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FROBENIUS FORMS ON WEAKLY PRIMITIVE AXIAL ALGEBRAS OF JORDAN TYPE

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In honor of our colleague and friend Vesselin Drensky on his 70th birthday

Abstract. In a previous paper we studied "weakly primitive axial algebras" with respect to more general fusion rules, for which at least one axis satisfies the fusion rules. In this continuation, a concise description is provided of the 2-generated algebras we obtained in that paper and we show that in certain cases, the algebras have a Frobenius form, a key tool in their study.

1. Introduction. This paper continues our study of "axial algebras", generated by "axes", i.e., semisimple idempotents. In [12] we considered noncommutative, nonassociative 2-generated algebras satisfying certain fusion rules to be reviewed below. The main result of this paper is to use those results to define a Frobenius form. This is significant because the structure of algebras becomes much more tractable when they are endowed with Frobenius forms, as exemplified in [2, Appendix A], and as we shall see in proving that $A_0(a)^2 \subseteq A_0(a)$ and $A_{\mu,\nu}(a)$ is in the radical of the form for $\mu \neq \nu$.

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Definitions and notation 1.1.
We review some definitions from [7] and [12] and add a few more.

(1) A always denotes a nonassociative algebra *(possibly not commutative)*, over a field $\mathbb{F}$ of characteristic $\neq 2$. In this paper $\lambda, \delta \in \mathbb{F} \setminus \{0, 1\}$ always.

(2) For $y \in A$, write L_y for the left multiplication map $z \mapsto yz$, and R_y for the right multiplication map $z \mapsto zy$. $A_\mu(L_y)$ denotes the μ-eigenspace of L_y, for $\mu \in \mathbb{F}$, which is permitted to be 0; similarly for $A_\nu(R_y)$. We let $A_{\mu,\nu}(y) = A_\mu(L_y) \cap A_\nu(R_y)$.

(3) A *left axis* a is an idempotent for which A is a direct sum of its left eigenspaces with respect to L_a. A *right axis* is defined analogously. A *(2-sided) axis* a is a left axis which is also a right axis, for which $L_a R_a = R_a L_a$. When a is an axis, if $S_L := S_L(a)$ is the set of left eigenvalues and $S_R := S_R(a)$ is the set of right eigenvalues, then A decomposes as a direct sum of subspaces

$$A = \oplus_{\mu \in S_L,\ \nu \in S_R} A_{\mu,\nu}(a). \tag{1.1}$$

The pairs of eigenvalues $(1, 1)$ (and possibly $(0, 0)$) play a particularly significant role. We write

$$S(a) := S_L(a) \times S_R(a) \setminus \{(0, 0), (1, 1)\}.$$

$A_{\mu,\nu}(a)$, for $(\mu, \nu) \in S$, is called a *two-sided eigenspace*. When a is understood, We write S for $S(a)$, and *we call a an S-axis*. We also write $A_\mu(a)$ for $A_{\mu,\mu}(a)$. For a subset $S' \subseteq S$, and an element $y \in A$, we write $y_{\mu,\nu}$ for the projection of y on $A_{\mu,\nu}(a)$, and we let

$$A_{S'}(a) := \textstyle\sum_{(\mu,\nu)\in S'} A_{\mu,\nu}(a), \text{ and } y_{S'} = \sum_{(\mu,\nu)\in S'} y_{\mu,\nu}.$$

Thus, A decomposes as a direct sum of eigenspaces

$$A = A_1(a) \oplus A_0(a) \oplus A_S(a), \tag{1.2}$$

(one could have $S(a) = \emptyset$), and for $y \in A$ we write

$$y = y_1 + y_0 + y_S,$$

where $y_i \in A_i(a)$, $i = 0, 1$, and $y_S \in A_S(a)$, so that $y_S = \sum_{(\mu,\nu)\in S} y_{\mu,\nu}$.

Let

$$S^\circ(a) = \{(\mu, \nu) \in S : A_{\mu,\nu}(a) \neq 0\}.$$
$$S^\dagger(a) = \{(\mu, \nu) \in S^\circ(a) : \mu \neq \nu\}.$$
$$S^{\mathrm{comm}}(a) = \{(\mu, \mu) : (\mu, \mu) \in S^\circ(a)\}.$$

Thus $S^\circ(a) = S^\dagger(a) \cup S^{\mathrm{comm}}(a)$.

Also define the *complementary set* $\tilde{S} := \{(\nu, \mu) : (\mu, \nu) \in S\}$.

(4) An S-axis a is *commutative* if $S^\dagger = \emptyset$. (Note that this implies that a commutes with all elements in A.) Otherwise we say that a is *noncommutative*.

(5) The *fusion rules* in this paper are comprised of the rules:

(a) $A_1(a) + A_0(a)$ is a subalgebra of A;

(b) $A_{\mu,\nu}(a)$ is a left and right $(A_1(a) + A_0(a))$-module.

(c) $A_{\mu,\nu}(a)A_{\mu',\nu'}(a) \subseteq \delta_{\mu,\mu'}\delta_{\nu,\nu'}(A_1(a) + A_0(a))$, $\forall(\mu, \nu), (\mu', \nu') \in S$, where δ denotes the Kronecker delta.

In particular, for each nonempty subset $S' \subseteq S$, the fusion rules imply a grading of A as a $\mathbb{Z}_2$-graded algebra, i.e.,

$$A = \overbrace{A_1(a) \oplus A_0(a) \oplus A_{S\setminus S'}(a))}^{(+)\text{-part}} \oplus \overbrace{A_{S'}(a)}^{(-)\text{-part}}. \tag{1.3}$$[1]

(6) An S-axis a satisfying the fusion rules is of *Jordan type* if

$$A_0(a)^2 \subseteq A_0(a). \tag{1.4}$$

(7) An axis a is *weakly primitive* if $A_1(a) = \mathbb{F}a$. An axis a is *left primitive* if $A_1(L_a) = \mathbb{F}a$. Similarly for *right primitive*. An axis which is both left and right primitive is called *primitive*.

