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CODIMENSION GROWTH OF FINITE DIMENSIONAL ALGEBRAS

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Abstract. We study polynomial identities of algebras over a field F of characteristic zero. Given an algebra A over F, one can associate the sequence of non-negative integers $\{c_n(A)\}, n = 1, 2, \ldots$, called codimension sequence of A, characterizing the number of its identical relations. We discuss asymptotic behavior of codimension sequence in case where A is a finite dimensional algebra.

1. Introduction. We study polynomial identities of finite dimensional algebras over a field F of characteristic zero. Given an algebra A over F, one can associate the sequence of non-negative integers $\{c_n(A)\}, n = 1, 2, \ldots$, called codimension sequence of A, characterizing the number of its identical relations. If there are no restrictions on the type of algebra then the behavior of $c_n(A)$ is quite diverse. For example, if $A = F\{X\}$ is an absolutely free F-algebra then

$$c_n(A) = \frac{1}{n}\binom{2n-2}{n-1} n! \ .$$

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If $A = F\langle X\rangle$ is free associative algebra then $c_n(A) = n!$. In case where $A = L(X)$ is free Lie algebra one has $c_n(A) = (n-1)!$. Recall that the sequence of non-negative real numbers $\{c_n\}$ has an overexponential growth if

$$\lim_{n\to\infty} \frac{c_n}{a^n} = \infty$$

for any $a > 1$. If $a^n < c_n < b^n \ \forall n = 1, 2, \ldots$, for some $1 < a < b$ then $\{c_n\}$ has an exponential growth. In case when

$$\lim_{n\to\infty} \frac{c_n}{a^n} = 0 \quad \forall a > 1 \quad \text{whereas} \quad \lim_{n\to\infty} \frac{c_n}{n^t} = \infty \ \forall t \in \mathbb{N}$$

we say that the sequence $\{c_n\}$ is of intermediate growth. If $c_n < Cn^t$ for some positive integers C and t then it is polynomially bounded. An example of overexponentially growing sequence gives us $(n!)^{\alpha n}, 0 < \alpha < 1$. All functions of this type can be realized as codimensions sequences of suitable algebras [4]. In [5] series of nonassociative algebras with intermediate codimension growth like $c_n(A) \sim n^{n^\beta}, 0 < \beta < 1$, was constructed. In [2] it was shown in case of polynomial codimension growth that $c_n(A) = \alpha n^q + O(n^{q-1})$, where q is a natural number for associative, Jordan or Lie algebra.

Main goal of present paper is to give short review of asymptotic behavior of codimension growth of finite dimensional algebras. Fundamental role for these purposes play the following results.

Theorem 1 ([1, Propstion 2.3], [6, Proposition 2]). *Let A be a finite dimensional algebra,* $\dim A = d$. *Then*

$$c_n(A) \le d^{n+1}. \tag{1}$$

Theorem 2 ([5, Theorem 2]). *Let A be a finite dimensional algebra,* $\dim A = d$. *Then either the sequnce $\{c_n(A)\}$ is polynomially bounded or*

$$c_n(A) \ge \frac{1}{n^2} 2^{\frac{n}{3d^3}} \tag{2}$$

asymptotically.

In particular, codimension sequences of finite dimensional algebras do not admit overexponential or intermediate growth. All necessary information on PI theory one can find in [7], [3].

2. Preliminaries. Recall now main notions. Let F be a field of characteristic zero and let $F\{X\}$ be a free (non-associative) algebra over F with an infinite set X of free generators. If A is some F-algebra then a polynomial $f = f(x_1, \dots, x_n) \in F\{X\}$ is said to be an identity of A if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in A$. The set $Id(A)$ of all identities of A forms an ideal of $F\{X\}$ stable under all endomorphisms of $F\{X\}$.

If P_n is a subspace of all multilinear polynomials on $x_1, \dots, x_n \in X$ then the intersection $P_n \cap Id(A)$ is in fact the set of all multilinear identities of A of degree n. Denote

$$P_n(A) = \frac{P_n}{P_n \cap Id(A)} \quad \text{and} \quad c_n(A) = \dim P_n(A).$$

The sequence $\{c_n(A)\}, n = 1, 2, \dots$, is called the codimension sequence of A. Codimension sequence is a significant numerical invariant of polynomial identities of A.

