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A GUIDE TO TROPICAL MODIFICATIONS

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Abstract. This paper surveys *tropical modifications*, a notion that has become part of the folklore of tropical geometry. Tropical modifications are used in tropical intersection theory, tropical Hodge theory, and the study of singularities. They admit interpretations in several contexts, including hyperbolic geometry, Berkovich spaces, and non-standard analysis.

The goal of the paper is to present several points of view, give references, and illustrate the usefulness of tropical modifications. We assume that the reader has already encountered tropical modifications and wants to understand them better.

The paper also contains two new contributions: an obstruction to the realizability of non-transversal intersections and a tropical version of Weil's reciprocity law.

Substance is by nature prior to its modifications.
... nothing is granted in addition to the understanding,
except substance and its modifications.
Ethics, Benedictus de Spinoza.

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This paper surveys *tropical modifications*, a notion that has become part of the folklore of tropical geometry. Tropical modifications are used in tropical intersection theory, tropical Hodge theory, and the study of singularities. They admit interpretations in several contexts, including hyperbolic geometry, Berkovich spaces, and non-standard analysis.

One cannot say better than in [12]: "*Tropical modifications ... can be seen as a refinement of the tropicalization process, and allows one to recover some information ... sensitive to higher order terms.*" Tropical modifications were defined by Mikhalkin in [52].

The goal of the paper is to present several points of view, give references, and illustrate the usefulness of tropical modifications. We assume that the reader has already encountered tropical modifications and wants to understand them better.

The paper also contains two new contributions: an obstruction to the realizability of non-transversal intersections, Theorem 2.3, and a tropical version of Weil's reciprocity law, Theorem 2.2. A generalization of tropical momentum is given in Section 2.8.

One of the main results is the subordination obstruction, Theorem 2.3: if two algebraic varieties lift a prescribed tropical intersection, then the valuation of their actual intersection divisor is not an arbitrary divisor rationally equivalent to the stable intersection. It must be subordinate to the stable-intersection divisor, meaning that it is cut out by a tropical function lying below the stable-intersection function and agreeing with it on the frozen locus.

As a preliminary introduction to tropical geometry, see [10], [11] and [55], where tropical modifications are also discussed. We are glad to mention other texts promoting modifications from different perspectives:

singularities and tangency: [12] (examples, construction of curves with inflection points), [15] (repairing the j-invariant of elliptic curves), [16] (tropical genus two curves from several perspectives), [48] (hyperelliptic curves) [29] (tropical quartic curves), [17] (tropical bitangent lines to a planar quartic), [24, 25] (enumeration of cuspidal curves), [45] (lifting tropical bitangents), [35, 34, 36] (multiplicity of tropical singular points),

realizability of intersections: [66] (intersection theory on tropical surfaces), [14] (lifting divisors, constructing curves without maps to an elliptic curve), [46] (studying tropical self-intersections), [57] (realizability of tropical linear subspaces), [74] (tropical $K3$ surfaces), [63] (tropical constructions of real reducible curve in the Hirzebruch surfaces and other real algebraic varieties), [7] (constructing tropical curves of big genus in $\mathbb{T}^n$ and other questions about realizability of

big Betti numbers), [30] (lifting non-proper tropical intersections),

maps between tropical curves: [19] (presenting metric graphs as tropical planar curves), [33, 20] (gonality of tropical curves), [72](a tropical analogue of the double ramification locus), [22] (maps to a tree from a modification of a genus g curve), [50, 76] (trigonal curves),

tropical cohomology theory, Hodge theory: [32] (tropical Dolbeault cohomology), [18] (Chern–Schwartz–MacPherson cycles of matroids), [4] (homology of tropical fans), [3] (Hodge theory for tropical varieties).

This paper is organized as follows. Section 1.1 defines tropical modifications via multivalued operations and discusses basic examples. We then prove several structural theorems and discuss applications. Section 4 gives intuitive interpretations of tropical modifications; a reader seeking motivation may start there and then return to Section 1.1. The paper concludes with several questions where tropical modifications may be useful.

1. Definitions and examples.

1.1. Tropical modification via the graphs of multivalued functions. Recall that the tropical semiring $\mathbb{T}$ is defined as

$$\mathbb{T} = \mathbb{R} \cup \{-\infty\}, \qquad \mathbb{T}^* = \mathbb{T} \setminus \{-\infty\} = \mathbb{R}.$$

We extend addition from $\mathbb{R}$ to $\mathbb{T}$ by the rule $-\infty + A = -\infty$ for all $A \in \mathbb{T}$. We extend the order relation from $\mathbb{R}$ to $\mathbb{T}$ by the rule $-\infty < A$ for all $A \in \mathbb{R}$. A convenient way to define tropical modifications is through the multivalued tropical addition introduced by Viro in [73].

Definition 1.1. *The set $\mathbb{T}$ is equipped with tropical multiplication (usual addition on $\mathbb{R}$), and tropical addition, defined as follows:*

- *the result of tropical multiplication of A and B is $A + B$,*
- *the result of tropical addition of A and B is $\max(A, B)$ if $A \neq B$, and*
- *the result of tropical addition of A and A is $\{x \in \mathbb{T} \mid x \leq A\}$.*

We may say, equivalently, that the operation max *is redefined to be multivalued in the case of equal arguments, i.e.,* $\max(A, A) = \{X \in \mathbb{T} \mid X \leq A\}$.

Definition 1.2. *On the dense torus $\mathbb{R}^n = (\mathbb{T}^*)^n$, a tropical Laurent monomial is a function*

$$X \mapsto A + i_1 X_1 + \cdots + i_n X_n, \qquad A \in \mathbb{R}, \quad (i_1, \ldots, i_n) \in \mathbb{Z}^n.$$

If we work on all of $\mathbb{T}^n$*, we restrict to ordinary tropical monomials with* $(i_1, \dots, i_n) \in \mathbb{Z}^n_{\geq 0}$*. This is the tropical analogue of the ordinary monomial* $aX_1^{i_1}X_2^{i_2}\cdots X_n^{i_n}$*. A tropical polynomial* f *is a tropical sum of a finite number of tropical monomials,*

$$f(X_1, \dots X_n) = \max_{(i_1,\dots,i_n)\in I}(A_{i_1\dots i_n} + i_1X_1 + i_2X_2 + \cdots + i_nX_n), \tag{1}$$

where I *is a finite subset of* $\mathbb{Z}^n$*,* $A_{i_1\dots i_n} \in \mathbb{T}^*$*, and the operation* max *is multivalued.*

With the above multivalued convention, a point X *belongs to the zero set of* f *if* $-\infty \in f(X)$*. Equivalently, in the usual single-valued notation,* X *is a point where the maximum in* (1) *is attained by at least two monomials. We say that* f *vanishes at* X*. A tropical hypersurface (as a set) is the zero set of a tropical polynomial on* $\mathbb{T}^n$*.*

Remark 1.1. Note that the zero set of F is the set where the maximum in the right-hand side of (1) is attained at least twice; this coincides with the standard definition of a tropical hypersurface [10]. We assign weights to the maximal-dimensional faces of the tropical hypersurface. On a codimension-one face where exactly two monomials with exponent vectors $u, v \in \mathbb{Z}^n$ dominate, the weight is the lattice length of $u - v$, i.e.

$$\gcd(u_1 - v_1, \dots, u_n - v_n).$$

For non-generic faces this is understood by refinement, or equivalently by the usual lattice-index definition. Refer to [52] for details.

Definition 1.3 (Tropical modification)**.** *Let* N *be a tropical hypersurface in* $\mathbb{T}^n$*, i.e., the zero set of a tropical polynomial* f *on* $\mathbb{T}^n$*. The modification of* $\mathbb{T}^n$ *along* N *is the set*

$$m_N(\mathbb{T}^n) = \{(X, Y) \in \mathbb{T}^n \times \mathbb{T} | Y \in f(X)\}, \tag{2}$$

i.e., the graph of the multivalued function f*. For a given tropical variety* $K \subset \mathbb{T}^n$*, a tropical subvariety* $K' \subset m_N(\mathbb{T}^n)$ *is called* **a** *modification of* K *if the natural projection* $p : \mathbb{T}^n \times \mathbb{T} \to \mathbb{T}^n$ *restricted to* K' *is a tropical morphism* $p : K' \to K$ *of degree one. We write* $K' = m_N(K)$ *in this case.*

Proposition 1.1 (cf. [52], 1.5 B,C)**.** *The set* $m_N(\mathbb{T}^n)$ *coincides with the zero set of the tropical polynomial*

$$f'(X_1, \dots, X_n, Y) = \max(f(X_1, \dots, X_n), Y) : \mathbb{T}^n \times \mathbb{T} \to \mathbb{T}.$$

Definition 1.4. *For an abstract tropical variety M and its subvariety $N \subset M$ defined as the zero set of a tropical function $f : M \to \mathbb{T}$, we define the tropical modification $m_N(M)$ of M along N as the graph of f in $M \times \mathbb{T}$. A subvariety $K' \subset m_N(M)$ is called a modification $m_N(K)$ of K along N if the natural projection $K' \to K$ is a tropical morphism of degree one.*

Modifications along non-principal divisors, which can be locally presented as zero sets, can be defined locally and then glued to a global abstract tropical variety in a natural way.

1.2. Modification reveals intersections. Consider two algebraic curves $C_1, C_2 \subset (\mathbb{C}^*)^2$ defined by equations

$$F_1(x, y) = 0, F_2(x, y) = 0,$$

respectively. Let us build the map

$$m_{C_2} : (x, y) \to \big(x, y, F_2(x, y)\big) \in (\mathbb{C}^*)^2 \times \mathbb{C}. \tag{3}$$

The set $m_{C_2}\big((\mathbb{C}^*)^2\big)$ is the graph of F_2, namely $z = F_2(x, y)$. The intersection $C_1 \cap C_2$ can be easily recovered as $m_{C_2}(C_1) \cap \{(x, y, 0)\}$. For the complex curves, this does not seem very interesting. Still, during the tropicalization process, the plane $(x, y, 0)$ goes to the plane $(X, Y, -\infty) = \{Z = -\infty\}$, and the intersection of tropical curves will be represented by certain rays going to minus infinity by Z coordinate, see examples later.

1.3. Tropical modifications as limits of amoebas. Look now at what happens in the limiting procedure. Recall that a tropical curve $C \subset \mathbb{T}^2$ is the tropical limit of a family $C_t \subset (\mathbb{C}^*)^2, t \in \mathbb{R}$ of plane algebraic curves if in the Gromov–Hausdorff sense, we have

$$C = \lim_{t \to \infty} \mathrm{Log}_t(C_t)$$

where we apply $\mathrm{Log}_t : \mathbb{C} \to \mathbb{T}$ coordinate-wise, i.e. in the two dimensional case

$$\mathrm{Log}_t(C_t) = \{(\log_t |x|, \log_t |y|) | (x, y) \in C_t\}.$$

Let F_t be the equation of C_t in $(\mathbb{C}^*)^2$.

Proposition 1.2. *Let F_t be a family of Laurent polynomials whose tropical limit is the tropical polynomial F defining C, and assume that no cancellation changes this tropicalization. Then the tropical limit of the graphs*

$$S_t = \{(x, y, F_t(x, y)) \mid (x, y) \in (\mathbb{C}^*)^2\} \subset (\mathbb{C}^*)^2 \times \mathbb{C}$$

is the tropical modification $m_C \mathbb{T}^2$.

Near a smooth point of C_t, the function F_t can be used as a local coordinate transverse to C_t. Hence the logarithm of F_t realizes arbitrarily negative values, producing the vertical part of the tropical modification. Therefore, after taking the logarithm near C, the function $\log_t |F_t(x,y)|$ will assume all the values of a neighborhood of $-\infty$.

In other words, for a general point $(X,Y) \in \mathbb{T}^2$ we have that $m_C(X,Y) = (X,Y,Z)$ where Z is computed as $Z = \lim \log_t |F_t(x_t,y_t)|$ and (x_t,y_t) are chosen such that $X = \lim \log_t |x_t|, Y = \lim \log_t |y_t|$. But if $(X,Y) \in C$, then (x_t,y_t) can be chosen to be close to C_t, and thus Z can assume any value from the interval $F(X,Y) \subset \mathbb{T}$.

Use notation (3).

Proposition 1.3. *Assume that $C_{1,t}$ and $C_{2,t}$ have tropical limits C_1 and C_2, that $C_{1,t}$ is not contained in $C_{2,t}$, and that the projection of the limiting graph to C_1 has degree one. Then the tropical limit of*

$$m_{C_{2,t}}(C_{1,t})$$

is a tropical modification of C_1 along C_2.

Note that $m_{C_2}\mathbb{T}^2$ depends only on C_2. Quite the contrary, for given tropical curves C_1, C_2 we **can** construct different families $C_{1,t}, C_{2,t}$ and the limit $\lim\limits_{t\to\infty} \mathrm{Log}_t m_{C_{2,t}}(C_{1,t})$ can be different, see numerous examples below.

1.4. Modification and non-archimedean valuation. We always suppose that an algebraic hypersurface comes with a defining equation. Instead of taking the limit of amoebas we can consider non-Archimedean amoebas of the varieties defined over the valuation fields.

