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ON THE SHAPE OF A COUNTEREXAMPLE TO THE TWO-DIMENSIONAL JACOBIAN CONJECTURE*

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To Vesselin Drensky with best wishes

Abstract. Polynomials $f,\ g \in \mathbb{C}[x,y]$ is a Jacobian pair if the Jacobian $\mathrm{J}(f,g) = 1$. The Jacobian conjecture (JC) formulated by O. H. Keller states that then $\mathbb{C}[f,g] = \mathbb{C}[x,y]$. In this paper further information on the shape of the Newton polygon of f if the pair $f,\ g$ is a counterexample to JC is obtained.

Introduction.

What is the Jacobian Conjecture

Take $f,\ g \in \mathbb{C}[x,y]$ where $\mathbb{C}$ is the field of complex numbers which satisfy $\mathrm{J}(f,g) = \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x} = 1$. Is it true that $x,\ y \in \mathbb{C}[f,g]$? This is a natural question because if $x,\ y \in \mathbb{C}[f,g]$ then $\mathrm{J}(f,g)$ is a non-zero constant by the chain rule of calculus. Geometric reformulation of this question also seems reasonable:

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is it true that a polynomial mapping of $\mathbb{C}^2$ to itself is globally invertible if it is locally invertible?

History

This question makes sense for polynomials with any number of variables and it was asked in 1939 by O. H. Keller (see [14]) though for polynomials with integer coefficients. He checked it for a birational map.

The case $n = 1$ is trivial, the first published proof for the case $n = 2$ appeared in 1955 (see [9]), and several other proofs, even for arbitrary n were published a bit later ([24, 25, 26, 10]).

The question still commanded some attention apparently because it has a very simple algebraic formulation and some subtlety: it is certainly wrong if the polynomials are replaced by rational or analytic functions but an answered question is not as attractive as an open problem.

Then all the proofs with exception of Engel's became wrong and at least for $n > 2$ the problem became open.

Finally it was Engel's turn. Apparently Abhyankar knew about mistakes, but it seems that only after Vitushkin's talk (see [29]) it became official.

Say, L. A. Campbell, who was first to show the if $C(x)/C(f)$ is Galois extension then it is birational, (see [6]) wrote the paper assuming that Engel's paper has no mistakes and added in proofs a remark with attribution to Vitushkin that it has (at least) two.

The paper [5] of Bass, Connell, and Wright which appeared in 1982 in the Bulletin(New Series) of the American Mathematical Society about ten years after *JC* became a problem again, contains a snapshot of what was known about JC including interesting historical remarks.

Thanks to this paper JC became very popular and now-days even has its own code 14R15 in MSC2020. Also Smale (see [27]) included it in his list of problems for the 21st century.

Usually there are proofs of JC on the arXiv which are never published. The last published wrong proof appeared in 2022 (see [4]).

Why JC is difficult

As these examples $f = e^x$, $g = ye^{-x}$; $f = x^2, g = \frac{1}{2}x^{-1}y$ show JC is not true for analytic or rational functions, so polynomiality is very essential and nobody figured out yet how to use it. If the mapping $\phi : (x, y) \to (f, g)$ given by a Jacobian pair f, g is proper, everything is easy. It is possible though for a mapping given by polynomials to have an infinite algebraic curve as the preimage of a finite arc of an algebraic curve: point at infinity can be mapped on

a finite point. Because of that, say, x as an algebraic function of f, g can have singular points and so the polynomial and the rational functions cases are not very different.

It is also hard to construct a counterexample. Several authors (see [12, 16, 30]) showed that if a counterexample exists the total degree of f or g must exceed 100.

What is done in this paper

There are different restrictions on potential counterexamples to JC (see [1, 2, 3, 7, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 28]). Among them are restrictions on the shape of the Newton polygons of potential counterexamples and the quasi-homogeneous forms which are supported by their edges. Here a new restriction on these forms is obtained. This gives some information on Newton polygons of potential counterexamples.