(8) An axis a is *special* if a is weakly primitive satisfying the fusion rules, such that for any idempotent b, the subalgebra $A = \langle\langle a, b\rangle\rangle$ satisfies $\dim A_0(a) \leq 1$.

(9) An axis b is of *Jordan type* η if b is an S-axis of Jordan type with one of the following:

(a) $S^{\mathrm{comm}}(b) = \{(\eta, \eta)\}$ for some $\eta \neq 0, 1$.

(b) $A{=}\langle\langle a, b\rangle\rangle$, $S^{\mathrm{comm}} = \emptyset$, $\eta \in \{0, 1\}$, $\dim A_\eta(b) = 2$, and $\dim A_{1-\eta}(b) = 1$.

[1] Fusion rules (a) and (b) are widespread, cf. [8], but (c) generalizes the definition of Jordan type in [4].

(10) If $S^{\mathrm{comm}}(b) = \emptyset$ and

(a) if b is weakly primitive then we say that b is of *Jordan type* $\emptyset$.

(b) if $A = \langle\langle a, b\rangle\rangle$, $\dim A_0(b) = 2 = \dim A_1(b)$, then we say that b is of *Jordan type* $\{0, 1\}$.

(11) For a given axis $a \in A$, any element $y \in A$ has a unique decomposition $y = y_1 + y_0 + y_{S^{\mathrm{comm}}(a)} + y_{S^\dagger(a)}$, where $y_1 \in A_1(a)$ and $y_0 \in A_0(a)$; when a is weakly primitive we write $y_1 = \varphi_a(y)a$ where $\varphi_a(y) \in \mathbb{F}$.

Lemma 1.2. *If a is an S-axis satisfying the fusion rules and $S' \subseteq S$, then $A_1(a) \oplus A_0(a) \oplus A_{S'}(a)$ is a subalgebra of A.*

Proof. By the fusion rules. □

Definition 1.3.

(1) *$X \subseteq A$ denotes a set of axes. $\langle\langle X\rangle\rangle$ denotes the subalgebra of A generated by X. If $A = \langle\langle X\rangle\rangle$, we say that X generates A; if $|X| = n$ we say that A is n-generated.*

(2) *An axial algebra A is a (weak) PAJ, if $A = \langle\langle X\rangle\rangle$, where X is a set of (weakly) primitive axes of Jordan type.*

(a) *A is of Jordan type η if each $a \in X$ has Jordan type η.*

(b) *A commutative PAJ is called a CPAJ.*

(3) *A set X of axes is homogeneous if $S^\circ(b) = S^\circ(c)$ or $S^\circ(b) = \tilde{S}^\circ(c)$, for $b, c \in X$.*

1.1. Review of the Classification of 2-generated weak PAJ's, from [12]. [12, Theorem A] characterized all 2-generated-CPAJ's $\langle\langle a, b\rangle\rangle$.

Example 1.4. Next, we list the possibilities for $A = \langle\langle a, b\rangle\rangle$, as classified in [12], where a is a special axis, and b is an axis satisfying the fusion rules. (It follow by [12, Lemma 3.7] that $\dim A_{\mu,\nu}(a) \leq 1$ for all $(\mu, \nu) \in S(a)$.)

In (iii)–(vi) below, both a and b are of Jordan type $\emptyset$, $\{\frac{1}{2}\}$, or $\{2\}$. In the last three examples, b is not weakly primitive. We write $S(a)$ and $S(b)$ to avoid ambiguity.

(i) The list of [4, Theorem 1.1]. We call these axial algebras *HRS algebras*, which are commutative. In each case, $S(a)$ and $S(b)$ have order ≤ 1. Both axes are primitive of Jordan type. If $S(a) = \{\eta\}, \eta \notin \{0, 1\}$, then necessarily $S(b) = \{\eta\}$ or $S(b) = \{1 - \eta\}$.

(ii) A 4-dimensional CPAJ, cf. [12, Example 4.5]. Let

$$A = \mathbb{F}a + \mathbb{F}b_0 + \mathbb{F}b_{2,2} + \mathbb{F}b_{\frac{1}{2},\frac{1}{2}}, \qquad b = a + b_0 + b_{2,2} + b_{\frac{1}{2},\frac{1}{2}},$$

$$b^2_{\frac{1}{2},\frac{1}{2}} = 0, \quad b^2_{2,2} = -\tfrac{1}{2}b_0, \quad b_{\frac{1}{2},\frac{1}{2}}b_{2,2} = 0.$$

$$b_0^2 = \tfrac{3}{2}b_0, \qquad b_0 b_{\frac{1}{2},\frac{1}{2}} = 0, \qquad b_0 b_{2,2} = -\tfrac{3}{2}b_{2,2}.$$

$$a = b + a_0 + a_{2,2} + a_{\frac{1}{2},\frac{1}{2}}, \quad a_0 \in A_0(b), \quad a_{\eta,\eta} \in A_\eta(b).$$

Hence the axis b is primitive and $S(a) = S(b) = \{(\frac{1}{2},\frac{1}{2}), (2,2)\}$.