If $\{c_n(A)\}$ is exponentially bounded, the lower and the upper limits

$$\underline{\exp}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n(A)}, \ \overline{\exp}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n(A)}$$

exist and called lower and upper PI-exponents, respectively. In case $\underline{\exp}(A) = \overline{\exp}(A) = a$ their common value a is called the (ordinary) PI-exponent of A, $a = \exp(A)$.

An existence and possible values of PI-exponent of concrete algebra is one of most significant problem of numerical PI theory.

3. Associative and Lie algebras, Lie supealgebras. In 1972 A. Regev have proved that codimension sequence is exponentially bounded for any (not only finite dimensional) associative PI-algebra [15]. At the end of 1980th S.Amitsur conjectured that PI-exponent exists an is a non-negative integer for any associative PI-algebra A. This conjecture was confirmed in [8]. Moreover, in [9] it was shown that $\exp(A) = \dim(A)$ if and only if A is simple provided that F is algebraically closed. For finite dimensional Lie algebras case the same result takes place [19].

In Lie superalgebras case the picture is more complicated. In particular an existence of PI-exponent of a finite dimensional superalgebra is unknown. Nevertheless, intermediate codimension growth is also impossible.

Theorem 3 ([20], [23])**.** *For any Lie superalgebra L, the codimension sequence $\{c_n(L)\}$ is either polynomially bounded or $c_n(L) \gtrsim \sqrt{2}^n$, asymptotically. If $\dim L < \infty$, then either $c_n(L) < Cn^t$ or $\overline{\exp}(L) \geq 2$.*

Unlike associative and Lie algebras Lie superalgebras admit fractional exponentially codimension growth. Recall the description of the algebras of type $b(t)$ (see [16] for details). Let $M_t(F)$ be the algebra of $t \times t$ matrices over F and let T be transpose involution. Then L is of type $b(t)$ if L is the Lie superalgebra of $2t \times 2t$ matrices over F of the type

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix},$$

where $A, B, C \in M_t(F)$, $B^T = B, C^T = -C$ and $trA = 0$. We write $L = L_0 \oplus L_1$ where

$$L_0 = \{\begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \mid A \in M_t(F), tr(A) = 0\},$$

and

$$L_1 = \{\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B^T = B, C^T = -C \in M_t(F)\}.$$

The elements of $L_0 \cup L_1$ are called homogeneous. Any $x \in L_0$ (or $x \in L_1$) has homogeneous degree $|x| = 0$ ($|x| = 1$, resp.). The algebra L is a Lie superalgebra with $\mathbb{Z}_2$-grading $L = L_0 \oplus L_1$ if we define a product $[\ ,\ ]$ of two homogeneous elements $x, y \in L$ as

$$[x, y] = xy - (-1)^{|x||y|}yx.$$

The algebra L is simple not only as a superalgebra but also as an algebra for any $t \geq 3$ and $\dim L = 2t^2 - 1$. We have

Theorem 4 ([10]). *Let L be a finite dimensional Lie superalgebra of type $b(t)$, $t \geq 3$. Then the PI-exponent of L exists and* $\exp(L) < 2t^2 - 1 = \dim L$.

It is unknown if for the algebras L of type $b(t)$ the inequality $2t^2 - 2 < \exp(L)$ holds (thought it seems very reasonable).

When $t = 2$, the Lie superalgebra L of type $b(2)$ is not simple, but the previous theorem can be improved in this case.

Theorem 5 ([10]). *Let L be a Lie superalgebra of type $b(2)$. Then the PI-exponent of L exists and*

$$6 < \exp(L) < 7 = \dim L.$$

Actually it was shown that $6.26 < \exp(L) < 6.47$.

On the other hand there are sufficiently many examples of positive solutions of Amitsur's problem.

An alternative to simple superalgebras are solvable ones. In this case there is the following result.