The Newton polygon of a polynomial in two variables is the convex hull of degree vectors of all monomials which appear in this polynomial, i.e. if $x^i y^j$ is a monomial of a polynomial in question, the point with coordinates (i, j) belongs to its Newton polygon.

Weight degree functions and lemma on independence. Define a weight degree function on $\mathbb{C}[x, y]$ as follows. First, take weights $w(x) = \alpha$, $w(y) = \beta$ where $\alpha, \beta \in \mathbb{Z}$, $\gcd(\alpha, \beta) = 1$ and put $w(x^i y^j) = i\alpha + j\beta$. For a $p \in \mathbb{C}[x, y]$ denote by $\mathrm{supp}(p)$ the support of p, i.e. the collection of all monomials appearing in p with non-zero coefficients, and define $w(p) = \max(w(a) | a \in \mathrm{supp}(p))$. A polynomial p can be written as $p = \sum p_i$ where p_i are forms homogeneous relative to w and the form p_w of the maximal weight of this presentation is called the *leading form* of p.

Lemma on independence. *Let p, $q \in \mathbb{C}[x, y]$ be a pair of algebraically independent polynomials and w be a non-zero weight degree function for which $p_w \notin \mathbb{C}$. Then there exists an $r \in \mathbb{C}[p, q]$ such that r_w and p_w are algebraically independent.*

We will deduce this lemma from the

Lemma on the leading monomial. *Consider subalgebra $\mathbb{C}[p, q]$ of $\mathbb{C}[x, y]$ where p, $q \in \mathbb{C}[x, y]$ and the semigroup $\Pi = \overline{\mathbb{C}[p, q]}$ of all leading monomials of elements of $\mathbb{C}[p, q]$ in w-lexicographic order: for $r \in \mathbb{C}[x, y]$ take r_w and then take the maximal monomial of r_w in lexicographic order $x >> y$. If p, $q \in \mathbb{C}[x, y]$ are algebraically independent then Π contains algebraically independent monomials.*

P r o o f. Proof is based on the notion of Gelfand-Kirillov dimension (see [11]). Consider all polynomials $F(x,y) \in \mathbb{C}[x,y]$ with the total degree not exceeding N. Denote by V_N the linear space of polynomials $F(p,q)$. Dimension $\dim(V_N) = \binom{N+2}{2}$ since the polynomials $\{p^i q^j,\ i+j \le N\}$ form a basis of this space. Assume now that all monomials in Π are algebraically dependent. Two monomials $x^{i_1}y^{j_1},\ x^{i_2}y^{j_2}$ are algebraically dependent only if the degree vectors $\langle i_1, j_1\rangle$ and $\langle i_2, j_2\rangle$ are collinear. Denote by π the monomial which corresponds to the shortest non-zero vector with non-negative integral coordinates proportional to the degree vectors of monomials from Π. Then $\Pi \subseteq \mathrm{Span}(\pi)$.

If $a \in \mathbb{C}[x,y]$ denote by $\overline{a}$ its leading monomial. Clearly $\overline{ab} = \overline{a}\,\overline{b}$. Because of that $\overline{p^i q^j} = \pi^{is+jt}$ where $\overline{p} = \pi^s,\ \overline{q} = \pi^t$. Hence there is a basis in V_N consisting of some of polynomials with the leading monomials $\{\pi^d,\ 0 \le d < N(s+t)\}$. Therefore $\binom{N+2}{2} \le N(s+t)$. But $\binom{N+2}{2} > \frac{N^2}{2}$ and $\frac{N^2}{2} > N(s+t)$ if $N > 2(s+t)$. □

Now we prove Lemma on independence.

If the leading forms of elements in $\mathbb{C}[p,q]$ are algebraically dependent with p_w then the leading monomials in w-lexicographic order are also algebraically dependent with the leading monomial of p_w which is not possible. □

Remark 1. If the leading forms of two polynomials are algebraically dependent then the leading monomials of these forms are also algebraically dependent.