(iii) [12, Example 4.13] $S^\circ(a) \subseteq \{(\mu,\nu) \in \mathbb{F}\times\mathbb{F} : \mu+\nu = 1\}$, with $(\frac{1}{2},\frac{1}{2}) \in S^\circ(a)$, and with $S^\dagger(a) \neq \emptyset$. Set

$$A = \mathbb{F}a + \mathbb{F}b_0 + \sum_{(\mu,\nu)\in S^\circ(a)} \mathbb{F}b_{\mu,\nu}$$

with $b_0 \neq 0$, and

$$b = a + b_0 + \textstyle\sum_{(\mu,\nu)\in S^\circ} b_{\mu,\nu}, \quad b^2_{\mu,\nu} = 0, \forall (\mu,\nu) \in S^\dagger(a), \quad b^2_{\frac{1}{2},\frac{1}{2}} = b_0,$$

$$b_{\mu,\nu}b_{\mu',\nu'} = 0, \text{ for } (\mu,\nu) \neq (\mu',\nu'),$$

$$b_0^2 = 0, \quad b_0 b_{\mu,\nu} = b_{\mu,\nu}b_0 = 0, \ \forall (\mu,\nu) \in S^\circ(a).$$

A is a PAJ of type $\frac{1}{2}$. Note that $\mathbb{F}a + \mathbb{F}b_0 + \mathbb{F}b_{\frac{1}{2},\frac{1}{2}} = B(\frac{1}{2}, 1)$, in the notation of [4]. Also, $b_0 A = A b_0 = 0$.

(iv) [12, Example 3.12(2)] $S^\circ(a) \subseteq \{(\mu,\nu) \in \mathbb{F}\times\mathbb{F} : \mu+\nu = 1\}$, $A = \mathbb{F}a + \sum_{(\mu,\nu)\in S} \mathbb{F}b_{\mu,\nu}$, and $S^\dagger(a) \neq \emptyset$, where $b = a + \sum_{(\mu,\nu)\in S^\circ(a)} b_{\mu,\nu}$, and $b^2_{\mu,\nu} = 0$ for each $(\mu,\nu) \in S^\circ(a)$. $Z := \sum \mathbb{F}b_{\mu,\nu}$ is a square zero ideal in A.

When $|S^\circ(a)| = 1$ we have [12, Example 2.2] in the case of a 2-generated algebra.

(v) The "S-exceptional algebra," [12, Example 3.18] Suppose $S^\circ(a) \subseteq \{(\mu,\nu) \in \mathbb{F}\times\mathbb{F} : \mu+\nu = 1\}$ with $S^\dagger(a) \neq \emptyset$, and let $A = \mathbb{F}a + \mathbb{F}b_0 + \sum_{(\mu,\nu)\in S^\circ(a)} \mathbb{F}b_{\mu,\nu}$, with $b_0 \neq 0$. We define multiplication on A according to the following rules, with the sums taken over all $(\mu,\nu) \in S(a)$, where $b = b_0 + \sum_{(\mu,\nu)\in S(a)} b_{\mu,\nu}$:

$$ab_{\mu,\nu} = \mu b_{\mu,\nu} = b_{\mu,\nu}b, \qquad b_{\mu,\nu}a = \nu b_{\mu,\nu} = b b_{\mu,\nu}, \qquad b_0^2 = b_0, \tag{1.5}$$

$$b_{\mu,\nu}b_{\mu',\nu'} = 0, \quad \forall (\mu,\nu), (\mu',\nu') \in S(a). \tag{1.6}$$

When $(\frac{1}{2},\frac{1}{2}) \in S(a)$, A is of Jordan type $\frac{1}{2}$; otherwise A is of Jordan type $\emptyset$.

(vi) [12, Example 4.12] Let $T \subseteq \{(\mu,\nu) \in \mathbb{F}\times\mathbb{F} : \mu+\nu=1\}$, and let $S^\circ(a) = T \cup \{(2,2)\}$, with $S^\dagger(a) \neq \emptyset$. Set

$$A = \mathbb{F}a + \mathbb{F}b_0 + \mathbb{F}b_{2,2} + \sum_{(\mu,\nu)\in T} \mathbb{F}b_{\mu,\nu},$$

with $b_0 \neq 0$, and

$$b = a + b_0 + \textstyle\sum_{(\mu,\nu)\in S^\circ} b_{\mu,\nu}, \quad b_{\mu,\nu}^2 = 0, \forall (\mu,\nu) \in T, \quad b_{2,2}^2 = -\tfrac{1}{2}b_0.$$

$$b_{\mu,\nu}b_{\mu',\nu'} = 0, \text{ for } (\mu,\nu) \neq (\mu',\nu').$$

$$b_0^2 = \tfrac{3}{2}b_0, \quad b_0b_{2,2} = b_{2,2}b_0 = -\tfrac{3}{2}b_{2,2}, \quad b_0b_{\mu,\nu} = b_{\mu,\nu}b_0 = 0, \ \forall (\mu,\nu) \in T.$$

Clearly $bb_{\mu,\nu} = \mu b_{\mu,\nu}$, and $b_{\mu,\nu}b = \nu b_{\mu,\nu}$, $\forall (\mu,\nu) \in T$.

a and b act symmetrically; both a and b are of Jordan type 2.

(vii) [12, Example 4.9] $A = \mathbb{F}a + \mathbb{F}b_0 + \sum_{(\mu,\nu)\in S^\circ(a)} \mathbb{F}b_{\mu,\nu}$, with $\mu+\nu=1$, for all $(\mu,\nu) \in S^\circ(a)$, and $b_0 \neq 0$. Let

$$b = a + b_0 + \sum_{(\mu,\nu)\in S^\circ} b_{\mu,\nu}.$$

$$b_{\mu,\mu}b_{\mu',\nu'} = 0, \quad \text{for all } (\mu,\nu),(\mu',\nu') \in S^\circ.$$

$$b_0^2 = b_0, \quad b_0b_{\mu,\nu} = b_{\mu,\nu}b_0 = 0, \ \forall (\mu,\nu) \in S^\circ.$$

b is an $\tilde{S}$-axis. b is not weakly primitive since $\dim A_1(b) = 2$, and b is of Jordan type 1. We get the commutative [12, Example 4.1] (which is not an HRS algebra) when $S^\circ(a) = \{(\frac{1}{2},\frac{1}{2})\}$.