Definition 1.5. *Let $M' \subset (\mathbb{K}^*)^n$ be a variety over a valued field $\mathbb{K}$. Let $N' \subset (\mathbb{K}^*)^n$ be a hypersurface defined by $f = 0$. Consider the graph*

$$\Gamma_f(M') = \{(x,z) \in (\mathbb{K}^*)^n \times \mathbb{K} \mid x \in M',\ z = f(x)\}.$$

The modification $m_N M$ of $M = \mathrm{Trop}(M')$ along $N = \mathrm{Trop}(N')$ is

$$\mathrm{Val}(\Gamma_f(M')) \subset \mathbb{T}^{n+1}.$$

The approach with limits of amoebas gives the same results as the approach with non-Archimedean amoebas.

Proposition 1.4. *Consider a tropical variety $M \subset \mathbb{T}^n$ and a tropical hypersurface N defined by a tropical polynomial F. Let $\mathbb{K}$ be the field of power*

series in t, converging for t in a neighborhood of $0 \in \mathbb{C}$, and $\mathrm{val} : \mathbb{K}^* \to \mathbb{R}$ *be its natural valuation (we use the convention*

$$\mathrm{val}\left(\sum_q a_q t^q\right) = -\min\{q : a_q \neq 0\},$$

so that $\mathrm{val}(a+b) \leq \max(\mathrm{val}(a), \mathrm{val}(b))$ *and* $\mathrm{val}(t+2t^2) = -1$*). Suppose that*

$$f \in \mathbb{K}[x_1, \ldots, x_n], f = \sum_{I \in \mathcal{A}} a_I x^I \text{ and } F = \max_{I \in \mathcal{A}}(\mathrm{val}(a_I) + I \cdot X) : \mathbb{T}^n \to \mathbb{T}$$

where $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$ are multi-indices. Let $M' \subset (\mathbb{K}^*)^n$ be an affine algebraic variety, and its non-Archimedean amoeba $\mathrm{Val}(M')$ *be M. For a small (by module) complex number ε, we can substitute t as ε. Using this substitution we define $M'_\varepsilon \subset \mathbb{C}^n$ and $f_\varepsilon \in \mathbb{C}[x_1, \ldots, x_n]$. Then, the three following objects coincide:*

- *the limit*
$$\lim_{\varepsilon \to 0} Log_\varepsilon\big(\{(x, f_\varepsilon(x)) \mid x \in (\mathbb{C}^*)^n\}\big),$$
- *non-Archimedean amoeba* $\mathrm{Val}(\{(x, f(x)) | x \in (\mathbb{K}^*)^n\}) \subset \mathbb{T}^{n+1}$ *of the graph of f.*
- *the tropical modification $m_N(\mathbb{T}^n)$.*

Additionally, two following objects coincide and equal to **a** *tropical modification $m_N(M)$:*

- *the limit* $\lim_{\varepsilon \to 0} Log_\varepsilon(\{x, f_\varepsilon(x) | x \in M'_\varepsilon\})$,
- *the non-Archimedean amoeba*
$$\mathrm{Val}\big(\{(x, f(x)) \mid x \in M'\}\big) \subset \mathbb{T}^{n+1}$$
of the graph of f on M'.

We repeat again that given **only** tropical curves $C_1, C_2 \subset \mathbb{T}^2$, in general it is not possible to uniquely "determine" the image of C_1 after the modification along C_2. That is why a modification of a curve along another curve is multivalued and rather can be used as *a method*. The strategy being usually applied is the following: given two tropical curves, we lift them in a non-Archimedean field (or

present them as limits of complex curves, that is the same), then we construct the graph of the function as above and take the non-Archimedean amoeba of the limit of amoebae. Depending on the conditions we imposed on lifted curves (be smooth or singular, be tangent to each other, etc.), we will have a set of possible results (which often can be described using simple combinatorial conditions) for modification of the first curve along the second curve; see examples below.

If C_1 intersects C_2 transversally, then $m_{C_2}(C_1)$ is uniquely determined. If not, there are the following conditions:

- one equality (via tropical momentum or tropical Menelaus theorem): the sum of the coordinates of all the legs of $m_{C_2}(C_1)$ going to minus infinity by Z-coordinate is fixed, see Proposition 2.4;
- one inequality (subordination of divisors): the valuation of the divisor of the intersection of lifted curves is *subordinate* to the stable intersection of C_1 and C_2 (Theorem 2.3).

Both conditions have higher dimensional analogs.

1.5. Examples. In this section we study examples of modification, treated as a method. The reader should not be scared by these horrific equations; they are reverse-engineered, starting from the pictures. All the calculations are quite straightforward.

We start by considering the modification of a curve along itself and discuss the ambiguity that appears in this case. Then, we consider how modifications resolve indeterminacy when the intersection of tropical objects is non-transversal. This example promotes the point of view that a tropical modification is the same as adding a new coordinate. So, if one changes coordinates, one can do it via repetitive modifications as in [25].

In the third example, a modification helps recover the position of the inflection point. Also, the usefulness of the tropical momentum and tropical Menelaus Theorem is demonstrated. The tropical Weil theorem that shortens the combinatorial descriptions of possible modification results is proved in Section 2.2.

In the fourth example we study the influence of a singular point on the Newton polygon of a curve. The identical method suits higher dimension and different types of singularities, but more needs to be done due to complicated combinatorics. In the same example, we describe how to find all possible valuations of the intersections of a line with a curve, knowing only their stable tropical intersection – the key tool here is the Vieta theorem. The same arguments may be applied for non-transversal intersections of tropical varieties of any dimension.

Example 1.1 (Modification along itself)**.** Consider a tropical horizontal line L, given by $\max(1, Y)$. This is the tropicalization of a line of the type $y = t^{-1} + o(t^{-1})$. Note that if we make a modification of a line along itself, then all its points go to the minus infinity (Figure 1, left). Indeed, if $F(x, y)$ is the equation of C, then the set of points $\{(x, y, F(x, y))|(x, y) \in C\}$ belongs to the plane $z = 0$, so

$$\mathrm{Val}(\{(x, y, F(x, y))|(x, y) \in C\}) \subset \{(X, Y, Z) \in \mathbb{T}^3 | Z = -\infty\}.$$

On the other hand, if we consider two different lines C_1, C_2 (with equations $y = t^{-1}$ and $y = t^{-1} + t^3$) whose tropicalization is L, then all the points in $m_{C_1}C_2$ have the valuation -3 of Z coordinate. Again, we see an ambiguity — even if L is fixed, we can take different lifts of L and have different results of the modification. On the other hand we can say that the canonical modification along itself is the result similar to Figure 1, left, i.e. we might define $m_C C$ as the projection of C to the plane $Z = -\infty$. Nevertheless, it is better to always keep this unambiguity in mind instead of giving a precise definition of $m_C C$. More about the question of intersections of two curves having the same tropicalization can be found in [46].

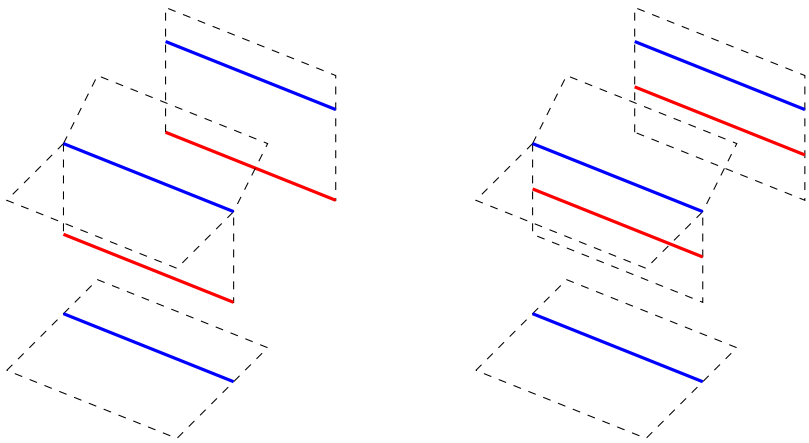


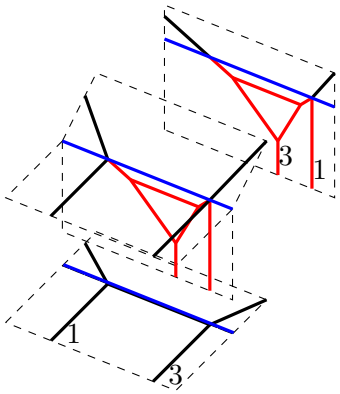
Fig. 1. Example of a modification of a line along itself. Let L_1, L_2 be defined by $y = t^{-1}, y = t^{-1} + t^3$ respectively. In each group of pictures, the bottom picture is the initial $\mathbb{T}^2$, the middle picture is $m_{L_1}\mathbb{T}^2$, the picture at the back is the projection to X, Z-plane. On the left we see the modification of L_1 along L_1, on the right we see the modification of L_2 along L_1. Red line is the result $m_{L_1}L_1$ (resp. $m_{L_1}L_2$) of the modification

Example 1.2 (Modification, root of big multiplicity, Figure 2a)**.** In this example we see two tropical curves with non-transverse intersection which hides tangency and genus. Consider the plane curve C, given by the following equation: $F(x,y)=0$,

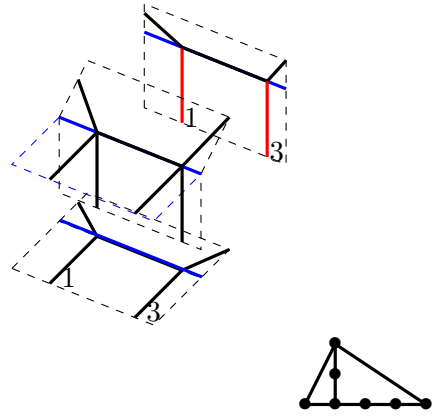
$$F(x,y)=(x-t^{1/3})^3(x-t^{-2})+t^{-4}xy^2+(t^{-4}+2t^{-5})xy+(t^{-5}+t^{-6})x.$$

Its tropicalization[1] is the curve, given by the set of non-smooth points of

$$\mathrm{Trop}(F)=\max(1,6+x,5+x+y,4+x+2y,5/3+2x,2+3x,4x).$$



(a) Initial picture is below. In the center we see the limit of the graphs of the logarithm of the functions $F_{2,t}$. On the picture behind we see the projection of the graph to the plane XZ. Numbers on the edges are the corresponding weights.



(b) Notation is the same as for the picture on the left. We see the result of the modification in the case when the stable intersection is the actual intersection. The Newton polygon of the curve C is depicted below.

Fig. 2. Example of a modification along a line

We want to know what is the intersection of C with the line L given by the equation $y+t^{-1}=0$. Tropicalizations of C and L are drawn on Figure 2a, below,

[1] One can think that we have a family of curves C_t (given by $F_t=0$) with parameter t and its tropicalization is the limit of amoebas $\lim\limits_{t\to 0}\mathrm{Log}_t(\{(x,y)|F_t(x,y)=0\})$, or that we have a curve C over Puiseux series $\mathbb{C}\{\{t\}\}=\mathbb{K}$ given by $\sum a_{ij}x^iy^j=0, a_{ij}\in\mathbb{K}$. Its non-Archimedean amoeba is given by the set of non-smooth points of the function $\max\limits_{ij}(\mathrm{val}(a_{ij})+ix+jy)$. Both ways lead to the same result.

as well as the Newton polygon of C. The intersection is not transverse, hence we do not know the tropicalization of $C \cap L$.

To deal with that, let us consider the map $m_L : (x, y) \to (x, y, y + t^{-1})$. On Figure 2a, in the middle, we see the tropicalization of the set $\{(x, y, y + t^{-1})\}$ and the tropicalization of the image of C under the map m_L. Let $G(x, z)$ be the equation of the projection of $m_L(C)$ on the xz-plane. So, $F(x, y) = 0$ implies that for the new coordinate $z = y + t^{-1}$ we have

$$G(x, z) = 0, G(x, z) = (x - t^{1/3})^3(x - t^{-2}) + t^{-4}xz + t^{-4}xz^2. \tag{4}$$

Therefore the curve $C' = pr_{xz}m_L(C)$ is given by the set of non-smooth points of

$$\max(1, 4 + X + Y, 4 + X + 2Y, 2 + 3X, 4X),$$

we see C' on the projection onto the plane XZ on the left part of Figure 2a. Notice that in order to recover the transversal intersection of non-Archimedean amoebas we did nothing else as a change of coordinates.

Remark 1.2. Consider the restriction of $\mathrm{Trop}(F)$ on the line $Y = 1$. We obtain $\max(1, 7 + X, 5/3 + 2X, 2 + 3X, 4X) = \max(1, 7 + X, 4X)$, whose locus of non-linearity corresponds to the stable intersection of our tropical curves. On the other hand, if we restrict F on the line $y + t^{-1} = 0$ and only then take the valuation, we obtain $\max(1, 3X+2, 4X)$ because $F(x, -t^{-1}) = (x-t^{1/3})^3(x-t^{-2})$, and we see that this agrees with the picture of the modifications.

Definition 1.6. *As we see in this example, a tropical curve in $\mathbb{T}^n$ typically contains infinite edges. We call them legs of a tropical curve. For each leg we have a canonical parametrization $(a_0 + p_0 s, a_1 + p_1 s, a_2 + p_2 s)$ where $a_i \in \mathbb{R}, p_i \in \mathbb{Z}, s \in \mathbb{R}, s \geq 0$, where the vector (p_0, p_1, p_2), the direction of the leg, is primitive.*

Now, on the tropicalization of C' we see a vertical leg of weight 3, whose Z-coordinate goes to $-\infty$. That happens because we have the tangency of order 3 between C and L, and z as a function of x has a root of order 3.

Note that this leg cannot mean that the point is a singular point of C, because the curve C (according to criteria of [47] or, more generally [35]) has no singular points, even though the tropicalization of C has an edge of multiplicity 3.