Dixmier formulae and the shape of supp(f). Consider now a Jacobian pair f, g. Polynomials f, g are algebraically independent. Indeed, if $f = f_0 + f_{1,0}x + f_{0,1}y + \dots,\ g = g_0 + g_{1,0}x + g_{0,1}y + \dots$ then $f_{1,0}g_{0,1} - f_{0,1}g_{1,0} = 1$ and their linear forms are algebraically independent. Hence $f - f_0,\ g - g_0$ are algebraically independent and $f,\ g$ are algebraically independent.

Now take a weight defined by $\alpha > 0,\ \beta > 0$ for which $w(f) > 0$ and f_w is not a monomial. If this is impossible then $\mathrm{supp}(f)$ contains a monomial $x^{d_x}y^{d_y}$ where $d_x = \deg_x(f),\ d_y = \deg_y(f)$.

Suppose that we found such a weight and $h = p(f,g)$ such that $\overline{h_w} = x^{i_1}y^{j_1}$ and $\overline{f_w} = x^{i_2}y^{j_2}$ are algebraically independent.

From $\mathrm{J}(x^{i_1}y^{j_1}, x^{i_2}y^{j_2}) = (i_1 j_2 - i_2 j_1)x^{i_1+i_2-1}y^{j_1+j_2-1}$ we see that two monomials are algebraically dependent only if their Jacobian is zero. It is also clear from this computation that $\overline{\mathrm{J}(p,q)} = \mathrm{J}(\overline{p}, \overline{q})$ if $\mathrm{J}(\overline{p}, \overline{q}) \neq 0$. Therefore h_w

and f_w are also algebraically independent.

Denote $\mathrm{J}(f,a)$ by $\partial(a)$ for $a \in \mathbb{C}[x,y]$. Since by the rules of calculus $\mathrm{J}(f,g^i) = ig^{i-1},\ \partial(p(f,g)) = \dfrac{\partial p}{\partial g}$ and $\partial^d(h) = 0$ for a sufficiently large d. Again by the rules of calculus $\mathrm{J}(a,b)_w = \mathrm{J}(a_w,b_w)$ if $\mathrm{J}(a_w,b_w) \neq 0$. This is because $w(\mathrm{J}(a,b)) = w(a) + w(b) - w(xy)$ if $\mathrm{J}(a_w,b_w) \neq 0$. Thus if $\partial_w(a) = \mathrm{J}(f_w,a)$ then $\partial_w^d(h_w) = 0$.

Remark 2. We checked that for any weight w derivative $d_w:\ d_w(p) = \mathrm{J}(f_w,p_w)$ is *locally nilpotent* on the algebra $\mathbb{C}[f,g]_w$ of the w leading forms of polynomials from $\mathbb{C}[f,g]$, i.e. that any form from this algebra is annihilated, i.e. becomes zero, after we apply ∂_w to it sufficiently many times.

Suppose that ϕ is w homogeneous and $\mathrm{J}(f_w,\phi) = 0$. If we look at the leading monomials $\overline{f_w} = x^{i_1}y^{j_1}$ and $\overline{\phi} = x^{i_2}y^{j_2}$ they are algebraically dependent. Therefore $z_2\langle i_2,j_2\rangle = z_1\langle i_1,j_1\rangle$ for some positive integers $z_1,\ z_2$ since vectors $\langle i_1,j_1\rangle$ and $\langle i_2,j_2\rangle$ are proportional. Hence ϕ^{z_2} and $f_w^{z_1}$ have the same leading monomial and we can find $c \in \mathbb{C}$ such that the leading monomial of $\psi = \phi^{z_2} - cf_w^{z_1}$ is smaller than $\overline{f_w}^{z_1}$. But then $\psi = 0$ because $\mathrm{J}(f_w,\psi) = 0$ and there is only one monomial "proportional" to $\overline{f_w}$ with weight equal to the weight of ϕ^{z_2} since $w(f) > 0$.

Hence $h_1 = \mathrm{J}(f_w,h_w) \neq 0$ because the leading monomials of f_w and h_w are algebraically independent. Since we know that $h_d = \partial_w^d(h_w) = 0$ we can find $h_i = \partial_w^i(h_w)$ for which $h_{i+1} \neq 0$ but $h_{i+2} = 0$.