(viii) [12, Example 4.2] Let $A = \mathbb{F}a + \mathbb{F}b_0 + \mathbb{F}b_{\mu,\mu}$, $b = \frac{1}{2}a + b_0 + b_{\mu,\mu}$,

$$b_{\mu,\mu}^2 = \tfrac{1}{4\mu}a, \quad b_0^2 = b_0 + \tfrac{\mu-1}{4\mu}a, \quad b_0b_{\mu,\mu} = \tfrac{1-\mu}{2}b_{\mu,\mu}.$$

$\dim A_0(b) = 1$ and $\dim A_1(b) = 2$, so again b is not weakly primitive but is of Jordan type 1. This is not an HRS algebra, so is not a PAJ in the sense of [11].

(ix) [12, Example 4.3] $A = \mathbb{F}a + \mathbb{F}b_0 + \mathbb{F}b_{\mu,\mu} + \mathbb{F}b_{\nu,\nu}$, where $\mu,\nu \in \mathbb{F}\setminus\{0,1\}$, $\mu \neq \nu$, so A is 4-dimensional. A is commutative, and multiplication in A is defined as follows.

$$a^2 = a, \quad ab_0 = 0, \quad ab_{\mu,\mu} = \mu b_{\mu,\mu}, \quad ab_{\nu,\nu} = \nu b_{\nu,\nu}.$$

To define $b^2_{\mu,\mu}$ and $b^2_{\nu,\nu}$ we use the following equation:

$$b^2_{\mu,\mu} + b^2_{\nu,\nu} = \tfrac{1}{4}a + \tfrac{1}{2}b_0, \quad \mu b^2_{\mu,\mu} + \nu b^2_{\nu,\nu} = \tfrac{1}{4}a.$$

We also let

$$b_0^2 = \tfrac{1}{2}b_0, \quad b_0 b_{\mu,\mu} = \tfrac{1-\mu}{2}b_{\mu,\mu}, \quad b_0 b_{\nu,\nu} = \tfrac{1-\nu}{2}b_{\nu,\nu}, \quad b_{\mu,\mu}b_{\nu,\nu} = 0.$$

$\dim A_0(b) = \dim A_1(b) = 2$, so b is not weakly primitive but is of Jordan type $\{0, 1\}$.

These are the only possibilities, by the main theorems A and B of [12].

Remark 1.5. In conclusion, in all of Examples 1.4,

(i) a is of Jordan type (i.e., (1.4) holds) in all examples except (viii), in which case b is not weakly primitive.

(ii) $\mu + \nu = 1$ and $A_{\mu,\nu}(a)^2 = 0$ for all $(\mu, \nu) \in S^\dagger(a)$.

(iii) By symmetry, and in view of (i), when b is weakly primitive, then b is of Jordan type.

(iv) $S^\dagger(b) = \tilde{S}^\dagger(a)$ or $S^\dagger(b) = S^\dagger(a)$.

(v) There are three commutative examples in which $\dim A_1(b) = 2$, namely (vii), and (viii), and (ix).

Remark 1.6. If a is a special axis, and b is an axis that commutes with a then $A = \langle\langle a, b\rangle\rangle$ is commutative, by [12, Lemma 3.7(ii)].

Corollary 1.7. *The only HRS algebras which extend to weak PAJ's which are not commutative are* $\mathbb{F}$, $\mathbb{F} \times \mathbb{F}$, $B(\frac{1}{2}, 1)$ *and its* 2*-dimensional quotient, and* $B(2, 1)$.

Remark 1.8. The algebras of Examples 1.4 all are not isomorphic one to the other. Indeed, the 2-dimensional algebras are all HRS or as in (iv) with $|S^\circ| = 1$. We may eliminate the commutative 3-dimensional algebras, since the only commutative non-HRS algebras are as in (vi) or (viii). The noncommutative algebras (iv), (v), and (vi) are clearly different from each other since $S^{\mathrm{comm}}(a)$ differ, and these algebras are different from the rest of the list. One axis of (vii) is not weakly primitive, differentiating it from (iv) and (v). (viii) is 3-dimensional commutative but not HRS, since one of its axes is not primitive.

This leaves (ii) and (ix). We show that for $\mu = \frac{1}{2}$ and $\nu = 2$, these examples are distinct. Towards this end, suppose on the contrary that they

are the same. Then change the name of b of example (ix) to c. Let A be the algebra of example (ix). A contains the axes a, b, c. Since $\dim A_\lambda(a) = 1$, for $\lambda \in \{1, 0\}$ and $\dim A_{\eta,\eta}(a) = 1$, for $\eta \in \{\frac{1}{2}, 2\}$, we must have $\mathbb{F}b_0 = \mathbb{F}c_0$ and $\mathbb{F}b_{\frac{1}{2},\frac{1}{2}} = \mathbb{F}c_{\frac{1}{2},\frac{1}{2}}$. However $b_0 b_{\frac{1}{2},\frac{1}{2}} = 0$ (by example (ii)) whereas $b_0 c_{\frac{1}{2},\frac{1}{2}} = \frac{1-\frac{1}{2}}{2} c_{\frac{1}{2},\frac{1}{2}}$ (by example (ix)); since $c_{\frac{1}{2},\frac{1}{2}} \neq 0$, this is a contradiction.

1.2. Miyamoto involutions. We modify the standard method of generating new axes. Let $a \in A$ be any S-axis. For $S' \subseteq S$, define the *Miyamoto map of A*

$$\tau_{a;S'} : y_1 + y_0 + \sum_{(\mu,\nu)\in S\setminus S'} y_{\mu,\nu} - \sum_{(\mu,\nu)\in S'} y_{\mu,\nu}.$$

Note that $\tau_{a;S'}(y) = y$ iff $y_{\mu,\nu} = 0$ for all $(\mu, \nu) \in S'$.