Theorem 6 ([23]). *Let L be a Lie superalgebra with nilpotent commutator subalgebra. If $[L,L]^{c+1}=0$, then the PI-exponent of L exists, is an integer and does not exceed $2c$. Moreover, for any integer $1\leq t\leq 2c$ there exists a Lie superalgebra L such that $[L,L]^{c+1}=0$ and $\exp(L)=t$.*

If L is an ordinary Lie algebra with $\dim L<\infty$ then solvability of L implies nilpotency of L^2 automatically, But in general super-case this is not true. So, the natural question arises: does PI-exponent of a finite dimensional solvable Lie superalgebra exist and is it an integer? In [21] new series of finite dimensional solvable Lie superalgebras with non-nilpotent commutator subalgebra were defined.

We use the following agreements. If A is a Lie superalgebra then we denote the product of elements of A by an ordinary commutator bracket $[x,y]$. If A is an associative algebra then $[x,y]=xy-yx$. If $A=A_0\oplus A_1$ is an associative algebra with $\mathbb{Z}_2$-grading and x and y are homogeneous elements from A then

$$[x,y]=xy-(-1)^{|x||y|}yx,$$

where $|x|$ is the homogeneous degree of x, that is, 0 or 1. We agree to omit the brackets in the case of left-normed arrangement, that is, $[x_1,\dots,x_{k+1}]=[[x_1,\dots,x_k],x_{k+1}]$ for all $k\geq 2$.

First, let R be an arbitrary associative algebra with involution $*:R\to R$. Consider the associative algebra $Q=M_2(R)$,

$$Q=\left\{\begin{pmatrix} A & B\\ C & D\end{pmatrix}\mid A,B,C,D\in R\right\},$$

and endow Q by the $\mathbb{Z}_2$-grading $Q=Q_0\oplus Q_1$ by setting

$$Q_0=\left\{\begin{pmatrix} A & 0\\ 0 & D\end{pmatrix}\right\},\quad Q_1=\left\{\begin{pmatrix} 0 & B\\ C & 0\end{pmatrix}\right\}.$$

It is well-known that Q with the product $[\cdot,\cdot]$ is a Lie superalgebra. Given an associative algebra R with involution, denote by R^+ and R^- the subspaces of symmetric and skew elements of R, respectively:

$$R^+=\{x\in R\mid x^*=x\},\ R^-=\{x\in R\mid x^*=-x\}.$$

Then the subspace

$$L=\left\{\begin{pmatrix} x & y\\ z & -x^*\end{pmatrix}\mid x\in R,y\in R^+,z\in R^-\right\},\tag{3}$$

is also Lie superalgebra with the same product as in R where

$$L_0 = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x^* \end{pmatrix} \right\}, \quad L_1 = \left\{ \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} \right\}.$$

Remark 1. For the Lie superalgebra L constructed above the following conditions are equivalent:

1) L is solvable,

2) L_0 is solvable Lie algebra,

3) R is Lie solvable,

4) maximal semisimple subalgebra of R is commutative.

Thus, the proposed construction gives us a wide class of finite-dimensional solvable Lie superalgebras with a non-nilpotent (as a rule) commutator subalgebra. As an example we can take algebra of upper triangular $t \times t$ matrices $R = UT_t(F)$, finite-dimensional incidence algebra or any associative subalgebra in $UT_t(F)$. We restrict ourselves to the case $R = UT_t(F)$.

Recall the description of two involutions on $UT_t(F)$. One of them $\circ : R \to R$ is the reflection along secondary diagonal. That is $e_{ij}^{\circ} = e_{t+1-j,t+1-i}$ (e_{ij} are the matrix units). It is defined for all $t \geq 2$. We will call it *orthogonal*. Another one is defined only for even t. Let $t = 2m$. Set

$$D = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix},$$

where E is the identity $m \times m$ matrix. Then the map $s : R \to R$,

$$X^s = DX^{\circ}D^{-1},$$

is also an involution on R. It is said to be *symplectic*. If we write X as

$$X = \begin{pmatrix} U & V \\ 0 & W \end{pmatrix},$$

then

$$X^s = \begin{pmatrix} W^{\circ} & -V^{\circ} \\ 0 & U^{\circ} \end{pmatrix}.$$

Definition. *Lie superalgebra* $(S(t), *), t \geq 2$, *is algebra* (3), *where* $R = UT_t(F)$ *and* $* = \circ$ *or* s *is the orthogonal or symplectic involution on* R.