Therefore $h_{i+1}^{z_3} = c_1 f_w^{z_4},\ c_1 \in \mathbb{C}^*,\ z_3,z_4 \in \mathbb{Z}^+$. We can assume that $\gcd(z_3,z_4) = 1$ and find z_5,z_6 for which $z_3z_5 + z_4z_6 = 1$. To make formulae look better denote f_w by ω, h_i by χ and h_{i+1} by ψ. Then $\psi^{z_3} = c_1\omega^{z_4}$ and for a rational function $\rho = \psi^{z_6}\omega^{z_5}$ its powers $\rho^{z_4} = \psi^{z_4z_6}\omega^{z_4z_5} = \psi^{z_4z_6}(c_1^{-1}\psi^{z_3})^{z_5} = c_1^{-z_5}\psi$ and $\rho^{z_3} = c_1^{z_6}\omega$. Since $z_3 > 0$ we see that ρ is a polynomial.

We can assume without loss of generality that $\omega = \rho^{z_3},\ \psi = \rho^{z_4}$ and that $\mathrm{J}(\rho^{z_3},\chi) = \rho^{z_4}$. Since $\mathrm{J}(\rho^{z_3},\chi) = z_3\rho^{z_3-1}\mathrm{J}(\rho,\chi)$ we have $\mathrm{J}(\rho,\chi) = c_2\rho^{z_4-z_3+1}$ and $\mathrm{J}(\rho,c_3\chi\rho^{z_3-z_4-1}) = 1$. Hence we found a rational function $\sigma = c_3\chi\rho^{z_3-z_4-1}$ such that $\mathrm{J}(\rho,\sigma) = 1$. For formulae we are going to obtain it is more convenient to take $\tau = \rho\sigma$ with property $\mathrm{J}(\rho,\tau) = \rho$.

Polynomial ρ and a rational function τ are $\alpha,\ \beta$ homogeneous and we can write $\rho = x^r p(z),\ \tau = x^t q(z)$ where $r,\ t$ are rational numbers, $z = x^{\frac{-\beta}{\alpha}}y$, and $p \in \mathbb{C}[z],\ q \in \mathbb{C}(z)$.

If $\alpha = 0$ then $z = x,\ \rho = y^r p(x),\ \tau = y^t q(x)$. We needn't consider this

case separately because we can make a substitution $x \to -y,\ y \to x$.

$$\begin{aligned}&\mathrm{J}(\rho,\tau)\\ &= (rx^{r-1}p(z)-\frac{\beta}{\alpha}x^{r-\frac{\beta}{\alpha}-1}yp'(z))x^{t-\frac{\beta}{\alpha}}q'(z)-x^{r-\frac{\beta}{\alpha}}p'(z)(tx^{t-1}q(z)-\frac{\beta}{\alpha}x^{t-\frac{\beta}{\alpha}-1}yq'(z))\\ &= rx^{r-1}p(z)x^{t-\frac{\beta}{\alpha}}q'(z) - x^{r-\frac{\beta}{\alpha}}p'(z)tx^{t-1}q(z) = x^r p(z).\end{aligned}$$

Since $t-\dfrac{\beta}{\alpha}-1=0$ because $w(\tau)=\alpha t = w(x)+w(y)=\alpha+\beta$

$$rpq' - tp'q = p, \tag{1}$$

where $w(\rho)=\alpha r$ and we have

$$w(\rho)pq' - w(\tau)p'q = \alpha p. \tag{2}$$

Hence $w(\rho)\dfrac{q'}{q} - w(\tau)\dfrac{p'}{p} = \dfrac{\alpha}{q}$ and $\ln(q^{w(\rho)}p^{-w(\tau)})' = \dfrac{\alpha}{q}$. Therefore

$$p^{w(\tau)} = q^{w(\rho)} \exp\left(\int -\frac{\alpha}{q}dz\right). \tag{3}$$

Since $w(\tau) > 0$ and $w(\rho) > 0$ a pole of q induces a pole of p in the same point. Hence $q \in \mathbb{C}[z]$ and $\tau \in \mathbb{C}[x,y]$. All roots of q are simple because $\exp\left(\int \dfrac{dz}{(z-\lambda)^d}\right)$ is not an algebraic function if $d>1$. If $-\dfrac{\alpha}{q} = \sum\limits_i \dfrac{d_i}{z-\lambda_i}$ then $p = \prod (z-\lambda_i)^{w(\rho)+d_i}$, i.e. all roots of p are roots of q.