Lemma 1.9. *For any S-axis a and any $y \in A$,*

(i) *$y_0, y_1, y_{\mu,\nu} \in \langle\langle a, y\rangle\rangle$, for each (μ, ν) in S.*

(ii) *If $\tau_{a;\{(\mu,\nu)\}}(b) = b$ for all $(\mu, \nu) \in S$, for a weakly primitive axis a, and an idempotent $b \neq a$, then $\dim\langle\langle a, b\rangle\rangle = 2$.*

Proof. (i) By [12, Lemma 1.7].

(ii) $b = \alpha_b a + b_0$. Thus $ab = \alpha_b a = ba$, proving the assertion. □

In accordance with the literature, a **Miyamoto involution** is a Miyamoto map which is an algebra automorphism.

Lemma 1.10. *Each Miyamoto map associated to an S-axis a satisfying the fusion rules is a Miyamoto involution.*

Proof. $\tau_{a;S'}$ is an automorphism, because of the 2-grading in (1.3). □

When a does not satisfy the fusion rules, $\tau_{a;S'}(b)$ need not be an idempotent, for an axis b!

Write $\mathrm{Aut}(A)$ for the automorphisms of an algebra A. For $\rho \in \mathrm{Aut}(A)$ we write y^ρ for $\rho(y)$, for any $y \in A$.

Lemma 1.11. *Let a be a weakly primitive S-axis satisfying the fusion rules, and take $S' \subseteq S$, and $\rho \in \mathrm{Aut}(A)$. Then*

(i) *The axis a^ρ has the same Jordan type S as a; a^ρ is special if and only if a is special.*

(ii) $\tau_{a^\rho;S'} = \tau^\rho_{a;S'} := \rho^{-1}\tau_{a;S'}\rho$.

P r o o f. (i) Note for $(\mu,\nu) \in S$, that

$$(*) \qquad x \in A_{\mu,\nu}(a^\rho) \iff x^{\rho^{-1}} \in A_{\mu,\nu}(a).$$

(ii) Let $\tau' = \tau_{a^\rho;S'}$ and $\tau = \tau_{a;S'}$. We show that $x^{\tau'} = \varepsilon x \implies x^{\tau^\rho} = \varepsilon x$, for $\varepsilon \in \{1,-1\}$. Clearly this implies $\tau' = \tau^\rho$.

Let $x \in A_1(a^\rho) \cup A_0(a^\rho) \cup A_{S\setminus S'}(a^\rho)$, (so that $x^{\tau'} = x$). Then by $(*)$, $x^{\rho^{-1}\tau} = x^{\rho^{-1}}$, so $x^{\tau^\rho} = x$. Also if $x \in A_{S'}(a^\rho)$, (i.e., $x^{\tau'} = -x$), then, by $(*)$, $x^{\rho^{-1}} \in A_{S'}(a)$, so $x^{\rho^{-1}\tau} = -x^{\rho^{-1}}$, so $x^{\tau^\rho} = x^{\rho^{-1}\tau\rho} = -x^{\rho^{-1}\rho} = -x$. □

Example 1.12. We make the following observations on Example 1.4(ii), our example of a 4-dimensional CPAJ,

(i) $\mathbb{F}b_{\frac{1}{2},\frac{1}{2}}$ is a square-zero ideal of A.

(ii) $A = \langle\langle b, b^{\tau_{a;S}}\rangle\rangle$ where $S = \{\frac{1}{2}, 2\}$, unless char $\mathbb{F} = 3$. Indeed let $B = \langle\langle b, b^{\tau_{a;S}}\rangle\rangle$. Then $b_{2,2} + b_{\frac{1}{2},\frac{1}{2}} \in B$, as is $b_0 + a$, so $-\frac{1}{2}b_0 = (b_{2,2} + b_{\frac{1}{2},\frac{1}{2}})^2 \in B$, implying $b_0 \in B$, and $-\frac{3}{2}b_{2,2} = b_0(b_{2,2} + b_{\frac{1}{2},\frac{1}{2}}) \in B$, implying $B = A$.

(iii) When char $\mathbb{F} = 3$, $\mathbb{F}b_0$ is an annihilating ideal of A, and the image $(\mathbb{F}\bar{b}_{2,2} + \mathbb{F}\bar{b}_{\frac{1}{2},\frac{1}{2}})/\mathbb{F}b_0$ is a square-zero ideal of $A/\mathbb{F}b_0$.

Given a set of axes X satisfying the fusion rules, define $\mathcal{G}(X)$ to be the group of automorphisms generated by all Miyamoto involutions associated with the axes in X, and $\mathrm{Cl}(X) = \{\tau(x) : x \in X,\ \tau \in \mathcal{G}(X)\}$.

Theorem 1.13 (generalizing [11, Theorem 2.3])**.** *Suppose A is an axial algebra generated by a set X of weakly primitive axes satisfying the fusion rules.*

(i) *A is spanned by the set* $\mathrm{Cl}(X)$.

(ii) *If $W \subseteq A$ is a subspace containing X such that $xv \in W$ and $vx \in W$ for all $v \in W$ and $x \in X$, then $W = A$.*

(iii) *Let $a \in A$ be an S-axis. Then for any $(\mu,\nu) \in S$ we have*

$$A_{\mu,\nu}(a) = \sum\{\mathbb{F}b_{\mu,\nu} : b \in \mathrm{Cl}(X)\}.$$

For example,

$$A_0(a) = \sum\{\mathbb{F}b_0 : b \in \mathrm{Cl}(X)\}.$$

$A^2_{\mu,\nu} = \sum\{\mathbb{F}b_{\mu,\nu}c_{\mu,\nu} : b, c \in \mathrm{Cl}(X)\}$.