For these series of superalgebras we have the following result concerning so-called graded PI-exponent (see [21]).

Theorem 7. *Let $L = (S(t), *)$ be a Lie superalgebra of the type $S(t)$, where $*$ is orthogonal or symplectic involution. Then the graded PI-exponent of L exists and*

- $\exp^{gr}(L) = 2t$ *if t is even;*
- $\exp^{gr}(L) = 2t - 1$ *if t is odd.*

4. Jordan algebras and superalgebras. First note that Jordan case is similar to associative or Lie ones.

Theorem 8 ([11, see Theorem 3])**.** *Let A be a finite-dimensional Jordan algebra over a field of characteristic zero. Then* $\exp(A)$ *exists and is a non-negative integer. Moreover if A is nilpotent, then* $\exp(A) = 0$*. If A is not nilpotent, then either* $\exp(A) \geq 2$ *or* $\exp(A) = 1$*, and $\{c_n(A)\}$ is polynomially bounded.*

Theorem 9 ([11, Corollary 1])**.** *If A is a finite-dimensional simple unitary noncommutative Jordan algebra over an algebraically closed field of characteristic zero, then* $\exp(A)$ *exists and equals* $\dim A$*.*

Jordan super-case is close to Lie super-case. There are examples of Jordan superalgebras with fractional PI-exponent.

We start with the recalling of Jordan superalgebra $H(M_{t|t}, trp)$ (see, for example, [14]). Denote it for shortness by $P(t)$. Then $P(t) = P_0 \oplus P_1$ is a subspace of $2t \times 2t$ matrix algebra $Mat_{2t}(F)$ where

$$P_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^T \end{pmatrix} \mid A \in M_t(F) \right\}, \; T : X \to X^T \quad \text{is the transpose involution}$$

and

$$P_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid \quad B^T = B, \; C^T = -C\,, \quad B, C \in M_t(F) \right\}.$$

Elements from $P_0 \cup P_1$ are called homogeneous. Any $x \in P_0$ (resp. $x \in P_1$) has the homogeneous degree $|x| = 0$ (resp. $|x| = 1$). Algebra $P(t)$ is a Jordan superalgebra with $\mathbb{Z}_2$-grading if we define the product $\{*, *\}$ of two homogeneous elements $x, y \in P(t)$ as

$$\{x, y\} = xy + (-1)^{|x||y|} yx.$$

It is easily seen that $P(t)$ has also a natural $\mathbb{Z}$-grading

$$P(t) = P^{(-1)} \oplus P^{(0)} \oplus P^{(1)},$$

where $P^{(0)} = P_0$,

$$P^{(-1)} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid \ C^T = -C\,, \quad C \in M_t(F) \right\},$$

$$P^{(1)} = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid \ B^T = B\,, \quad B \in M_t(F) \right\}.$$

Theorem 10 ([17, Theorem 1])**.** *The graded PI-exponent of simple Jordan superalgebra $H(M_{t|t}, trp)$ exists and is less or equal to $t^2 + t\sqrt{t^2-1}$. In particular,* $\exp^{gr}(H(M_{t|t}, trp))$ *is strictly less than* $2t^2 = \dim H(M_{t|t}, trp)$.

Taking into account the inequality

$$c_n(A) \leq c_n^{gr}(A),$$

(see [1]) and results of [4] we get an upper bound for the ordinary PI-exponent.

Theorem 11 ([17, Theorem 2])**.** *The ordinary PI-exponent of the algebra $H(M_{t|t}, trp)$ exists and*

$$\exp(H(M_{t|t}, trp)) \leq t^2 + t\sqrt{t^2-1}.$$

In particular, $\exp(H(M_{t|t}, trp))$ *is strictly less than* $2t^2 = \dim H(M_{t|t}, trp)$.