Thus if $q = q_1 z + q_0,\ q_1, q_0 \in \mathbb{C}^*$ then $\tau = x^t(q_1 x^{\frac{-\beta}{\alpha}}y + q_0) \in \mathbb{C}[x,y]$. This is possible only if $t,\ \dfrac{\beta}{\alpha} \in \mathbb{Z}$. In this case $f_w = cx^{a_f}(y+q_0x^{b_f})^{d_f}$ or $f_w = cy^{a_f}(x+q_0y^{b_f})^{d_f}$ if $\alpha = 0$ and q has one root. In both cases we can make an automorphism of $\mathbb{C}[x,y]$ which will make f_w a monomial.

Assume now that q has at least two roots. Then $\sum\limits_i d_i = 0$ and p has a root with multiplicity larger than $w(\rho)$.

If $\alpha \ge 0$ and $\beta \ge 0$ then there are four options for τ because $w(\tau) = w(xy)$. They are
1) $\tau = cxy$; 2) $\tau = cx(y+c_1x^l),\ l \ge 0$; 3) $\tau = c(x+c_1y^l)y,\ l \ge 0$;
4) $\tau = c(x+c_1y)(y+c_2x)$.
The corresponding f_w are

1) $f_w = cx^iy^j$; 2) $f_w = cx^i(y + c_1x^l)^j$; 3) $f_w = c(x + c_1y^l)^iy^j$;
4) $f_w = c(x + c_1y)^i(y + c_2x)^j$ and $i \neq j$ because $\mathrm{J}(f_w, \tau) = 0$ if $i = j$.

In the last three case we can make automorphisms of $\mathbb{C}[x, y]$ which will replace τ with cxy and f_w with cx^iy^j.

Automorphisms in the case 2) and 3) decrees $\deg_x(f) + \deg_y(f)$ if $l > 0$, so after several automorphisms changing the α, β leading forms we will get f_k for which all α, β leading forms with $\alpha > 0,\ \beta > 0$ are monomials, the case we mentioned in the beginning of this section. Once again, $\mathrm{supp}(f_k)$ contains a monomial $x^{d_x}y^{d_y},\ d_x \neq d_y$ where $d_x = \deg_x(f_k),\ d_y = \deg_y(f_k)$.

From now on we will be looking at counterexamples where $\mathrm{supp}(f)$ has such a monomial. Since we can make an automorphism $x \to y,\ y \to -x$ we will also assume that $d_x = m < d_y = n$. Additionally, if $w_{1,0}(f) = x^mp(y)$ we can make a substitution $y \to y - c$ such that the order of polynomial $p(y - c)$ is larger than m because $p(y)$ has a root with multiplicity larger than m, as was just mentioned above. Let us call such an f reduced.

Divisibility Lemma. The pairs of homogeneous polynomials ρ, τ which satisfy relation $\mathrm{J}(\rho, \tau) = \rho$ played very important role in our considerations. Pairs like this were researched by Dixmier (see [8]) in his seminal paper devoted to the first Weyl algebra. In this section we will find a restriction on such pairs.