P r o o f. The proof is exactly as in the proof of [11, Theorem 2.3]. For example, we get (i) by noting that $ab = \alpha_b a + \sum_{(\mu,\nu)\in S(a)} \frac{\mu}{2}(b - \tau_{a;\mu,\nu}(b))$ for any $a, b \in \mathrm{Cl}(X)$. Then repeat the remainder of the proof of [11, Theorem 2.3], using (μ, ν) instead of (λ, δ). □

Corollary 1.14. *Suppose $A = \langle\langle X\rangle\rangle$, where each $x \in X$ is an axis satisfying $S^\dagger(x) = \emptyset$. Then A is commutative.*

P r o o f. Each axis $x \in \mathrm{Cl}(X)$ satisfies $S^\dagger(x) = \emptyset$, so $xy = yx$, for $x, y \in \mathrm{Cl}(X)$. Since $\mathrm{Cl}(X)$ spans A, A is commutative. □

2. The existence of a Frobenius form.

Definition 2.1. *A Frobenius form on an algebra is a symmetric bilinear form which is associative, i.e., satisfying $(xy, z) = (x, yz)$ for all x, y, z.*[2]

Lemma 2.2. *Let A be an algebra with a Frobenius form, and let $z \in A$. Denote by $\mathfrak{R}$ the radical of the form. Then*

(i) ([13, Lemma 2.1]) *If $u \in A_{\mu,\nu}(z)$ and $v \in A_{\mu',\nu'}(z)$, then $(u, v) = 0$, unless perhaps $\mu = \nu'$, and $\mu' = \nu$.*

(ii) *If $A_{\mu,\nu}(z)A_{\nu,\mu}(z) = 0$, then $(A_{\mu,\nu}(z), A_{\nu,\mu}(z)) = 0$, unless perhaps $\mu = \nu = 0$.*

(iii) *If $A_{\mu,\nu}(z)^2 \subseteq \mathbb{F}z + A_0(z)$, $(z, z) \neq 0$, and $\mu \neq \nu$, or $(\mu, \nu) = (0, 0)$, then $A_{\mu,\nu}(z)^2 \subseteq A_0(z)$.*

(iv) *If z is an axis, and $A_{\mu,\nu}(z)A_{\nu,\mu}(z) = 0$, for each $(\mu, \nu) \in S^\dagger(z)$, then $\sum_{(\mu,\nu)\in S^\dagger(z)} A_{\mu,\nu} \subseteq \mathfrak{R}$.*

(v) *If A is spanned by idempotents and $Az = 0$, then $z \in \mathfrak{R}$.*

P r o o f. (i) $\mu(u, v) = (zu, v) = (v, zu) = (vz, u) = \nu'(u, v)$ so $(u, v) = 0$, unless $\mu = \nu'$. By symmetry $(u, v) = 0$, unless $\mu' = \nu$.

(ii) Let $u \in A_{\mu,\nu}(z)$ and $v \in A_{\nu,\mu}(z)$. Then $\mu(u, v) = (zu, v) = (z, uv) = 0$, so if $(u, v) \neq 0$, $\mu = 0$. By symmetry if $(u, v) \neq 0$, then $\nu = 0$.

(iii) Let $u, v \in A_{\mu,\nu}(z)$, then $(z, uv) = \mu(u, v) = 0$, by (1), so $uv \in A_0(z)$.

(iv) This follows at once from (i) and (ii).

(v) For any idempotent b we have $(b, z) = (b^2, z) = (b, bz) = (b, 0) = 0$. □

[2] Any associative bilinear form on a commutative axial algebra is symmetric, and thus a Frobenius form, as seen in [3, Proposition 3.5].

Thus, (1.4) is a necessary condition for a Frobenius form. But it superfluous in 2-generated PAJ's, as seen in Remark 1.5, and in type η as seen in [9, Theorem 3.2]. This leads us to ask whether an algebra generated by special axes is necessarily a PAJ? We shall see this is often the case in Corollary 2.7.

The purpose of this section is to prove the following theorem (extending [6, Theorem 4.1, p. 407]), recalling φ_a from Definition 1.1(11):

Theorem 2.3. *Suppose the weak PAJ A is generated by a homogeneous set X of special axes.*

Then A admits a Frobenius form, $(\ ,\ )$, which satisfies $(a, y) = \varphi_a(y)$, and hence $(a, a) = 1$, for all axes $a \in \mathrm{Cl}(X)$ and $y \in A$. Furthermore $(xy, z) = (yx, z)$, for $x, y, z \in A$.

To prove the theorem, we shall see that it is enough to obtain a set of axes $\widehat{X}$ spanning A, for which $\varphi_b(c) = \varphi_c(b)$ for all $b, c \in \widehat{X}$. Indeed, by Theorem 1.13(i), and Lemma 2.4 below, $\widehat{X} = \mathrm{Cl}(X)$ satisfies this property.

Proposition 2.4. *$\varphi_a(b) = \varphi_b(a)$ for any special S-axes $a, b \in A$, under any of the following conditions:*

(i) *$ab = ba = 0$, in which case $\varphi_a(b) = \varphi_b(a) = 0$;*

(ii) *a and b do not commute;*

(iii) *$\langle\langle a, b\rangle\rangle$ is commutative, of dimension 3, and a, b are of the same type.*

(iv) *$\langle\langle a, b\rangle\rangle$ is commutative, of dimension 4, and a, b are of the same type.*

P r o o f. (i) The projections are both 0.

(ii) The only relevant cases are Example 1.4(iii),(iv),(vi) in which $\varphi_a(b) = \varphi_b(a) = 1$, and Example 1.4(v) in which $\varphi_a(b) = \varphi_b(a) = 0$.

(iii) This appears in [4].

(iv) By Example 1.4(ii),(vi), by symmetry of the presentation. □

The other algebras of Example 1.4 are not relevant, since the axis b is not weakly primitive.