If $t = 2$ then we have the following.

Theorem 12 ([17, Theorem 3])**.** *Let P be a simple Jordan algebra of the type $H(M_{2|2}, trp)$. Then the graded PI-exponent* $\exp^{gr}(P)$ *has a fractional value. Moreover,*

$$\exp^{gr}(P) = 4 + 2\sqrt{3}.$$

Theorem 13 ([17, Theorem 4])**.** *PI-exponent of simple Jordan superalgebra $H(M_{2|2}, trp)$ is strictly less than the dimension of $H(M_{2|2}, trp)$. Moreover,* $\exp(H(M_{2|2}, trp))$ *is not an integer and*

$$7.19 < 10 \cdot 3^{-\frac{3}{10}} \leq \exp(H(M_{2|2}, trp)) \leq 4 + 2\sqrt{3} < 7.47\ .$$

5. General nonassociative case. As it was mentioned in the Introduction codimension growth of a finite-dimensional algebra is exponentially bounded and cannot be of intermediate type (see Theorem 2). We can clarify the lower bound of exponential growth for unitary algebras.

Theorem 14 ([22]). *Let A be a finite dimensional unitary algebra then either $c_n(A)$ is polynomially bounded or $\underline{\exp}(A) \geq 2$.*

The existence of PI-exponent of an arbitrary finite-dimensional algebra is an open question. Nevertheless for some special classes of algebras solution is positive. For example, if the algebra is simple then the answer is positive.

Theorem 15 ([10, Theorem 3]). *Let A be a finite dimensional simple algebra over a field of characteristic zero. Then $\exp(A) = \lim\limits_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and $\exp(A) \leq \dim A$.*

As it was shown earlier for associative, Lie and Jordan algebras the equality $\exp(A) = \dim A$ holds if and only if A is simple provided that ground field is algebraically closed. In [13] it was shown that the difference between $\dim A$ and $\exp(A)$ may be as big as you want.

Let A_m, $m \geq 3$, be the algebra of dimension m with the basis a_s, $s = -1, 0, \ldots, m-2$. We define the multiplication in such a way that the element a_0 will be the unit that is for any s holds $a_s a_0 = a_0 a_s = a_s$. For $i, j \neq 0$, we set $a_i a_j = a_{i+j}$ provided that $i \geq j$ and $-1 \leq i + j \leq m-2$. In all other cases, that is if $i + j < -1$, or $i + j > m - 2$, or $i < j$ the product $a_i a_j$ is equal to zero. Note that for $m \geq 3$ the algebra A_m is unitary and simple.

Theorem 16 ([13]). *PI-exponents of algebras A_m exist and satisfy inequalities*

$$3 = \exp(A_3) < \cdots < \exp(A_m) < \exp(A_{m+1}) < \cdots < 4\,.$$

On the other hand the equality $\exp(A) = \dim A$ does not guarantee the simplicity of A. Consider the following series of algebras.

Given an integer $m \geq 2$, denote by B_m the linear space with the basis $b, a_1, \ldots, a_m$. Multiplication on B_m is defined as

$$b^2 = b,\ ba_1 = a_1,\ a_1^2 = a_1,\ a_i a_{i+1} = a_{i+1}\ \forall\ 1 \leq i \leq m-2,\ a_{m-1}a_m = a_1.$$

if m is odd. If m is even then also set $ba_2 = a_1$. All remaining products assumed to be zero. Note that $\dim B_m = m + 1$ and B_m is not simple. In particular, subspace $Span\{a_1, \ldots, a_m\}$ is two-sided ideal.

Theorem 17. *[18] For any $m \geq 3$ the following equalities hold*

$$\exp(B_m) = m + 1 = \dim B_m.$$

Possible values of PI-exponents of finite dimensional algebras give us the following result.

Theorem 18 ([12, Corollary 7.2]). *For any real $1 < \alpha < \beta$ there exists a finite-dimensional algebra A such that $\alpha < \exp(A) < \beta$. In particular, the set of real points*

$$\{\exp(R) |\ \dim R < \infty\}$$

is dense in the infinite interval $(1; \infty)$.

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