Lemma on divisibility. *If $J(\rho, \tau) = \rho$ where $\rho,\ \tau$ are w homogeneous, $w(\rho) > 0,\ w(\tau) > 0$ and ρ is not a monomial then $w(\rho)$ doesn't divide $w(\tau)$.*

P r o o f. Since ρ is not monomial, τ is also not monomial (recall that if τ is a monomial then ρ is also a monomial). Because of that $\overline{\tau}$ and $\overline{\rho}$ are algebraically dependent. If $\dfrac{w(\tau)}{w(\rho)} = k \in \mathbb{Z}$ then we can find $c \in \mathbb{C}$ for which $\overline{\tau - c\rho^k} < \overline{\tau}$. Since $\mathrm{J}(\rho, \tau - c\rho^k) = \rho$ these implies that $\overline{\tau - c\rho^k}$ and $\overline{\rho}$ are algebraically dependent. But this is impossible since $w(\rho) > 0$. □

Lemma on dependence. The leading $1, 1$ forms of f and g are algebraically dependent if $f,\ g$ is a counterexample to JC. Indeed, if $\mathrm{J}(f_w, g_w) \neq 0$ then $\mathrm{J}(f_w, g_w) = 1$. If $\mathrm{J}(f_w, g_w) = 1$ then supports of these forms contain x and y, say, $\mathrm{supp}(f_w) \ni x,\ \mathrm{supp}(g_w) \ni y$.

Since these forms are $1,\ 1$ homogeneous $f_w = c_1x + c_3y,\ g_w = c_2y + c_4x$ and $f = c_1x + c_3y + c_5,\ g = c_2y + c_4x + c_6$, certainly not a counterexample to JC and not a reduced f. But for other weights dependence is also true.

Lemma on dependence. *The leading $\alpha,\ \beta;\ \alpha+\beta>0$ forms of f and g are algebraically dependent if a pair $f,\ g$ is a counterexample to JC.*

P r o o f. If these forms are independent then $\mathrm{J}(f_w,g_w)=1$. Hence the supports of these forms must contain x and y, e.g. $\mathrm{supp}(f_w)\ni x,\ \mathrm{supp}(g_w)\ni y$. Since $\alpha>0$ or $\beta>0$ this means that either $\deg_x(f(x,0))=1$ or $\deg_y(g(0,y))=1$.

For the argument sake, assume that $\alpha>0$ and then $f(x,0)=f_0+f_{1,0}x$. The form f_w has the maximal weight in w decomposition of f. If $\beta>0$ then $f=f(0,y)+f_{1,0}x$ and we do not have a counterexample because we can make an automorphism after which f will become $f_{1,0}x$. If $\beta=0$ then $f=f_0(y)+xf_1(y),\ g=g_0(y)$ and $f_1(y)g_0(y)'=1$ since $\mathrm{J}(f,g)=1$. Hence $f_1(y)\in\mathbb{C}^*,\ \deg_y(g)=1$, no counterexample again. So, $\beta<0$ is the remaining case.

We have $\alpha i_1+\beta j_1=\alpha,\ \alpha i_2+\beta j_2=\beta$ if $x^{i_1}y^{j_1},\ x^{i_2}y^{j_2}$ are the leading monomials of f_w and g_w and $i_1j_2-j_1i_2=0$ if $\mathrm{supp}(f_w)$ or $\mathrm{supp}(g_w)$ has more than one monomial. Since $\alpha(i_1-1)+\beta j_1=0,\ \alpha i_2+\beta(j_2-1)=0$ we have $(i_1-1)(j_2-1)-j_1i_2=0$. Therefore $1-i_1-j_2=0$ and either $i_1=1,\ j_2=0$ or $i_1=0,\ j_2=1$ and $j_1i_2=0$.

If $i_1=1,\ j_2=0$ then $i_2=0$ because otherwise $\alpha i_2>\beta$ and $g_w=g_{0,1}y$ since $\beta<0$ and $w(1)=0$. Also $j_1=0$ since $\mathrm{J}(xy^{j_1},y)=y^{j_1}$ and $f_w=f_{1,0}x$.

If $i_1=0,\ j_2=1$ then $j_1=0$ because otherwise $\beta j_1<\alpha$ and $f_w=f_{1,0}x$ since $\alpha>0$ and $w(1)=0$. Also $i_2=0$ since $\mathrm{J}(x,x^{i_2}y)=x^{i_2}$ and $g_w=g_{0,1}y$.

