\mathbb{C})$ by modular forms of weight 3 for $i = 1$ and $i = 2$. This result is strongly related to the fact that the bi-elliptic minimal model $Y_3 = E \times E/G_3$, $G_3 \simeq (\mathbb{Z}_3, +)$ of $X_i = (\mathbb{B}/\Gamma_i)'$, $1 \leq i \leq 2$ has a 3-torsion canonical bundle. Bearing in mind that $X_3 = (\mathbb{B}/\Gamma_3)'$ and $X_4 = (\mathbb{B}/\Gamma_4)'$ have bi-elliptic minimal model $Y_{3,3} = E \times E/G_{3,3}$, $G_{3,3} \simeq (\mathbb{Z}_3 \times \mathbb{Z}_3, +)$ with a 3-torsion canonical bundle, we raise the following

Conjecture 16. *Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification with minimal Haar measure $\mathrm{vol}(X) = \mathrm{vol}(\mathbb{B}/\Gamma) = \frac{8\pi^2}{3}$, which is not isomorphic to Hirzebruch's example $X_0 = (\mathbb{B}/\Gamma_0)'$ from [10]. Then the Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ of $\mathbb{B}/\Gamma$ admits a regular embedding in $\mathbb{P}^5(\mathbb{C})$ by Γ-modular forms of weight 3.*

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Received September 30, 2025
Accepted December 15, 2025