Thus, this new tropicalization restores the multiplicity of the intersection. We see that the modification of the plane (i.e. amoeba of the set $\{(x, y, y+t^{-1})\}$) is defined, but in codimension one this procedure shows multiplicities of roots and more unapparent structures such as hidden genus squashed initially onto an

edge. One can think that this cycle was close to intersection, but after a change of coordinates it becomes visible on the picture of the amoeba of C'. More examples of this kind can be found in [15, 16].

Remark 1.3. Nevertheless, for a general choice of representatives in Puiseaux series for these two tropical curves $\mathrm{Trop}(C), \mathrm{Trop}(L)$, after modification we will have Figure 2b, which represents stable intersection of the curves.

Example 1.3 (Modification, inflection point, momentum map)**.** We consider a curve and its tangent line at an inflection point. Suppose, that the intersection of their tropicalizations is not transverse. How can we recover the presence of the inflection point?

We consider a curve C with the equation $F(x, y) = 0$ where

$$\begin{aligned} F(x,y) =& y + t^{-3}xy + (t^{-1} + 4 + 6t + 4t^2 + t^3)x^2 + (-t^{-3} - 3 - t - t^2)xy^2 \\ &+ (t^{-2} - t^{-1} - 2 + t^2 + t^3)x^2y + x^2y^2, \end{aligned}$$

and a line L with the equation $y = 1 + tx$. The equation of the curve is chosen just in such a way that the restriction of F on the line L is $t^2(x-1)^3(x-t^{-1})$, i.e. the point $(1, 1+t)$ is the inflection point of the curve and L is tangent to C at this point.

Tropicalization of the curve is given by the following equation:

$$\mathrm{Trop}(F) = \max(y, x + y + 3, 2x + 1, 2x + y + 2, x + 2y + 3, 2x + 2y). \tag{5}$$

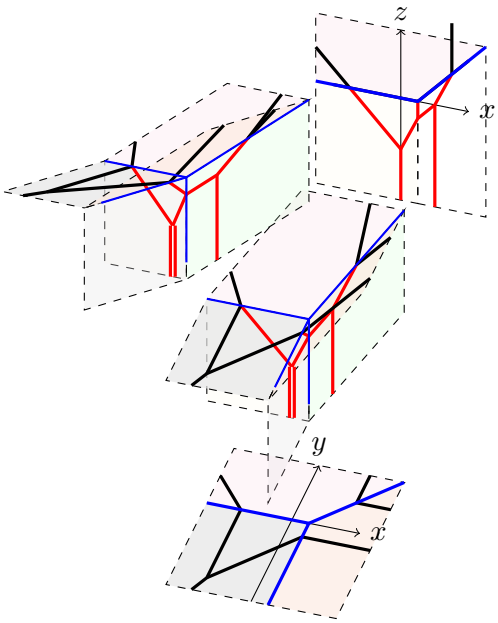
On Figure 3a we see the non-Archimedean amoeba of the image of the curve under the map $(x, y) \to (x, y, y - 1 - tx)$.

In order to find X-coordinates of the possible legs we can apply the tropical momentum: see Figure 3b.

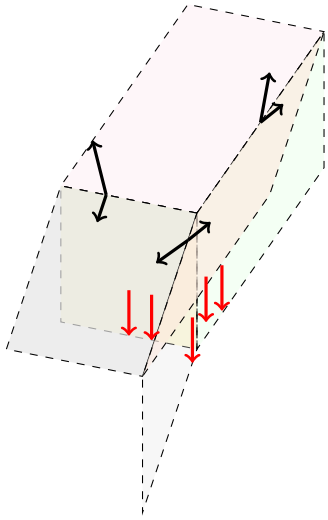
Definition 1.7. *The momentum of a leg* $(A_0 + P_0 s, A_1 + P_1 s, A_2 + P_2 s)$ *with respect to a point* (B_0, B_1, B_2) *is the vector product* $(A_0 - B_0, A_1 - B_1, A_2 - B_2) \times (P_0, P_1, P_2)$.

We will prove a (simple) theorem that *the sum of the moments of the legs, counted with their weights, is zero.* Note, that in our case, all the legs we do not know are of the form $(X_0, Y_0, Z_0 - s)$, because they are vertical. Refer to Figure 3b. So, we take the vertex O of the tropical plane, and sum up the vector products $OX_i \times X_iY_i$ where X_iY_i are black legs (that we already know) and red legs (which are all vertical). Computation gives us

$$(-4, 0, 0) \times (-1, 1, 1) + (-4, 0, 0) \times (0, -1, 0) + (0, -1, 0) \times (-1, -1, 0)$$



(a) In the center we see a modification of the picture below, its XZ-projection is on the right, on the left we see it from a different perspective. Doubled red lines graphically represents a line with weight three



(b) Application of the generalized tropical Menelaus Theorem: we know the direction of the infinite black rays emanating from the tropical curve (in the center on the left), therefore an application of this theorem gives the sum of X- and Y-coordinates of red legs, going vertically to the bottom (these legs present exactly the intersection of two considered curves)

Fig. 3. Example of modification in the case of inflection point. The point $(0,0)$ on the bottom picture is the tropicalization of the inflection point. We modified the black curve along the blue curve, red parts are the parts becoming visible after the modification

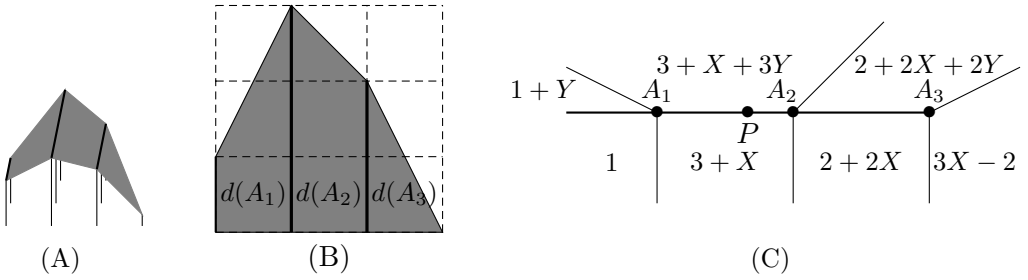
$$+ (0,-1,0) \times (1,0,1) + (2,2,2) \times (1,0,1) + (2,2,2) \times (0,1,1)$$
$$+ (X,0,0) \times (0,0,-1) + (0,Y,0) \times (0,0,-1) + (Z+1,Z,0) \times (0,0,-1) = 0,$$

i.e. $(1,-2,0) + (Y+Z+1, X+Z, 0) = 0$, where X stands for the sum of the X-coordinates of the vertical legs situated under the line $(1-s,0,0)$, Y stands for the sum of the Y-coordinates of the vertical legs under the line $(1,-s,0)$, Z stands for the sum of the Y-coordinates of the vertical legs under the line $(1,s,s)$.

On the left picture we see where the red legs are situated. But, since modification of a tropical curve C along a tropical curve C' is not canonically defined[2], then, for example, a modification of C could differ from C just by adding vertical legs at four vertices of the C: this would correspond to the stable intersection (which is always realizable in the sense that there exists a curve in Puiseux series, such that the valuation of their intersection is the stable intersection).

Example 1.4. Singular point, its unique position, and possible liftings of intersection. Consider a curve C' defined by the equation $G(x,y)=0$, where

$$\begin{aligned} G(x,y) = {} & t^{-3}xy^3 - (3t^{-3}+t^{-2})xy^2 + (3t^{-3}+2t^{-2}-2t^{-1})xy \\ & - (t^{-3}+t^{-2}-2t^{-1}-3t^2)x + t^{-2}x^2y^2 - (2t^{-2}-t^{-1})x^2y \\ & + (t^{-2}-t^{-1}-3t^2)x^2 + t^{-1}y - (t^{-1}+t^2) + t^2x^3. \end{aligned}$$



(A) (B) (C)

Fig. 4. The extended Newton polyhedron $\widetilde{\mathcal{A}}$ of the curve C' is drawn in (A). The projection of its faces gives us the subdivision of the Newton polygon of C'; see (B). The tropical curve $\mathrm{Trop}(C')$ is drawn in (C). The vertices A_1, A_2, A_3 have coordinates $(-2,0),(1,0),(4,0)$. The edge A_1A_2 has weight 3, while the edge A_2A_3 has weight 2. The point P is $(0,0)=\mathrm{Val}((1,1))$

Let us make the modification along the line $y=1$. For that we draw the graph of the function $z(x,y)=y-1$.

Note that we can easily find the number (with multiplicities) of the vertical legs. Indeed, each edge from A_1, A_2, A_3 going up in direction (i,j) becomes after the modification a ray going in the direction (i,j,j). Therefore, the total momentum of the vertical legs is the sum of Y-parts of momenta of the edges going up from A_1, A_2, A_3, that is, 3. Then, if we know that after the modification

[2]If the intersection $C \cap C'$ is transverse, then the modification is uniquely defined.

our curve has a leg of weight 3, then its unique position can be found from the generalized tropical Menelaus theorem. So, in this case (the points $(1,1)$ is of multiplicity 3 for the curve) the pictures after the modifications is as on Figure 3, If $\mathrm{Val}(C') = C$, but we do not have the other restricting condition, then the picture after the modification can be as in Figure 5, right top, or right bottom, both cases can be realized.

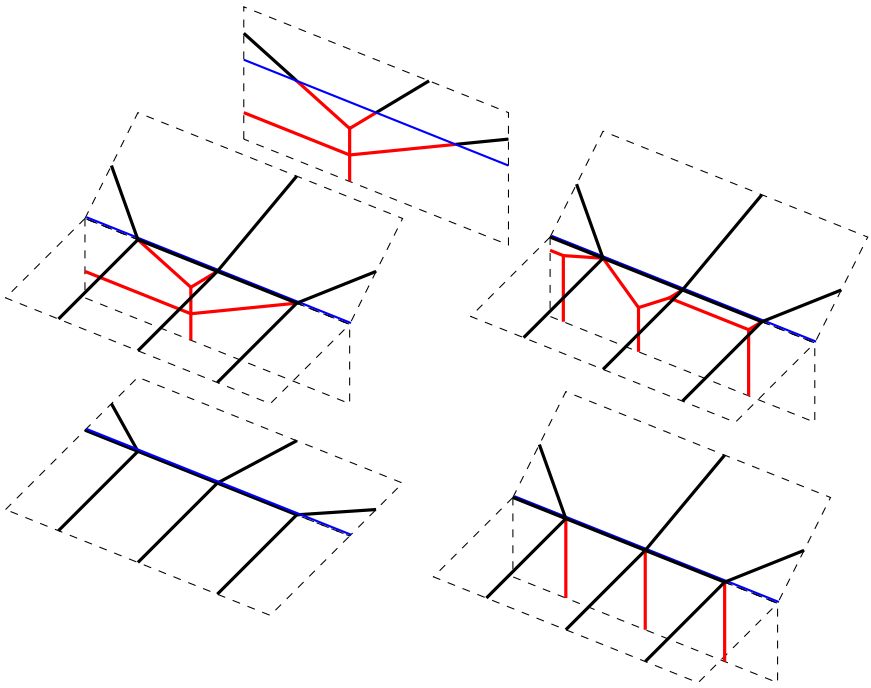


Fig. 5. Refer to Example 1.4. Left bottom picture represents a curve C. On top of it, a modification of it is depicted, with the projection of the latter on the XZ-plane. On the right side we see two other possible modification of C

2. Structural theorems about tropical modification.

2.1. Multiplicity of an intersection. We will recall here certain facts about intersections of tropical curves.

The *multiplicity* $m(P)$ of the point P of the transverse intersection of two lines in directions $(u_1,u_2),(v_1,v_2) \in \mathbb{Z}^2, gcd(u_1,u_2) = gcd(v_1,v_2) = 1$ is $|u_1v_2 - u_2v_1|$.

Given two tropical curves $A, B \subset \mathbb{T}^2$ we define their *stable intersection* as follows. Let us choose a generic vector v. Then we consider the curves $T_{tv}A$ where

$t \in \mathbb{R}, t \to 0$ and T_{tv} is translation by the vector tv. For a generic small positive t, the intersection $T_{tv}A \cap B$ is transversal and consists of points $P_i^t, i = 1, \ldots, k$ with multiplicities $m(P_i^t)$.

Proposition 2.1. *Suppose that a horizontal edge E of a tropical curve C contains a point P. Suppose that on the dual subdivision of the Newton polygon for C the vertical edge $d(E)$ is dual to E. Let the endpoints of E be A_1, A_2 and two faces $d(A_1), d(A_2)$ adjacent to $d(E)$ have no other vertical edges. Let the sum of widths in the horizontal direction of the faces $d(A_1), d(A_2)$ be equal to m. Then the stable intersection of E with a horizontal line through E is m.*

Proof. Refer to Example 1.4 and Figure 2b. Let L be a tropical line containing E, and assume that the vertex of L does not coincide with the endpoints of E. Making the modification along the line L we see that the sum S of vertical components of edges going upward from A_1, A_2 equals the sum m of the y-components of them.