Thus for an $\alpha,\ \beta$ weight where $\alpha>0>\beta$ we have $f_w=f_{1,0}x$ and $g_w=g_{0,1}y$.

If $f=f(0,y)+f_{1,0}x$ we don't have a counterexample. So $\mathrm{supp}(f)$ contains monomials $x^iy^j,\ i>0,\ j>0$. Among these monomials we can choose a monomial $x^{i_1}y^{j_1}$ for which the ratio $\dfrac{i_1-1}{j_1}$ is minimal. Then if we take $\alpha=j_1,\ \beta=1-i_1$ the leading form of f will contain monomials x and $x^{i_1}y^{j_1}$.

Since $\alpha>0>\beta$ the $\mathrm{supp}(g)$ doesn't contain monomials $x^i,\ x^iy$ with $i>0$. Hence if $\mathrm{supp}(g)$ doesn't contain monomials $x^iy^j,\ i>0,\ j>1$ then $g=g(0,y)$ and we don't have a counterexample because then $f_xg_y=1$ and $g=g_0+g_{0,1}y$. We can choose in the $\mathrm{supp}(g)$ a monomial $x^{i_2}y^{j_2}$ with $i_2>0,\ j_2>1$ and minimal ratio $\dfrac{i_2}{j_2-1}$. Then if we take $\alpha=j_2-1,\ \beta=-i_2$ the leading form of g will contain monomials y and $x^{i_2}y^{j_2}$.

Therefore for the weight which corresponds to the smaller ratio $\mathrm{supp}(f)\ni x$, $\mathrm{supp}(g)\ni y$ and at least one of these supports contains another monomial. Since this is impossible, Lemma is proved. □

Remark 3. We can see from this Lemma that g is reduced if f is reduced.

Remark 4. The proof of Lemma shows that $\deg_x(f(x,0)) > 1$. Indeed, assume that $\deg_x(f(x,0)) = 1$. Then we can find a polynomial $p(t) \in \mathbb{C}[t]$ such that $\tilde{g}(x,0) = 0$ for $\tilde{g} = g - p(f)$. Of course, $\mathrm{J}(f,\tilde{g}) = 1$. We can write $f = f_0 + f_{1,0}x + f_1(x)y + \ldots,\ \tilde{g} = \tilde{g}_1(x)y + \tilde{g}_2(x)y^2 + \ldots$. Then $1 = f_{1,0}\tilde{g}_1'$ and $\tilde{g}_1 = f_{1,0}^{-1}y$. After that we can find two forms f_w and $\tilde{g}_w$ such that $\mathrm{supp}(f_w) \ni x,\ \mathrm{supp}(\tilde{g}) \ni y$ and at least one of these supports contains another monomial.

Right edges and the left leading edge. Since f is reduced, i.e. for any $\alpha > 0,\ \beta > 0$ leading form of f is $x^n y^m$, points of the Newton polygon $\mathcal{N}(f)$ of f belong to the rectangle with vertices $(0,0),\ (m,0),\ (m,n),\ (0,n)$.

There are two edges e_0 and e_1 with the vertex (m,n) not necessarily parallel to the coordinate axes, extension of e_0 intersects the y axis and extension of e_1 intersects the x axis. We will call them left and right leading edges accordingly.

If $w(f) > 0$ for a weight $\alpha,\ \beta > 0,\ \alpha + \beta > 0$, then we can use Dixmier formulae to find $\alpha,\ \beta$ homogeneous polynomial τ.