P r o o f o f T h e o r e m 2.3. First note that by Theorem 1.13(i) and by Lemma 2.4 for $b, c \in \mathrm{Cl}(X)$, $\varphi_b(c) = \varphi_c(b)$. As in the proof of [11, Theorem 6.3], we start by defining a symmetric bilinear form $(\cdot\,, \cdot)$ on a base $\mathcal{B} \subseteq \mathrm{Cl}(X)$. We put $(a, b) = \varphi_a(b)$, for all $a, b \in \mathcal{B}$, which we extend by linearity to A to get the symmetric bilinear form $(\cdot\,, \cdot)$. We proceed by a sequence of steps.

(1) $(a\,,\,y) = \varphi_a(y)$, for a given axis $a \in \mathcal{B}$ and all $y \in A$. Indeed, for $y = \sum_{b\in\mathcal{B}} \alpha_b b$, since φ_a is linear,

$$(*) \qquad \varphi_a(y) = \varphi_a\left(\sum_{b\in\mathcal{B}} \alpha_b b\right) = \sum_{b\in\mathcal{B}} \alpha_b(a,b) = (a, \sum_{b\in\mathcal{B}} \alpha_b b) = (a,y).$$

Let now $a \in A$ be a special axis having the same type as the axes in X. Then $\varphi_a(b) = \varphi_b(a) = (a,b)$, for all $b \in \mathcal{B}$. Thus, as in $(*)$ we get,

1. $(a\,,\,y) = \varphi_a(y)$, for every special axis $a \in A$ having the same type as the axes in X, and all $y \in A$; in particular $(a\,,\,a) = 1$.

2. $(\cdot\,,\,\cdot)$ is invariant under any automorphism of A. In particular $(\cdot\,,\,\cdot)$ is invariant under any automorphism in $\mathcal{G}(X)$.

Let $\psi \in \mathrm{Aut}(A)$, and $a \in \mathrm{Cl}(X)$. If

$$u = \varphi_a(u)a + u_0 + \sum_{(\mu,\nu)\in S(a)} u_{\mu,\nu}$$

is the decomposition of $u \in A$ with respect to a, then the decomposition of u^ψ with respect to the primitive axis a^ψ is $u^\psi = \varphi_a(u)a^\psi + u_0^\psi + \sum_{(\mu,\nu)\in S(a)} u_{\mu,\nu}^\psi$. Hence $\varphi_{a^\psi}(u^\psi) = \varphi_a(u)$, and so, by (2), $(a^\psi, u^\psi) = (a,u)$. Hence by linearity, also $(v,u) = (v^\psi, u^\psi)$, for all $v,u \in A$. Before concluding, we need a lemma inspired by Lemma 2.2(i).

Lemma 2.5. *For every $a \in \mathrm{Cl}(X)$, the different eigenspaces of a are orthogonal with respect to $(\cdot\,,\,\cdot)$.*

P r o o f. Clearly, if $u \in A_0(a) + A_S(a)$, then $(a,u) = \varphi_a(u) = 0$. Hence $A_1(a) = \mathbb{F}a$ is orthogonal to both $A_0(a)$ and to $A_S(a)$. Assume $(0,0) \neq (\mu,\nu) \neq (\mu',\nu')$. If $u \in A_{\mu,\nu}(a)$ and $v \in A_{\mu',\nu'}(a)$, take $\tau := \tau_{a,S(a)\setminus\{(\mu,\nu)\}}$. Then, by (3) above, $(u,v) = (u^\tau, v^\tau) = (-u,v)$, so $(u,v) = 0$. □

Now, as in [11], it is easy to show associativity of the form by checking eigenspaces. Let $a \in \mathrm{Cl}(X)$. First we note that $(ay,z) = (y,az)$, for all $y,z \in A$. Indeed, this is obvious if y,z are in the same component with respect to a, and if $y \in A_{\mu,\nu}(a)$ and $z \in A_{\mu',\nu'}(a)$ for $(\mu,\nu) \neq (\mu',\nu')$, then both sides are in $(A_{\mu,\nu}(a), A_{\mu',\nu'}(a)) = 0$. Thus by linearity $(ay,z) = (y,az)$, for all $y,z \in A$. By linearity again, $(xy,z) = (y,xz)$, for all $x,y,z \in A$.

Similarly $(yx,z) = (y,zx)$. Now $(xy,z) = (y,xz) = (xz,y) = (x,yz)$. □

Remark 2.6.

(i) For any special axis a, $(a,a) = 1$ is equivalent to $(a,y) = \varphi_a(y)$, for all $y \in A$, by Lemma 2.2(i).

This implies that the Frobenius form in Theorem 2.3 is uniquely defined under the condition that $(a,a) = 1$, for all $a \in \mathrm{Cl}(X)$. Indeed, if $y = \sum \gamma_i a_i$ then

$$(y,z) = \sum_i (\gamma_i a_i, z) = \sum_i \gamma_i (a_i, z) = \sum \gamma_i \varphi_{a_i}(z).$$

(ii) One might wonder whether the condition of homogeneity is needed. We know that all 2-generated weak PAJ's of Example 1.4 are homogeneous, but we do not know how to prove the theorem without a homogeneous generating set of axes, and with 3-generated algebras we do not know how to modify the axes to get a homogeneous set.

Corollary 2.7. *Under the hypotheses of Theorem 2.3, a special axis a satisfying $(a,a) \neq 0$ is of Jordan type.*

P r o o f. By Lemma 2.2(iii). □

3. Applications to noncommutative PAJ's. In view of Lemma 2.2(i),(ii), $\sum_{(\mu,\nu)\in S^\dagger} A_{\mu,\nu}(a)$ is in the radical of the Frobenius form (if such a form exists).