Then the sum of the vertical components of the downward edges equals S by the balancing condition for tropical curves. The sum of the y-components of the edges incident to a vertex v is exactly the width, in the $(1, 0)$-direction, of the face $d(v)$ dual to v in the Newton polygon. □

Definition 2.1 (cf. [64])**.** *For each connected component X of $A \cap B$, we define the local stable intersection of A and B along X as $A \cdot_X B = \sum_i m(P_i^t)$ for t close to zero, where the sum runs over $\{i | \lim_{t\to 0} P_i^t \in X\}$. For a point $Q \in A$, we define $A \cdot_Q B$ as $A \cdot_X B$, where X is the connected component of Q in the intersection $A \cap B$.*

Proposition 2.2 ([12] Proposition 3.11, see also [61] Corollary 12.12)**.** *For two algebraic curves $C_1, C_2 \subset (\mathbb{K}^*)^2$ we consider a compact connected component X of the intersection $\mathrm{Trop}(C_1) \cap \mathrm{Trop}(C_2)$. Then,*

$$\sum_{x \in C_1 \cap C_2, \mathrm{Val}(x) \in X} m(x) = \mathrm{Trop}(C_1) \cdot_X \mathrm{Trop}(C_2)$$

where $m(x)$ is the multiplicity of the point x in the intersection $C_1 \cap C_2$.

Proof. Consider the equation $F(x, y) = 0$ of C_2. We construct the non-Archimedean amoeba $m_{C_2}C_1$ of $\{(x, y, F(x, y) \mid (x, y) \in C_1)\}$. Then

$$\mathrm{Trop}(C_1) \cdot_X \mathrm{Trop}(C_2)$$

is the sum of the weights of the vertical legs of $m_{C_2}C_1$ under X. The latter is equal to $\sum_{x\in C_1\cap C_2, \mathrm{Val}(x)\in X} m(x)$. $\square$

Remark 2.1. For non-compact connected components of the intersection we only have an inequality $\sum_{x\in C_1\cap C_2, \mathrm{Val}(x)\in X} m(x) \le \mathrm{Trop}(C_1)\cdot_X \mathrm{Trop}(C_2)$. It can be upgraded to equality by considering intersections of C_1, C_2 "at infinity", in the appropriate compactification of the torus, see [67].

For further discussion about the multiplicity of singular points in the tropical world, see [35].

2.2. Tropical Weil reciprocity law and the tropical momentum map. The aim of this section is to establish another fact in tropical geometry, which is the tropical analogue of a classical fact in algebraic geometry.

Weil reciprocity law can be formulated as

Theorem 2.1. *Let C be a smooth compact complex curve, and let f, g be nonzero meromorphic functions on C whose divisors have disjoint supports. Then*

$$\prod_{x\in C} f(x)^{\mathrm{ord}_x(g)} = \prod_{x\in C} g(x)^{\mathrm{ord}_x(f)}.$$

Here, if u is a local parameter at x and

$$f = au^n + \cdots, \qquad a \neq 0,$$

then $\mathrm{ord}_x(f) = n$.

The products in this theorem are finite because $\mathrm{ord}_g x, \mathrm{ord}_f x$ equal to zero everywhere except finite number of points.

If $f = au^n + \cdots$ and $g = bu^m + \cdots$ in a local parameter u at x, define the tame symbol

$$[f, g]_x = (-1)^{nm}\frac{a^m}{b^n}.$$

Then Weil reciprocity is equivalently

$$\prod_{x\in C} [f, g]_x = 1.$$

Example 2.1. If $C = \mathbb{C}P^1$ and f, g are polynomials

$$f(x) = A\prod_{i=1}^{n}(x - a_i), g(x) = B\prod_{j=1}^{m}(x - b_j) \tag{6}$$

with $a_i \neq b_j$, then

$$\prod_{x\in C} g(x)^{\mathrm{ord}_x(f)} = \prod_{i=1}^{n} g(a_i) = B^n \prod_{i=1}^{n}\prod_{j=1}^{m}(a_i - b_j),$$

and

$$\prod_{x\in C} f(x)^{\mathrm{ord}_x(g)} = \prod_{j=1}^{m} f(b_j) = A^m \prod_{j=1}^{m}\prod_{i=1}^{n}(b_j - a_i).$$

The remaining factor is exactly compensated by the tame symbol at infinity.

Khovanskii studied various generalizations of the Weil reciprocity law and reformulated them in terms of logarithmic differentials [40, 41, 42]. The final formulation is for toric surfaces and resembles a tropical balancing condition, which is indeed the case. The symbol $[f, g]_x$ is related with Hilbert character and link coefficient, and is generalized by Parshin residues. Mazin [49] treated them in geometric context of resolutions of singularities.

In order to study what happens after a modification we consider a tropical version of Weil theorem. We need to define tropical meromorphic function and $\mathrm{ord}_f x$, see also [55].

Definition 2.2 ([52])**.** *For a piecewise-linear function f with integer slopes on a tropical curve C, define*

$$\mathrm{ord}_x(f) = \sum_{e\ni x} \mathrm{slope}_e(f),$$

where the slopes are taken in the outgoing directions from x. Points with $\mathrm{ord}_x(f) > 0$ *are zeros and points with* $\mathrm{ord}_x(f) < 0$ *are poles.*

Example 2.2. The function $f(x) = \max(0, 2x)$ on $\mathbb{T}P^1 = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ has a zero of multiplicity 2 at 0, i.e. $\mathrm{ord}_f(0) = 2$, and a pole of multiplicity 2 at $+\infty$, i.e. $\mathrm{ord}_f(+\infty) = -2$.

Theorem 2.2 (A proof is in Section 2.4)**.** *Let C be a compact tropical curve and let $f, g : C \to \mathbb{R}$ be finite-valued piecewise-linear functions with integer slopes. Then*

$$\sum_{x\in C} f(x)\mathrm{ord}_x(g) = \sum_{x\in C} g(x)\mathrm{ord}_x(f).$$

Word-by-word repetition of the reasoning in Example 2.1 proves this theorem in the case $C = \mathbb{T}P^1$. Indeed, after adding a constant and, if necessary, a

linear term, a one-variable tropical polynomial can be written in the form

$$c + \sum_i \max(A_i, X),$$

where the A_i are its tropical roots, counted with multiplicity.

For the general statement there are many proofs (and one can proceed by studying piece-wise linear functions on a graph, see [37]), we give here the shortest one (and also using tropical modifications), via so-called *tropical momentum*.

Suppose that C is a planar tropical curve. We list all the edges $E_1, \dots, E_k$ of C, suppose that their directions are given by primitive (i.e. non-multiple of another integer vector) integer vectors $v_1, \dots, v_k$. Suppose that each edge E_i has weight m_i and if E_i is infinite, then the direction of v_i is chosen to be "to infinity" (there are two choices and for us the orientation of v_i will be important). Let A be a point on the plane. Let us choose a point $B_i \in E_i$ for each $i = 1, 2, \dots, k$.

Definition 2.3 ([75]). *Tropical momentum of an edge E_i of C with respect to the point A is given by $\rho_A(E_i) = m_i \cdot \det(v_i, AB_i)$.*

Definition 2.4. *For a point $A \in \mathbb{R}^2$ define $\rho_A(C)$ as $\sum_E \rho_A(E)$ where E runs over all infinite edges of C.*

Lemma 2.1 ([75]). *If a tropical curve C has only one vertex, then $\rho_A(C) = \sum_{i=1}^{k} \rho_A(E_i) = 0$ for any point A on the plane.*

Proof. First of all, $\sum_{i=1}^{k} \rho_A(E_i)$ does not depend on the point A, because if we translate A by some vector u, then each summand in $\rho_A(C)$ will change by $\det(v_i, u) \cdot w_i$ and the sum of changes is zero because of the balancing condition. Therefore, $\rho_A(C) = 0$, because we can place A in the vertex of this curve. □

Lemma 2.2 (Moment condition in [75], also it appeared in [54] under the name Tropical Menelaus Theorem). *For an arbitrary plane tropical curve $C \subset \mathbb{R}^2$ and any point $A \in \mathbb{R}^2$ the equality $\rho_A(C) = 0$ holds.*

Proof. Note that the total momentum for a curve is the sum of momenta for all vertices, because a summand corresponding to an edge between two vertices will appear two times with different signs. So, this lemma follows from the previous one. □

Definition 2.5. *We consider a balanced tropical curve $C \subset \mathbb{R}^3$ with finitely many unbounded edges. Let $E_1, E_2, \dots, E_n$ be its infinite edges. We define the momentum of C with respect to A as*

$$\rho_A(C) = \sum_{i=1}^{n} m_i \, v_i \times \overrightarrow{AB_i}.$$

where $\times$ stands for the vector product, v_i is the primitive vector (in the direction "to infinity") of an edge E_i, m_i is the weight of E_i, and B_i is a point on E_i.

Proposition 2.3 (Generalized Tropical Menelaus theorem)**.** *For a balanced tropical curve $C \subset \mathbb{R}^3$ and any point $A \in \mathbb{R}^3$, one has*

$$\rho_A(C) = 0.$$

P r o o f. We proceed as in the planar case. We show that $\rho_A(C)$ does not depend on A because of the balancing condition. Indeed, if C has only one vertex, then the claim is trivial. In general case we sum up the tropical momentum by all the edges, and the terms for internal edges appear two times with different signs, which concludes the proof. □

An application of this theorem can be found in Example 1.3.

2.3. Application of the tropical momentum to modifications.

Example 2.3. Consider the graph of a tropical polynomial

$$f(X) = \max(A_0, A_1 + X, \dots, A_n + nX).$$

Suppose that we know only A_0 and A_n. Definitely, the positions of the tropical roots of f may vary, being dependent on the coefficients of f. Nevertheless, we can apply the tropical Menelaus theorem for the graph of f. We will calculate the momentum with respect to $(0, 0)$. This graph has one infinite horizontal edge with momentum A_0 and one edge of direction $(1, n)$ with the momentum $-A_n$. Also, for each root $P_i \in \mathbb{T}$ of f we have an infinite vertical edge with the momentum $-P_i$. Application of the tropical moment theorem gives us $\sum P_i = A_0 - A_n$, which is simply a tropical manifestation of Vieta's theorem — the product of the roots p_i of a polynomial $\sum_{i=1}^{n} a_i x^i$ is a_0/a_n.

Example 2.4. Let C be a planar tropical curve, such that all its infinite edges are horizontal or vertical. Consider first and second coordinates X, Y on C as two tropical functions. Denote these functions $f = X, g = Y$. Then,

Theorem 2.2 says that $\rho_{(0,0)}C = 0$, because a tropical root of f is represented by a horizontal leg of C, and the value of g at this root is exactly the Y-coordinate of this leg.

On Figure 2b, 3a, *a priori* we know only the sum of the directions of the edges with endpoints on the modified curve. We know that there is no horizontal infinite edges (in these examples). In general, it is possible, if the intersection of our two tropical curves is non-compact. Therefore by Weil theorem (or tropical Menelaus Theorem, it is the same) we know the sum of X-coordinates of the vertical infinite edges. Thus the sum of the weights for red vertical edges equals the sum of the vertical components of the black edges in the Figure 3b.

Lemma 2.3. *If we modify along a horizontal line, then the total vertical slope of the infinite vertical edges lying over this line equals the total horizontal width of the region in the Newton subdivision dual to the corresponding connected component of the intersection of the line with the curve.*

P r o o f. The argument is the same as in the proof of Proposition 2.1. □

Lemma 2.4. *If the stable intersection of* $\mathrm{Trop}(C)$ *with a horizontal line* L *is equal to* m, $\mathrm{Trop}(C) \cap L$ *is compact, and there exists a point* $q \in C$ *with* $\mu_q(C) \geq m$ *and* $\mathrm{Val}(q) \in \mathrm{Trop}(C) \cap L$, *then we can uniquely recover the position of* $\mathrm{Val}(q)$.

P r o o f. Indeed, consider a lift l of L which passes through q. If we make the modification along l, we obtain a leg of $m_L(\mathrm{Trop}(C))$ under $\mathrm{Trop}(C) \cap L$ of the weight at least m. Since the stable intersection $\mathrm{Trop}(C) \cap L$ is equal to m, there is only one leg under $\mathrm{Trop}(C) \cap L$. Therefore, the tropical momentum theorem gives us the unique position of this leg (of course, it is evident via balancing — we know all the infinite edges of a tropical curve except one, therefore the coordinates of this last edge can be found via the balancing condition). □

Proposition 2.4 (see [12], Proposition 4.5)**.** *For each compact connected component* C *of* $C_1 \cap C_2$ *the sum of* X *coordinates (and the sum of* Y*-coordinates) of the valuations of the intersection points of* C_1, C_2 *with valuations in* C *can be computed from the local behavior of* C_1 *and* C_2 *near* C.

Indeed, the tropical Menelaus theorem gives the sum of the momenta of all legs of $m_{C_2}C_1$ whose Z-coordinate tends to $-\infty$.

2.4. Proof of the tropical Weil theorem. We now prove the tropical Weil theorem. Given two tropical meromorphic functions f, g on a tropical curve

C, we would like to define the map

$$C \to \mathbb{T}^2, \qquad x \mapsto (f(x), g(x)),$$

and then apply the tropical Menelaus theorem; cf. Example 2.4. Here we have to use tropical modification, because *a priori*, the image of tropical curve under the map $(f, g) : C \to \mathbb{T}^2$ with f, g tropical meromorphic functions, is **not** a plane tropical curve: balancing condition is not satisfied near zeroes and poles of f and g, we need to add legs there. Formally, we have to consider a modification C' of C, and then extend f, g on it. Then, if the roots and poles of f, g will be only at 1-valent vertices, then the image of the map $C' \to \mathbb{T}^2$ will be a planar tropical curve.

Definition 2.6. *We call a triple (C, f, g) of a tropical curve C and two meromorphic function $f, g : C \to \mathbb{T}P^1$ on it admissible if all the zeroes and poles of f, g are located at different one-valent vertices of C.*

Lemma 2.5. *Given a triple (C, f, g) of a tropical curve C and two meromorphic functions $f, g : C \to \mathbb{T}P^1$ on it, we can always extend the functions f, g on the modification $D = m_{div(g)} m_{div(f)} C$ of C, such that the obtained triple (D, f', g') is admissible and*

$$(7) \quad \sum_{x \in C} f(x) \operatorname{ord}_x(g) - \sum_{x \in C} g(x) \operatorname{ord}_x(f) = \sum_{x \in D} f'(x) \operatorname{ord}_x(g') - \sum_{x \in D} g'(x) \operatorname{ord}_x(f').$$

Proof. We perform tropical modifications of C in order to have all zeros and poles of f, g at the vertices of valency one. Namely, for a point p such that p is in the corner locus of f we add to C an infinite edge l emanating from p. We define f on l as the linear function with integer slope such that the sum of slopes of f over the edges from p is zero, i.e. $f(x) = f(p) - \operatorname{ord}_f p \cdot x$ where x is the coordinate on l such that $x = 0$ at p and then x grows. We define g on this edge as the constant $g(p)$. We perform this operation for all roots and poles of f. Then, we do the same procedure along the divisor of g. If p belongs to both divisors, we add one infinite leg l and extend f and g linearly on l so that, under the map (f, g), this leg has primitive direction proportional to $(\operatorname{ord}_p(f), \operatorname{ord}_p(g))$. With this choice the balancing defects of both f and g at p are moved to the new one-valent end. □

Proof of the tropical Weil theorem. By the lemma above we may suppose that the triple (C, f, g) is admissible. Now f, g define a map $C \to \mathbb{T}^2$ and the image is a tropical curve $D = \{(f(x), g(x)) | x \in C\}$: indeed, at every vertex of the image the balancing condition is satisfied; all one-valent

vertices go to infinity by one of the coordinates. Now it is easy to verify that $g(x) \cdot \operatorname{ord}_f(x)$ is a term in the definition of the momentum of D with respect to $(0,0)$: if $\operatorname{ord}_f(x) \neq 0$, then D has a horizontal infinite edge, and its Y-coordinate is $g(x)$. Finally,

$$\sum_{x \in C} f(x) \operatorname{ord}_x(g) - \sum_{x \in C} g(x) \operatorname{ord}_x(f) = \rho((0,0)) = 0. \tag{8}$$ □

Remark 2.2. If f, g come as limits of complex functions f_i, g_i, having $\operatorname{ord}_{f_i}(p_i) = k, \operatorname{ord}_{g_i}(p_i) = m, \lim p_i = p$, then the tropical limit of the family $\{(f_i(x), g_i(x)) | x \in C_i\}$ will not have vertical (with multiplicity k) and horizontal (with multiplicity m) leg from a common divisor point p of f and g, but will have one leg of direction (k, m). Nevertheless, because of the tropical Menelaus theorem or the balancing condition, it has no influence on Eq. (8).

2.5. Stable intersections, realizable intersections. One may ask if the only obstruction for a modification is the generalized tropical Menelaus theorem. As we will see in this section, not at all.

Let us start with a variety $M' \subset \mathbb{K}^n$ and a hypersurface $N' \subset \mathbb{K}^n$ and their non-Archimedean amoebas $M, N \subset \mathbb{T}^n$. We suppose that the intersection of M with a tropical hypersurface N is not transverse. We ask: what can the non-Archimedean amoeba of the intersection $M' \cap N'$ look like?

First of all, as a divisor on M (or N) it should be rationally equivalent to the divisor of the stable intersection of M and N, as it has been shown for the case of curves in [56]. In the general case it follows from the results of this section.

It is easy to find additional necessary conditions. Let us restrict the equation F of N' to M' and tropicalize the resulting data. We obtain a tropical function $f = \operatorname{Trop}(F)$ whose behavior in a neighborhood of $N \cap M$ is fixed, whereas its behavior on all of M is not.

2.6. A realizability obstruction for tropical intersections. The next result is one of the main points of the paper. It gives an obstruction to realizability of tropical intersection divisors. Stable intersection gives the divisor obtained when no cancellation occurs in the lifted equations. However, if the algebraic lifts have cancellations, the valuation of the intersection may move inside the stable-intersection component. The theorem below says that this movement is one-sided: the resulting divisor must be subordinate to the stable intersection.

Definition 2.7. *Let M be an abstract tropical variety, and let $\iota : M \to \mathbb{T}^n$ be an embedding realizing M as a tropical subvariety of $\mathbb{T}^n$. Let f be a tropical function on $\mathbb{T}^n$. We define the pullback $\iota^*(f)$ on M by $\iota^*(f) = f \circ \iota$. We say that $\iota^*(f)$ is frozen at $p \in M$ if, in a neighborhood of $\iota(p)$ in $\mathbb{T}^n$, the tropical polynomial $f = \mathrm{Trop}(F)$ is represented by a single monomial. Equivalently, $\iota(p)$ is not in the corner locus of f.*

Note that, in general, the slopes of f along $\iota(M)$ do not coincide with the slopes of $\iota^*(f)$ on M; see Example 2.5. From now on we consider tropical functions which have frozen points, the motivation is explained in the following definition.

Definition 2.8. *A principal divisor P on an abstract tropical variety M is called subordinate to a principal divisor Q (we write $P \prec Q$), which is defined by a tropical meromorphic function f with frozen points, if P can be defined by a tropical meromorphic function h, which satisfies $h \leq f$ everywhere and $h = f$ at the points where f is frozen.*

Thus a subordinate divisor is obtained by replacing the tropical function f by a smaller tropical meromorphic function h, while keeping $h = f$ on the frozen locus. In the intersection problem, f is the pullback of the tropical equation defining the stable intersection, and h records the possible cancellations in a lift.

Remark 2.3. On a compact connected tropical curve, after fixing the frozen locus, the condition is invariant under adding the same constant to both defining functions. In higher-dimensional or noncompact settings, one should regard the frozen locus and the chosen defining function f, up to the appropriate additive ambiguity, as part of the data.

Example 2.5. Refer to Example 1.4. Let us start from the tropical curve M given by

$$\max(3+X+3Y, 3+X+2Y, 3+X+Y, 3+X, 2+2X+2Y, 2+2X+Y, \\ 2+2X, 1+Y, 1, 3X-2)$$

and a horizontal line N given by $\max(Y, 0)$. We want to understand the valuations of possible intersections of $M' \cap N'$ where $\mathrm{Trop}(M') = M, \mathrm{Trop}(N') = N$.

We can choose the equation for M' in the form

$$F(x,y) = (t^{-1}+\alpha_0+t^{-1}y)+x(t^{-3}+\alpha_1+t^{-3}y^3)+x^2(t^{-2}+\alpha_2+t^{-2}y)+x^3(t^2+\alpha_3),$$

where $\mathrm{val}(\alpha_0) < 1, \mathrm{val}(\alpha_1) < 3, \mathrm{val}(\alpha_2) < 2, \mathrm{val}(\alpha_3) < -2$. It is clear, that for any $A \leq 1, B \leq 3, C \leq 2$ by choosing y of the form $1+\alpha, \mathrm{val}(\alpha) < 0$ and then

with careful choice for $\alpha_1, \alpha_2, \alpha_3$ we can obtain (see Figure 6)

$$f(X) = \mathrm{Val}(F(x, 1+\alpha)) = \max(A, B+X, C+2X, -2+3X).$$

In this example the set $X \geq 4$ on N is frozen for $\mathrm{Trop}(F)$, that is why we have a choice for the constant term A. If the intersection is a compact set (as in Example 1.2), then the constant term is also fixed. Note that for the stable intersection our tropical function is $\mathrm{Trop}(F)(X,0) = \max(1, 3+X, 2+2X, -2+3X)$ and $f(X) \leq \mathrm{Trop}(F)(X,0)$ at every point.

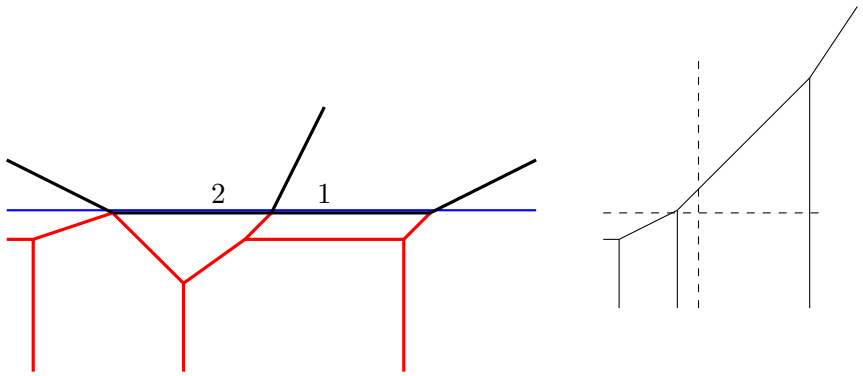


Fig. 6. Refer to Example 2.5. On the left figure we see the vertical part of the modification of the curve given by $F(x,y) = (-t^{-1} + t^{5/3} + t^{-1}y) + x(t^{-3}y - (t^{-3} + t^{-5/6}) + x^2(t^{-2}y - t^{-2} + t^{-3/2}) + x^3t^2$ along the line $y = 1$. On the right figure, we see the tropicalization of the restriction of F on $y = 1$, i.e., the function $\max(3X - 2, 2X + 1.5, X + 5/6, -5/3)$

Now we prove the following theorem, whose proof consists only of a reformulation in the language of tropical modifications and staring to the pictures; see Remark 1.2 as an illustration.

Fix an abstract tropical variety M, its tropical embedding $\iota : M \to \mathbb{T}^n$, and a tropical hypersurface $N \subset \mathbb{T}^n$, given by a tropical polynomial f. As we know, the pullback of the divisor of the stable intersection of $\iota(M)$ with N is given by $\iota^*(f)$. Note that we supply the function $\iota^*(f)$ with frozen points, according to Definition 2.7.

Theorem 2.3 (Subordination obstruction for lifted intersections)**.** *Let $M' \subset (\mathbb{K}^*)^n$ be a pure-dimensional algebraic variety, and let $N' \subset (\mathbb{K}^*)^n$ be the hypersurface defined by $F = 0$. Assume that $F|_{M'}$ is not identically zero on any irreducible component of M', and that $N' \cap M'$ is a proper Cartier divisor on M'.*

Let $M = \mathrm{Trop}(M')$, let $N = \mathrm{Trop}(N')$, and suppose that M is identified

with an abstract tropical variety via an embedding

$$\iota : M \to \mathbb{T}^n.$$

Let $f = \mathrm{Trop}(F)$, and equip $\iota^*(f)$ with the frozen locus from Definition 2.7. Then the pullback to M of the divisor

$$\mathrm{Val}(N' \cap M')$$

is subordinate to the stable-intersection divisor

$$\mathrm{div}(\iota^* f).$$

In other words, every realizable tropical intersection divisor lies below the stable intersection in the sense of Definition 2.8.

Consequently, if a divisor rationally equivalent to the stable intersection is not subordinate to it, then it cannot be obtained as the valuation of an intersection of algebraic lifts with the prescribed tropicalizations.

P r o o f. Recall that $f = \mathrm{Trop}(F), f : \mathbb{T}^n \to \mathbb{T}$. Let us make the modification of $\mathbb{T}^n$ along N. Look at the image $m_f(M)$ of M under this map. Clearly, the valuation of the set $\{(x, F(x)) | x \in M'\}$ belongs to $m_f(M)$. Define

$$h(X) = \max\{\mathrm{val}(F(x)) \mid x \in M',\ \mathrm{Val}(x) = X\}.$$

Then the graph of h is contained in the tropical modification $m_f(M)$, and $h(X) \leq \iota^* f(X)$ for all $X \in M$.

At points $X \in M$ where a unique monomial of F realizes the maximum defining f, no cancellation is possible, hence $h(X) = \iota^* f(X)$. These are precisely the frozen points in the sense used above.

Therefore the pullback of $\iota^*(\mathrm{Trop}(F|_{M'}))$ is at most $\iota^*(f)$ everywhere, and $\iota^*(\mathrm{Trop}(F|_{M'})) = \iota^*(f)$ at the points where $\iota^*(f)$ is frozen. Therefore the divisor of h on M is subordinate to the divisor of $\iota^*(f)$, which is the pullback of the stable intersection divisor. □

The graph of h can lie below the graph of $\mathrm{Trop}(F)|_M$, because cancellations may occur after substituting points of M' into F, which is invisible when we consider $\mathrm{Trop}(F)$ as a function on $\mathbb{T}^n$. Recall that if the image of the valuation map val is $\mathbb{T}$, then we know that $\mathrm{Trop}(F)(X)$ is the maximum of $\mathrm{val}(F(x))$ with $\mathrm{Val}(x) = X$. On the other hand, $\mathrm{Trop}(F_{M'})(X)$ for $X \in M$ is the maximum of $\mathrm{val}(F(x))$ with $\mathrm{Val}(x) = X$ and $x \in M'$. Clearly, the latter maximum is at most the former maximum.