Remark 3.1. Let $A = \langle\langle X \rangle\rangle$, where X is a set of axes satisfying the fusion rules, such that A possesses a Frobenius form and $a, b \in X$. Let

$$Z_a := \sum_{(\mu,\nu)\in S^\dagger(a)} A_{\mu,\nu}(a).$$

(i) $Z_a \subseteq \mathfrak{R}$, by Lemma 2.2(iv);

(ii) If $A = \langle\langle a,b \rangle\rangle$ with a, b special, then $Z_a = Z_b$ and $Z_a^2 = 0$, as seen by looking at the list in Example 1.4. Hence Z_a is an ideal of A, and A/Z_a is commutative.

(iii) $Z_a^2 a = 0$, by Lemma 2.2(iii). We noted in Example 1.4(iii) that one need not have $Az = 0$ implying $z = 0$.

Let us address the following question:

If X is a set of axes satisfying the fusion rules, is $Z_a^2 = 0$?

Lemma 3.2. *To prove $Z_a^2 = 0$ for axial algebras generated by an arbitrary number of special axes, it is enough to verify the assertion for algebras generated by three special axes.*

Proof. In view of Theorem 1.13, it suffices to prove $b_{\mu,\nu}c_{\mu,\nu} = 0$ for axes b, c. □

3.1. Noncommutative examples. We conclude with some illustrative noncommutative examples which are not 2-generated, generalizing the "uniform" and "exceptional" PAJ's of [12].

Example 3.3 ([12, Example 2.2])**.** Suppose $\mu_{i,j} + \nu_{i,j} = 1$, for $i, j \in I$. Define the algebra generated by a set of idempotents $E = \{x_i : i \in I\}$, satisfying $x_i x_j = \nu_{i,j} x_i + \mu_{i,j} x_j$ for all $i \neq j \in I$. All axes are special of Jordan type $\emptyset$.

When all the $\mu_{i,j}$ are equal, and all the $\nu_{i,j}$ are equal, we get the "uniform algebra" of [12, Example 2.2].

For the next two examples, take sets $S_{i,j}^\dagger \neq \emptyset$ comprised of pairs $(\mu_{i,j}, \nu_{i,j}) : i \in I$ for which $\mu_{i,j} + \nu_{i,j} = 1$, and $S^\dagger = \cup_{i,j} S_{i,j}^\dagger$. (To avoid complications in notation, one strictly should take $\mu_{i,j}, \nu_{i,j} \neq \frac{1}{2}$, although the constructions also work in that case.) Let $X = \{x_i : i \in I\}$ and $Y = \{y_i : i \in I\}$.

Example 3.4 (Generalizing [12, Example 3.18])**.** Define

$$A = \sum_{k \in I} \mathbb{F} x_k + \sum \mathbb{F} y_k + \sum_{i,j} \sum_{(\mu_i, \nu_j) \in S^\dagger} \mathbb{F} z_{\mu_i, \nu_j},$$

where we define multiplication in A according to the following rules, with the sums taken over all the $(\mu_i, \nu_j) \in S^\dagger$:

$$x_i y_j = \sum \mu_{i,j} z_{\mu_{i,j}, \nu_{i,j}}, \qquad y_i x_j = \sum \nu_{i,j} z_{\mu_{i,j}, \nu_{i,j}}, \tag{3.1}$$

$$x_i z_{\mu_{i,j}, \nu_{i,j}} = \mu_{i,j} z_{\mu_{i,j}, \nu_{i,j}} = z_{\mu_{i,j}, \nu_{i,j}} y_j, \qquad z_{\mu_{i,j}, \nu_{i,j}} x_i = \nu_{i,j} z_{\mu_{i,j}, \nu_{i,j}} = y_j z_{\mu_{i,j}, \nu_{i,j}}, \tag{3.2}$$

$$z_{\mu_{i,j}, \nu_{i,j}} z_{\mu_{i',j'}, \nu_{i',j'}} = 0, \quad \forall i, i', j, j' \in I. \tag{3.3}$$

Each axis x_i of X is special of Jordan type 0. By symmetry, each y_i is special of Jordan type 0. We specialize to a more manageable example when all the $\mu_{i,j}$ are the same and all the $\nu_{i,j}$ are the same. We specialize further to [12, Example 3.21] when $|S^\dagger| = 1$.

Example 3.5 (Generalizing [12, Example 4.12])**.** Let $S^\circ = S^\dagger \cup \{(2,2)\}$. Let $A = \sum \mathbb{F}x_i + \sum \mathbb{F}y_{i,j;2} + \sum_{i,j\in I} \mathbb{F}z_{\mu_{i,j},\nu_{i,j}}$, and we put

$$y_{i,j;0} = y_j + y_{i,j;2} - \sum z_{\mu_{i,j},\nu_{i,j}},$$

and we define multiplication in A according to the following rules, taken over all $(\mu_{i,j},\nu_{i,j}) \in S^\dagger$:

$$z_{\mu_{i,j},\nu_{i,j}} z_{\mu_{i',j'},\nu_{i',j'}} = 0, \quad \forall i,j,i',j' \in I,$$

$$y_{i,j;0}^2 = \tfrac{3}{2} y_{i,j;0}, \quad y_{i,j;0}y_{i,j;2} = y_{i,j;2}y_{i,j;0} = -\tfrac{3}{2} y_{i,j;2},$$

$$y_{i,j;0} b_{\mu,\nu} = b_{\mu,\nu} y_{i,j;0} = 0, \ \forall (\mu,\nu) \in S^\dagger,$$

$$x_i z_{\mu_i,\nu_j} = z_{\mu_i,\nu_j} y_j = \nu b_{\mu,\nu} \qquad z_{\mu_i,\nu_j} x_i = y_j z_{\mu_i,\nu_j} = \mu_j z_{\mu_i,\nu_j}, \ \forall (\mu,\nu_j) \in S_{i,j}^\dagger.$$

The axes x_i and y_j act symmetrically; both x_i and y_j are special of Jordan type 2.

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