Corollary 2.1 (Non-subordination implies non-realizability)**.** *Let D be a divisor on M rationally equivalent to the stable-intersection divisor* $\mathrm{div}(\iota^* f)$*. If D is not subordinate to* $\mathrm{div}(\iota^* f)$*, then D cannot be the valuation of $N' \cap M'$ for algebraic lifts M' and N' with the prescribed tropicalizations.*

Example 2.6. The following example illustrates the obstruction in Theorem 2.3. Refer to Figure 7. We have the stable intersection $A + B + C + D$ of the curves given by $\max(0, Y)$ and $\max(0, X, 2X-1, 3X-3, 4X-6, X+Y, 2X+Y-1, 3X+Y-3)$. The divisor $A + B' + C' + D$ is rationally equivalent to the stable-intersection divisor $A+B+C+D$. However, the rational function moving $A+B+C+D$ to $A+B'+C'+D$ forces the corresponding tropical function to exceed $\iota^*(f) = \max(0, X, 2X-1, 3X-3, 4X-6)$ on the frozen region $X \geq 4$. Hence this divisor is not subordinate to the stable intersection, and Theorem 2.3 implies that it is not realizable as the valuation of an intersection of lifts. This example shows that our conditions are stronger than in [56].

Note that to realize a divisor subordinate to the stable intersection as a tropicalization of intersection one needs to tune the coefficients of F to have all necessary cancellations. For the case of a planar curve this can be done along trees, [69]. If a curve contains cycles then new conditions appear.

2.7. Interpretation with chips. In the case of curves we can represent a divisor on a curve as a collection of chips. In the last subsection we proved Theorem 2.3 which says that any realizable intersection is subordinate to to the stable intersection. So, one might ask for a method to produce **all** the subordinate divisors to a given divisor (though, it is possible that not all of them are realizable as the valuation of an intersection).

Let us start with the stable intersection of two tropical curves, this intersection is a divisor (collection of chips) on the first curve. Then we allow the following movement: pushing continuously together two neighbor chips on an edge, with equal speed. We do not allow the opposite operation — when we slide continuously two points apart from each other (so, the operation in Figure 7 does not provide a subordinate to $A + B + C + D$ divisor).

This corresponds to the following: we look at the modification of the first curve along the second curve, given by a tropical polynomial $\mathrm{Trop}(F)$. By decreasing the coefficients of the monomials in $\iota^*(f)$ on C, one by one, we can obtain any function less than $\iota^*(f)$.

This reasoning can be applied to the intersection of any two tropical varieties if one of them is a complete intersection. We restrict the equations of the second variety on the first, which gives us a stable intersection, then we have a

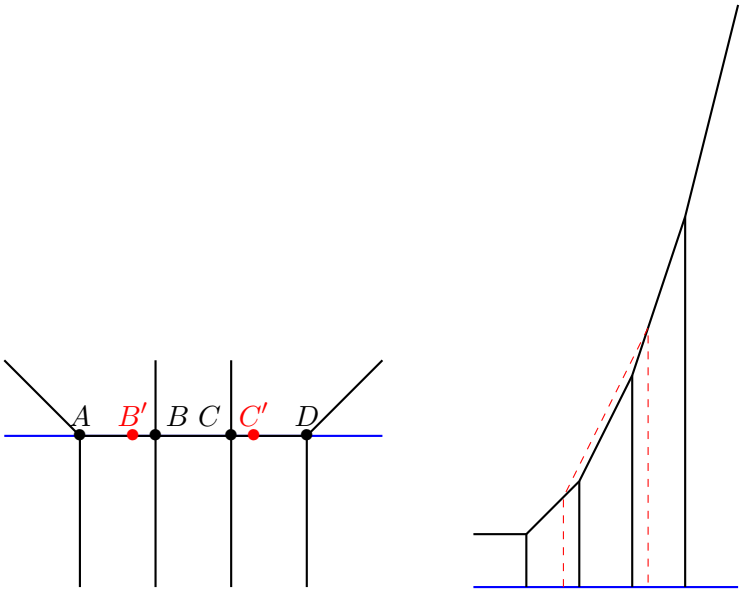


Fig. 7. The divisor $A + B' + C' + D$ is not realizable as the valuation of intersection of the lifts of the curves defined by $\max(0, Y)$ and $\max(0, X, 2X - 1, 3X - 3, 4X - 6, X + Y, 2X + Y - 1, 3X + Y - 3)$. On the right we see the function which carries this rational equivalence out, it is bigger than the function for the stable intersection and so violates the subordination condition of Theorem 2.3

situation similar to Definition 2.8, and, as above, by decreasing the coefficients of pullbacks of tropical polynomials we can obtain all the subordinate to the stable intersection divisors.

Example 2.7. Consider the function $\max(0, X-1, 2X-3)$. This function defines the divisor on $\mathbb{T}^1$ with two chips, one at $X = 1$ and the second at $X = 2$. When we decrease the coefficient in the monomial $X - 1$, these chips are moving closer. For example, the function $\max(0, X-1.3, 2X-3)$ defines the divisor with chips at the points with the coordinates 1.3 and 1.7.

Remark 2.4. Note that if the stable intersection is not compact, then we need to add a chip at infinity (or to treat infinity as a point with one chip). Now let A, B be two chips, A is at infinity and B is on the leg of V going to A. Then, the operation "pushing together A, B" moves only B towards infinity (and A remains unchanged at infinity). This corresponds to decreasing the constant term in Example 2.5.

Example 2.8. Big order tangency with only two degrees of freedom.

([12], Lemma 3.15). We consider a line $y - \alpha x - \beta = 0, \mathrm{val}(\alpha) = 0, \mathrm{val}(\beta) = 0$ and a curve $a_0 + a_1 y + a_2 x y^l = 0$ with $\mathrm{val}(a_0) = 0, \mathrm{val}(a_1) = 0, \mathrm{val}(a_2) = 0$.

Clearly, we have non-transversal intersection, we can perform substitution $y = \alpha x + \beta$, that gives $a_0 + a_1(\alpha x + \beta) + a_2 x(\alpha x + \beta)^l = (a_0 + a_1\beta) + x(a_1\alpha + a_2\beta^l) + \sum_{i=2}^{l+1} a_2 \beta^{l+1-i}\alpha^{i-1}x^i$. The contraction may only appear at two coefficients: the coefficient before x and the constant term. So we have only two degrees of freedom. Let us present the intersection points as chips. By changing the coefficients α, β, a_i we change the intersection, so we can look at how the chips move. So, when $\mathrm{val}(a_0 + a_1\beta) < \mathrm{val}(a_0)$, this corresponds to the movement in Remark 2.4, one chip moves towards infinity while the others do not move at all. Also we can push two chips together by decreasing the valuation of $a_1\alpha + a_2\beta^l$. Note that $l - 2$ chips at the point $(0, 0)$ are unmovable.

Here we have only two degrees of freedom because we have only two degrees of freedom in the equation $a_0 + a_1 y + a_2 x y^l = 0$.

Question 2.1. Motivated by the above example, we give the following suggestions, which seem to be reasonable in the question of the realizability of intersections. Suppose that we have a tropical line and a tropical curve defined by a tropical polynomial f. While defining $\iota^*(f)$ we keep track of all the monomials m_i of f and then in Definition 2.8 we allow g to contain only monomials of the type $\iota^*(m_i)$. I.e. if $f = \max(a_{ij} + iX + iY)$, then we only allow g of the type $\max(c_{ij} + \iota^*(x^i y^j))$ with $c_{ij} \leq a_{ij}$ which coincides with f on the frozen set of f. We explain why we concentrate on the case when one of the curves is a line. Normally, we can perturb the coefficients of the equations of both curves. If one of the curves is a line, we can always suppose that its equation is fixed. For the general case, one should expect that apart from $\iota^*(f)$ on M we can find another thin structure, which is responsible for the deformation of the equation of M being immersed to $\mathbb{T}^n$, something like "a pull-back of the normal bundle", coming from the map ι. As we mentioned before, this program can be carried for the case of planar curves [69].

Example 2.9. Difference between a leg of big weight and a root. Take the curve C given by $F = 0$ where $F(x, y) = 1 + (t^{-1} + t)x + (2t^{-1} + t^2 + t^4)x^2 + (t^3 + 2t^4)x^3 + t^{-1}xy + 2t^{-1}x^2y$ and intersect it with the line given by $t^5 x + y + 1 = 0$.

Performing the tropical modification along the line we see that the resulting curve has a leg of weight three going to $-\infty$. But it is not a root of multiplicity three! If we substitute $y = -1 - t^5 x$ to the equation, we will see that the obtained polynomial $1 + tx + t^2x^2 + t^3x^3$ has three roots with the valuation 1,

but they do not coincide. But if we consider the curve C' given by the equation $F = 0, F(x,y) = 1+(t^{-1}+3t)x+(2t^{-1}+3t^2+t^4)x^2+(t^3+2t^4)x^3+t^{-1}xy+2t^{-1}x^2y$, we see that $\mathrm{Trop}(C) = \mathrm{Trop}(C')$ and C' has a tangency of order three with the line.

The same example can be constructed for a similar Newton polygon

$$\mathrm{ConvHull}\{(0,0)-(1,1)-(n,1)-(n+1,0)\},$$

where we also can obtain the tangency of the order $n+1$.

Question 2.2. Suppose that the intersection of a tropical line with a tropical curve is a segment. Is it always possible to make a modification in order to have a leg of the weight equal to the local stable intersection (Definition 2.1)? If yes, is it always possible to find the coefficients for the equations in order to have a tangency of the order equal to the stable intersection? Also, we can ask this question for any two curves with non-transverse intersections.

Due to combinatorial restrictions in tropical terms, sometimes we can see that it is impossible to have a singular point with high multiplicity on a curve. Note that even in this case, we can have a leg of big multiplicity after the modification; see Example 2.9.

2.8. Digression: a generalization of the tropical momentum. A natural generalization of the vector product (or cross product) in $\mathbb{R}^3$

$$(x_1,y_1,z_1)\times(x_2,y_2,z_2) = (y_1z_2 - z_1y_2,\ z_1x_2 - x_1z_2,\ x_1y_2 - y_1x_2) \tag{9}$$

is the following. Given k vectors $v_1, v_2, \dots, v_k \in \mathbb{R}^n, k \le n$ we consider the vector consisting of all $k \times k$ minors of the $k \times n$ matrix whose rows are $v_1, \dots, v_k$. We call this vector of minors *generalized cross product of* $v_1, v_2, \dots, v_k$.

Consider a tropical variety $V^k \in \mathbb{T}^n, k < n$. Let us choose a basis in each face of V of codimension one and zero, i.e. for a face F we choose a basis in the lattice associated with the integer affine structure of this face. For each face G of codimension one in V and the faces $F_1, F_2, \dots, F_l$ of codimension zero, containing G, we choose vectors $v_G(F_i)$ which participate in the balancing condition along G. Now we can define the sign $s_G(F) \in \{+1,-1\}$ to be $+1$ if the basis in G with added vector $v_G(F)$ at the last place gives the same orientation in F as the previously chosen basis in F, and -1 otherwise.

Definition 2.9. *Let $\mathfrak{G}(V)$ be the abelian group generated formally by all the faces G of V of codimension one. Now we will describe relations in it. For a face $F \subset V$ of maximal dimension define $m(F) \in \mathfrak{G}(V)$ to be the sum*

$\sum_{G\subset F} s_G(F)\cdot G$. *For each bounded face* $F\subset V$ *of maximal dimension we add the relation* $m(F)$ *to* $\mathfrak{G}(V)$.

Example 2.10. Compare Definition 2.10 with the proof of Lemma 2.2. For the case of planar tropical curve C the group $\mathfrak{G}(V)$ is generated by all the vertices of C. Then, each internal edge of C gives the relation that its ends are equal. Therefore, in that case, the group $\mathfrak{G}(V)$ is $\mathbb{Z}$ with generator $\mathbb{1}$, and for each unbounded F, we have $m(F)=\mathbb{1}$.

Definition 2.10. *Let* A *be a point in* $\mathbb{T}^n$. *Pick a face* F *of* V *of codimension zero and let* B *be a point in* F. *Then, define* $r_A(F)$ *as the generalized cross product of the vector* AB *and the vectors in the basis in* F. *Note that* $r_A(F)$ *does not depend on* B. *Finally, define*

$$\rho_A(F)=r_A(F)\otimes_{\mathbb{Q}} m(F)\in \mathbb{R}^{\binom{n}{k+1}}\otimes_{\mathbb{Q}}\mathfrak{G}(V). \tag{10}$$

Proposition 2.5. *For any point* $A\in\mathbb{T}^n$ *we have* $\sum_F \rho_A(F)=0$, *where* F *runs over all the unbounded faces of* V *of the maximal dimension.*

P r o o f. The structure of the proof is the same as in Lemma 2.1. Let us only show that $\sum \rho_A(F)$ does not depend on the point A. Indeed, for each face $G\subset V$ of the codimension **one** we consider the terms in $\sum \rho_A(F)$ which contain G. It is easy to see, that thanks to the balancing condition along G and our choice of signs, the sum of these terms is zero. □

Question 2.3. It seems that in a general situation, if V is a tropical curve, then, again, $\mathfrak{G}(V)$ is $\mathbb{Z}$. On the other hand, it seems that if the dimension of V is at least two, then $\mathfrak{G}(V)$ is freely generated by the unbounded faces of V of codimension 1. Also, it would be nice to state an analog of the tropical Weil theorem in this new context and find its classical algebraic counterpart.

3. Applications of a tropical modification as a method.

The term "modification" is used here in two related senses: first, as a well-defined *operation* on tropical varieties, as in [52]; and second, as a *method* for revealing the behavior of other varieties in an infinitesimal neighborhood of N. Namely, performing the modification of M along $N\subset M$, we will see how M changes, but the objects of codimension one in M may behave differently, depending on their behavior near N. We will clarify this distinction with examples.

3.1. Inflection points. For a smooth plane curve, an inflection point is a smooth point at which the tangent line has contact order at least 3. For a nonsingular real plane curve of degree d, Klein's bound says that the number of real inflection points is at most $d(d-2)$, and this bound is sharp. The question attacked in [12] is *which topological types of planar real algebraic curves admit the maximal number of real inflection points?* Using Viro's patchworking method, a classical construction of algebraic curves, the authors construct examples. For this purpose, they study the possible local pictures of tropicalizations of inflection points. The property to be verified is tangency, but intersection of tropical curve with a tangent line at some point in most cases is not transversal and it is not visible what is the actual order of tangency. To see that, the authors do tropical modifications.

3.2. The category of tropical curves. For the treatment of this question with tropical harmonic maps see [1, 2]. G. Mikhalkin (lectures, 2011) defines the morphisms in the category of tropical curves as all the maps, satisfying the balancing and Riemann-Hurwitz conditions (see, for example [8]) and subject to the *modifiability* condition:

Definition 3.1. *A morphism $f : A \to B$ of tropical curves A, B is said to be modifiable if for any modification B' of B there exists a modification A' of A and a lift f' of f which makes the obtained diagram commutative.*

Remark 3.1. The modifiability condition ensures that a morphism came as a degeneration of maps between complex curves (see Section 4.1).

Sketch of a proof. After a number of modifications we may have the map f' contracting no cycles. Then we construct a family of complex curves B_i such that $\lim B_i = B'$ in the hyperbolic sense (see section 4.1). Finally, since f' should come as a tropicalization of a covering, the complex curves A_i with $\lim A_i = A'$ are constructed as coverings $f_i : A_i \to B_i$ over B_i where the combinatorics (ramification profiles, local degrees at points of tori contracting to tropical edges) of f_i is prescribed by f'. Balancing and Riemann-Hurwitz conditions follow. □

3.3. Realization of a collection of lines and (4,d)-nets. Which configuration of lines and points in $\mathbb{P}^2$ with given incidence relation are possible? That is a classical question and even for seemingly easy data the answer is often not clear.

Definition 3.2. *A $(4, d)$-net in $\mathbb{P}^2$ is four collections by d lines each of them, such that exactly four lines pass through any point of intersection of two*

lines from different collections, all these four lines are from different collections.

It is not clear whether a $(4, d)$-net exists for $d \geq 5$. In [28] the authors proved, using tropical geometry, that there exists no $(4, 4)$-net.

One of the key ingredients is the following: if some net exists in the classical world, then it exists in the tropical world. The problem appears: if we have more than three tropical lines through a point on a plane, then the intersection of two of them will be non-transversal. However, thanks to modifications we always can have transversal intersections, but probably in the space of bigger dimensions. For that, we just do modifications along lines that have non-transversal intersection. After these modifications, all intersections become transversal and the modified lines go to infinity. Then, let us think about the following theorem, announced by the authors of [28], from the point of view of modifications:

Question 3.1. If some combinatorial data (required dimensions of intersections of linear spaces) can be realized in $\mathbb{P}^k$ by a collection of linear spaces, does there exist a collection of tropical linear spaces that realize the same combinatorial data in $\mathbb{TP}^{k'}$ with $k' \geq k$?

Indeed, consider this realization in $\mathbb{P}^k$. By passing to the tropical limit we obtain a tropical configuration, but the intersection dimensions may increase. Then, by doing the modifications, we want to repair the correct dimensions. Is it always possible to achieve?

3.4. A point of big multiplicity on a planar curve. In its most general form, this question could be formulated as follows: given a cohomological class a of subvariety S in a bigger variety, how many singularities S may have? For example, is it possible for a surface of degree 4 in $\mathbb{C}P^4$ to have four double points and three two-fold lines?

There are several reasons why tropical geometry may provide tools for such questions. We will demonstrate these tools in the case of curves, where this deed has been already done. Combinatorics of a planar tropical curve is encoded in the subdivision of its Newton polygon. A singular point of multiplicity m influences a part of the subdivision of area of order m^2 [35], which is in accordance with the order of the number of linear conditions $\left(\frac{m(m+1)}{2}\right)$ that a point of multiplicity m imposes on the coefficients of the curve's equation. For a general treatment of the tropical singularities, see [35], [36] and Chapters 1,2 in [34].

In this section, we will only demonstrate how to apply modification techniques in this problem, though we will obtain a weaker estimation – but still of order m^2.

The idea is the following: if a curve C has a point p of multiplicity m, then for each curve D, passing through p, the local intersection of C and D at p is at least m. The multiplicity of a local intersection of C and D can be estimated from above by studying the connected component, containing $\mathrm{Val}(p)$, of the stable intersection $\mathrm{Trop}(C) \cap \mathrm{Trop}(D)$ for the non-Archimedean amoebas of C and D, see Theorem 2.2.

Here is method: we take the polynomial F defining D, and use the fact that the image of C under the map $m_D : (x, y) \to (x, y, F(x, y))$ intersects the plane $z = 0$ with multiplicity at least m. That implies **existence** of a modification of $\mathrm{Trop}(C)$ along $\mathrm{Trop}(D)$, which has a leg of weight m going in the direction $(0, 0, -1)$, exactly under the point $\mathrm{Val}(p)$. The latter modification is obtained just by taking the non-Archimedean amoeba of $m_D(C) \subset m_D(\mathbb{P}^2)$.

Now we reduce the problem to its combinatorial counterpart: is it possible for two given tropical curves, that after the modification along the second, the first curve will have a leg of weight m, which projects exactly on the given point $\mathrm{Val}(p)$? After some work with intrinsically tropical objects, we will get an estimate of this point's influence on the Newton polygon of the curve.

We are not going to consider this problem in the full generality, so we will have a close look at the simplest interesting example.

Proposition 3.1. *Suppose that a horizontal edge E of a tropical curve C contains a point $\mathrm{Val}(p)$ where p is of multiplicity m for a curve C' such that $\mathrm{trop}(C') = C$. Denote by $d(E)$ the vertical edge in the dual subdivision of the Newton polygon which is dual to E. Let the endpoints of E be A_1, A_2 and two faces $d(A_1), d(A_2)$ adjacent to $d(E)$ have no other vertical edges. Then the sum of widths of the faces $d(A_1), d(A_2)$ is at least m, so their total area is at least $m^2/2$.*

P r o o f. Suppose that p is of multiplicity m for C'. Let us take a line D through p, such that $\mathrm{Trop}(D)$ contains inside its vertical edge the point $\mathrm{Val}(p)$. Clearly the local intersection $\mathrm{Trop}(C') \cap \mathrm{Trop}(D)$ is one point, and the multiplicity of this point should be at least m. That immediately implies that the weight of E is at least m. Hence the lattice length of $d(E)$ is at least m.

Let us look at the dual picture in the Newton polygon. Two faces $d(A_1), d(A_2)$ adjacent to the vertical edge have the sum of width in the $(1, 0)$ direction at least m (by Proposition 2.1), $d(E)$ has length m, so the sum of the areas of $d(A_1), d(A_2)$ is at least $m^2/2$. □

Remark 3.2. Note that if the stable intersection of $\mathrm{Trop}(C)$ with the horizontal line is m, then we can uniquely determine the position of the valuation

of the singular point, see Lemma 2.4.

What to do if there is a usual horizontal line L, a part of C, through $\mathrm{Val}(p)$? We perform the modification along this horizontal line L. If a part of the curve goes to the minus infinity, this indicates that, along the corresponding initial degeneration, the equation of D appears as a factor. If D is an actual component of C', then one can divide the equation of C' by the equation of D. That means that the Newton polygon of C has two parallel vertical sides. The components of the modification which do not go to the minus infinity do not contribute to the singularity.

However, it is possible that $d(A_1), d(A_2)$ have other vertical sides besides $d(E)$. Let $\mathfrak{E}$ be the stable intersection of $\mathrm{Trop}(C)$ and the horizontal line; clearly $E \subset \mathfrak{E}$. Now, let us compute the sum of the areas of the faces $d(V)$ corresponding to vertices V of $\mathrm{Trop}(C)$ on $\mathfrak{E}$. It is possible that more than two faces correspond to one singular point, if the edge with the singular point has an extension, see again Example 1.4.

Suppose that a tropical curve has edges $A_1A_2, A_2A_3, \ldots, A_{k-1}A_k$ and $A_1, A_2, \ldots, A_k$ are situated on a horizontal interval $A_1A_k = \mathfrak{E}$. Suppose that p, point of multiplicity m, is on the edge A_sA_{s+1}. Making a modification along **a line** containing A_1A_k in its horizontal ray we estimate only the common width of faces corresponding to $A_1, A_2, \ldots A_k$, which gives no good estimate for the sum of areas of $d(A_i)$.

But we can make a modification along a quadric.

Lemma 3.1. *In the above hypothesis, the sum of areas of all faces*

$$d(A_1), d(A_2), \ldots, d(A_k)$$

is at least $m^2/4$.

P r o o f. Let $a_i = \omega_{(1,0)}(d(A_{s+i})), i \geq 1$ be the width of i-th face (i.e. $d(A_{s+i})$) on the right, $b_i = \omega_{(1,0)}(d(A_{s-i})), i \geq 0$ be the width of i-th face (i.e. $d(A_{s-i})$) on the left. Let c_i be the lattice length of i-th vertical edge on the right (i.e. $c_i = \omega_{(0,1)}d(A_{s+i}A_{s+i+1}), i \geq 1$), d_i be the length of the i-th vertical edge on the left (i.e. $d_i = \omega_{(0,1)}d(A_{s-i}A_{s-i+1}), i \geq 1$). Then, let $\sum_{i=1}^{k} a_i = A_k, \sum_{i=1}^{k} b_i = B_k$. With the same calculations as above, making the modification along a piece of a quadric with vertices on $A_{s-j}A_{s+1-j}$ and $A_{s+i}A_{s+1+i}$ we get $A_i + c_i + B_j + d_j \geq m$ for all pairs i, j. Denote

$$A = \min_i(c_i + A_i), \qquad B = \min_j(d_j + B_j),$$

so $A + B \geq m$.

Then, $c_i \geq A - A_i, d_j \geq B - B_j$. Sum S of areas can be estimated as

$$2S \geq (m+c_1)A_1+\sum(A_{i+1}-A_i)(c_i+c_{i+1})+(m+d_1)B_1+\sum(B_{i+1}-B_i)(d_i+d_{i+1})$$

$$2S \geq (m + A - A_1)A_1 + \sum(A_{i+1} - A_i)(A - A_i + A - A_{i+1})+$$

$$(m + B - B_1)B_1 + \sum(B_{i+1} - B_i)(B - B_i + B - B_{i+1}) \geq$$

$$A_1(m - A) + A^2 + B_1(m - B) + B^2 \geq m^2/2.$$

So, $S \geq m^2/4$. □

4. Intuitions and interpretations.

La science toujours progresse et jamais ne faillit,
toujours se hausse et jamais ne dégénère,
toujours dévoile et jamais n'occulte.
Anonyme

This section explains why tropical modification is a natural notion and gives several interpretations in different contexts. A reader primarily interested in definitions, examples, and theorems may proceed directly to the previous sections and return here only for motivation and references.

Tropical modifications were introduced in the seminal paper [51] as the main ingredient in the tropical equivalence relation. Namely, two tropical varieties are *equivalent* (tropical counterpart of birational isomorphism) if they are related by a chain of tropical modifications and reverse operations. For the full definition of an abstract tropical variety, refer to [55] and [53].

The underlying idea is as follows. Recall that a tropical variety V can be decomposed into a disjoint union of a compact part V_c and a non-compact part V_∞, and $V = V_c \cup V_\infty$. Moreover, V retracts on V_c. Then, the set V_∞ consists of "tree-like" unions of pieces of affine hyperplanes. We call these parts *legs* in the one-dimensional case and *leaves* in general situation. For tropical curves, V_∞ is a union of half-lines. For example, for a tropical elliptic curve (see Figure 8, left side) the set V_c is the ellipse, and V_∞ is the set of trees growing on the ellipse.

Remark 4.1. On a tropical rational[3] variety V, each point may be chosen as V_c, see Figure 8, right side.

[3]Rational tropical varieties are the contractible ones, as a topological space. They are not well studied even in small dimensions. For example, there exist algebraic three-dimensional cubic hypersurfaces which are not rational. It is not known whether we can see this tropically, because all tropical cubic surfaces are contractible.

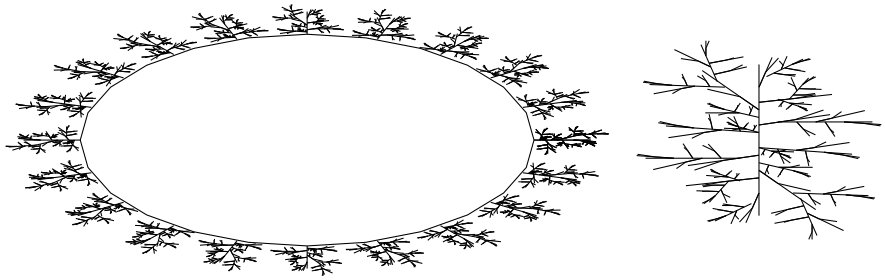


Fig. 8. On the left side we see a tropical elliptic curve V which is a part of the analytification of an elliptic curve. The ellipse is V_c and the union of tree-like pieces is V_∞. On the right side we see a tropical rational curve V, which is equal to V_∞. Each point x of V can serve as V_c, because V contracts onto any of its point $x \in V$

Consider the tropical limit V of algebraic varieties $W_i \subset (\mathbb{C}^*)^n$, i.e. $V = \lim\limits_{i\to\infty} \mathrm{Log}_{t_i}(W_i)$, where we apply the map $\mathrm{Log}_{t_i} : \mathbb{C}^* \to \mathbb{R}, x \to \log_{t_i} |x|$ coordinate-wise and $\{t_i\}_{i=1}^{\infty}$ is a sequence of positive numbers, tending to $+\infty$. In this case the set V_∞ encodes the topological way of how W_i approach some compactification of $(\mathbb{C}^*)^n$. For the moment, the particular choice of the compactification does not matter[4].

Besides, for i big enough, the Bergman fan $B(W_i) := \lim\limits_{t\to\infty} \mathrm{Log}_t(W_i)$ of W_i is equal to $\lim\limits_{t\to\infty} \frac{1}{t} V$. The latter limit is obtained by contracting the compact part V_c of V, so the Bergman fan can be restored by V_∞. Note, that V came here with a particular immersion to $\mathbb{R}^n$.

Example 4.1. If curves $W_i, i = 1, 2, \dots$ in $(\mathbb{C}^*)^2$ all have branches with asymptotic (s^k, s^l) with a local parameter $s \to \infty$, then the tropical limit V of this family lies in $\mathbb{R}^2$, and V has the infinite leg (half-line) in the lattice direction (k, l).

Let us suppose that we have an algebraic map $f : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$, and f is in general position with respect to the family $\{W_i\}$, i.e. for each i big enough, the image $f(W_i)$ is birationally equivalent to W_i. Let V' be the tropical limit of the family $\{f(W_i)\}$. One can prove that V'_∞ differs from V_∞ by adding new half-planes and contracting other half-planes. These half-planes grow along the tropicalization of zeros and poles of f on W_i (exactly as in Definition 1.3). This consideration suggests the ideas of *modification* and, subsequently, *tropical*

[4]For a fixed compactification, see the notion of sedentarity in [68] and [10], p. 44.

equivalence. The name "modification" was borrowed from complex analysis, and tropical modification is sometimes called "tropical blow-up".

In Section 3.2 we see how the notion of modifications allows us to define the *category of tropical curves*. This category keeps track of birational isomorphism in the category of complex algebraic curves. See also §2.2, where making modifications for curves simplifies a proof to some extent.

Alternatively, tropical geometry can be thought of as the study of skeletons of analytifications of algebraic varieties; see Figure 8, where the analytification of an elliptic curve is shown on the left and the analytification of $\mathbb{P}^1$ on the right. The analytification X^{an} of a variety X is the set of all seminorms on functions on X. Each point $x \in X$ defines such a seminorm by measuring the order of vanishing of a function at x. In Figure 8, these points are represented by the ends of leaves; equivalently, these valuations represent norms of "zero" radius. The analytification of an elliptic curve is the injective limit of all modifications of its tropicalization: we add a leg at every point of a circle, then add a leg at every point of the new space, and so on.

For the sake of shortness, we refer the reader to a nice introduction in Berkovich spaces, with a bit of pictures [5],[71] and to [6] to see how it has been applied to tropical geometry (also, see on the page 7 in [6], using of log reminds hyperbolic approach).

We can obtain a tropical variety V as the non-Archimedean amoeba of an algebraic variety W over a non-Archimedean field. This approach (see section §4.2) finally suggests the same idea of equivalence up to modification, because the analytification W^{an} is the injective limit of all "affine" tropical modifications (i.e. along only principal divisors) of V (see [60]). Berkovich proved that W^{an} retracts on a finite polyhedral complex, so V_c is a deformation retract of W^{an}. Even better, the metric on W^{an} agrees with the metric on V for the case of curves[5] ([6]). For elliptic curves V_c will be a circle in both tropical and analytical cases, and its length is prescribed by the j-invariant of the considered curve ([15]).

This connection between tropical geometry and analytic geometry leads to the questions of *lifting* or *realizability*, i.e. what could be the intersection of two varieties X, Y if we know the intersection of their tropicalizations? If their tropicalizations $\mathrm{Trop}(X), \mathrm{Trop}(Y)$ intersect transversally, the answer is relatively simple, see [58]. If the intersection of $\mathrm{Trop}(X), \mathrm{Trop}(Y)$ is non-transverse, then we can lift *the stable intersection* of these tropical varieties, see [59],[61].

This raised the following question: to what extent the only condition for a

[5]That should be true for varieties of any dimension, modulo integer affine transformations, but no proof has appeared yet. For the skeletons in higher dimensions see [26, 27].

divisor on a curve to be realizable as an intersection is to be rationally equivalent to the stable intersection (cf. [56], Conjecture 3.4)?

Tropical modification (as a *method*) helps dealing with such questions. It is known that being rationally equivalent to the stable intersection is not enough. We consider other existing obstructions (in fact, equivalent to Vieta theorem) for what can happen in non-transverse tropical intersections, and prove, for that occasion, the tropical Weil reciprocity law by using the tropical momentum Lemma 2.2.

Consequently, modifications are used in tropical intersection theory ([66, 67]), to define the intersection product. Nevertheless, one must use modifications along non-Cartier divisors (Examples 1.1.37, 3.4.18 in [67], for moduli space of five points on rational curve) and even along non-realizable subvarieties – for a proof that they are non-realizable as tropical limits.

As we stated before, one should think that a tropical modification along X reveals asymptotical behavior of objects near X. We can find an analogy in non-standard analysis: the tropical line is the hyperreal line, the modification at a point is an approaching this point with *an infinitesimal telescope*, see Figure 10 and Section 4.2. In order to define tropical Hopf manifolds one should also use the modifications to study certain germs [65].

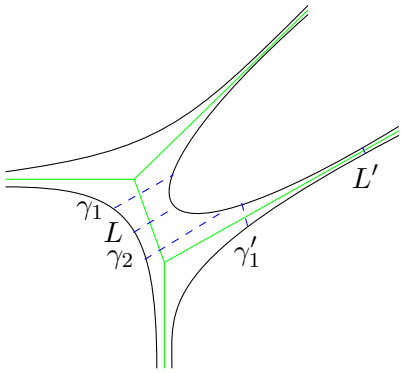
Given a surface with hyperbolic structure, we can make a puncture at x. This changes the hyperbolic structure and x goes, in a sense, to "infinity". A tropical curve can be obtained as a degeneration of hyperbolic structures, and making a puncture at x results as the modification at the limit of x, see Section §4.1.

A modification can be described as a graph of a function, if we use the convention about multivalued addition, brought in tropical geometry by Oleg Viro ([73]), see Section 1.1.

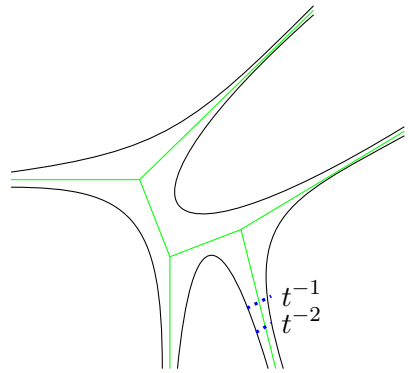
The other applications of tropical modification as a *method* are following. Passing to tropical limit squashes a variety, and some local features become invisible. In order to reveal them back we can do a modification (whence also this metaphor "look in an infinitesimal microscope"). For example, modifications allow us to restore transversality between lines if we have lost it during tropicalization (§3.3), then it allows us to see (-1)-curves on del Pezzo surfaces ([62]). Methods of lifting non-transverse intersections leads us to use modifications in questions about singularities: inflection points – [12], singular points – [47]. As an example (Section 3.4), we use modification in the study of singular points of order m (but obtain weaker results than in [35]).

4.1. Hyperbolic approach and moduli spaces. Consider a tropical curve C given as the tropical limit of complex curves C_i. From the point of view of hyperbolic geometry, a modification of C at a point $x \in C$ amounts to creating a puncture x_i on C_i, with the condition that $x_i \to x$. To explain this we need to know how to directly construct tropical curves via limits of Riemann surfaces with hyperbolic structure on them, without any immersions[6].

For details on how tropical geometry can be built on the basis of hyperbolic geometry, see [44]. Here we briefly sketch the construction.



(a) Blue dashed lines γ_1, γ_2 depict the collar of a geodesic L, γ_1' is a part of L''s collar.



(b) Modification subdivides an old edge and adds a new edge of infinite length. The lengths of the circles around the puncture are indicated, cf. with Figure 10

Fig. 9. We draw the tropical limits of Riemann surfaces, and a surface close to the limit. Modification adds a puncture to each curve in the family and a leg to the tropical curve

The approach proposed by L. Lang uses the collar lemma [13]. This lemma says that any closed geodesic of length l has a collar of width $\log(\coth(l/4))$ and, more importantly, that collars of distinct sufficiently short closed geodesics do not intersect; see Figure 9. It is also important that shorter geodesics have wider collars, and, intuitively, a puncture has a collar of infinite width.

Thus, given a family of curves C_i (of the same genus), we consider a fixed pair-of-pants decomposition by geodesics L_i. The tropical curve is constructed as follows: its vertices are in one-to-one correspondence with the pair-of-pants, each shared boundary component between two pairs-of-pants correspond to an

[6]Usually people consider curves C_i in toric variety X and then they consider degeneration of complex structures on X.

edge of the tropical curve, and the collar lemma furnishes us with the length of the edges of the tropical curve as the logarithms (with base t, and $t \to \infty$ as the hyperbolic structure degenerates) of widths of the collars of L_i's. Compare this approach with [9].

What will happen if we make a puncture? A puncture is the limit of small geodesic circles. Cutting out a disk with radius t^{-n} adds a leaf of finite length n, as it is seen from the above description. Therefore, cutting out a point results in adding an infinite edge, i.e. a modification.

That explains why a permanent using of graphs for moduli space problems is actually useful ([43], cf. [39]). Tropical curves describe the part of boundary of a moduli space, and modification corresponds to marking a point (read [21] to see the hyperbolic view on moduli space problems), which are punctures from the hyperbolic point of view (see applications to moduli space of points [53]). Tropical differential forms are also defined in this manner while taking a limit of hyperbolic structure [55].

4.2. Non-standard analysis. Non-standard analysis appeared as an attempt to formalize the notion of "infinitesimally small" variables (see §4 of [70] for a nice and short exposition).

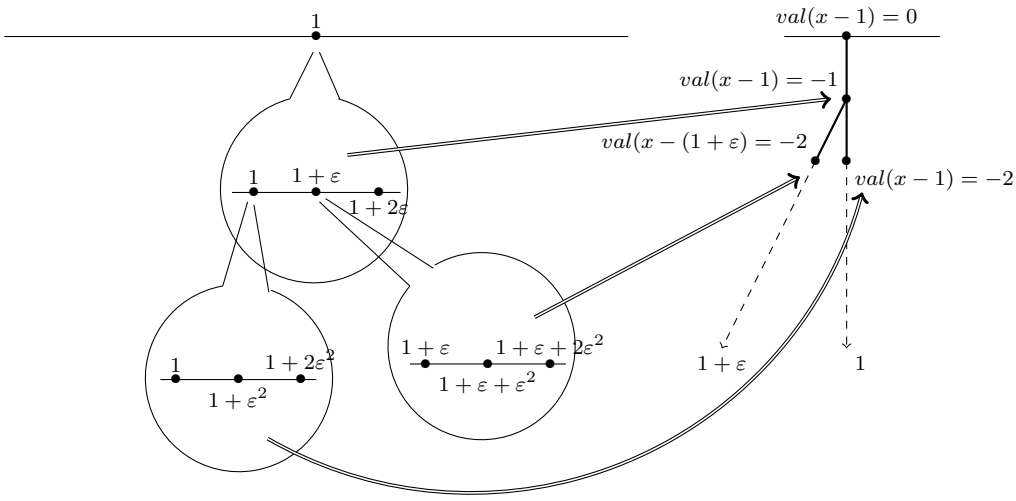


Fig. 10. Similarity in the pictures while using an infinitesimal microscope (left) and the tropical modification at points 1 and $1+\varepsilon$ (right)

There is a way to understand tropical geometry via nonstandard analysis (cf. §1.4 [31]). Figure 10 shows that tropical modifications are similar to

"infinitesimal microscope" for the hyperreal line in the terminology of [38], and this interpretation in computational sense is the same as for Berkovich spaces: doing modification at the point $x = 1$ on a curve is adding a leg to the tropical curve, which ranges points according their asymptotical distance to $x = 1$, i.e. $\mathrm{val}(x - 1)$, these pictures are also similar to the hyperbolic ones; see Figure 9. Dotted lines represent directions to the endpoints of the analytifications, and we have a similar type of branching at all points in Figure 8.

It is worth to note that there are still no applications of this point of view, neither in tropical geometry, nor in non-standard analysis. However, Berkovich spaces can be understood as a modern version of non-standard analysis, and tropical modification has applications there.

5. Questions. An important feature of tropical geometry is that it builds a bridge between geometric objects, such as hyperbolic surfaces, and discrete objects, such as p-adic valuations and non-Archimedean analytic spaces. Since tropical modifications dwell in both realms, we expect them to be useful in future applications.

Question 5.1. Explore tropical modifications over the phase-tropical numbers [73].

For example, in [23] tropical geometry was used to construct a system of polynomial equations with only 5 monomials and 6 positive solutions, and the authors used non-transverse intersections. Looking at questions of such type with a modification tool over phase tropical numbers looks promising.

Question 5.2. Study the tangency between tropical curves and tropical surfaces in $\mathbb{T}^3$.

For example, one can ask how many surfaces of a given degree are tangent to a given collection of lines (or twisted cubics) in $\mathbb{T}^3$.

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