If e_0 is horizontal and $\alpha = 0,\ \beta = 1$ then $f_w = y^n p(x),\ \tau = yq(x)$ and $f_w = \mathrm{J}(f_w,\tau) = y^n(p'q - npq')$. That is $p'q - npq' = p$. If $p = p_0x^m + \cdots + p_m$ and $q = q_0x^{d_q} + \cdots + q_{d_q}$ then the coefficient of $p'q - npq'$ with x^{m+d_q-1} is $p_0q_0(m - nd_q) \neq 0$ since $m < n$. Hence $m + d_q - 1 = m$ and $d_q = 1$. Thus $q = q_0x + q_1$ and $(q_0x+q_1)p' - nq_0p = p$. Therefore $\ln(p)' = \dfrac{nq_0+1}{q_0}(x+q_0^{-1}q_1)^{-1}$, $p = p_0(x+q_0^{-1}q_1)^{\frac{nq_0+1}{q_0}}$, and $\dfrac{nq_0+1}{q_0} = m$. After a substitution $x \to x - q_0^{-1}q_1,\ y \to y$ the polynomial p will become a monomial and e_0 will collapse to a vertex.

So, the new left edge will be slanted and we will get new forms $f_w = y^r p(z),\ \tau = y^t q(z),\ z = xy^{\frac{-\alpha}{\beta}}$ for the weights $\alpha < 0,\ \beta > 0,\ \alpha + \beta > 0$.

Here is equation (2) for these forms:

$$(\alpha m + \beta n)pq' - (\alpha + \beta)p'q = \beta p.$$

The coefficient with z^{m+d_q-1} is
$(\alpha m + \beta n)d_q - (\alpha + \beta)m > (\alpha m + \beta m)d_q - (\alpha + \beta)m = (\alpha + \beta)m(d_q - 1) > 0$
if $d_q > 1$. So, as above, $d_q = 1,\ q = q_0z + q_1$ and $\tau = y^t(q_0xy^{\frac{-\alpha}{\beta}} + q_1) \in \mathbb{C}[x,y]$. Since the leading monomial of f_w is $x^m y^n$, the leading monomial of τ is xy, also $q_1 = 0$ because otherwise $t \in \mathbb{Z}$ but since $t + \dfrac{-\alpha}{\beta} = 1$ and $0 < \dfrac{-\alpha}{\beta} < 1$ this is impossible.

We can conclude that $f_w = x^m y^n$. Therefore the left edge is not a vertex only if $\alpha + \beta \leq 0$. We cannot get rid of the case $\alpha + \beta = 0$ using Dixmier formulae

because $\mathrm{J}(y^i p(xy), xy) = -iy^i p(xy)$ and it will take us too far a field to prove that this edge also collapses to a vertex, so we just assume that e_0 has slope larger than one. Interested reader can look this up at [7] or [17].

If $w(f) > 0$ and $\alpha + \beta < 0$ then τ is a rational function rather than a polynomial and it is not clear how to collapse e_0.

Newton polygons of f and g. The right leading edge e_1 may end at the x axis and then there are no more right edges. If it doesn't, there is the next right edge e_2 and so on, until the edge e_k which has a vertex on the x axis. From Remark 4 this vertex cannot be $(1, 0)$.

The boundary of the Newton polygon $\mathcal{N}(f)$ of f consists of right edges $e_1,\ e_2, \dots, e_k$ and left edges $e_0,\ e_{-1}, \dots, e_{-l}$. Each of these edges contains the support of the leading form of f for the corresponding weight. Denote the weight which corresponds to e_i by w_i and f_{w_i} by f_i.

The Newton polygon of g belongs to the rectangle similar to the rectangle containing $\mathcal{N}(f)$ with the vertex (M, N). If $\lambda = \dfrac{N}{n}$ we can assume that λ is neither integer nor the reciprocal of an integer: in the first case we can replace g by a polynomial with smaller total degree, in the second case we can replace f by a polynomial with smaller total degree.

Denote by $E_1, \dots$ the right edges of $\mathcal{N}(g)$, and by $E_0, \dots$ the left edges of $\mathcal{N}(g)$.

Lemma on similarity. *If we add to Newton polygons of f and g the origin then the Newton polygon of g is homothetic to the Newton polygon of f with the coefficient λ.*

P r o o f. Denote by g_i the leading forms g_{w_i}. By Lemma on dependence $\mathrm{J}(f_i, g_i) = 0$. Denote by ξ_i the polynomial of minimal degree power of which is f_i. Let us call ξ_i *the maximal root of* f_i. The same considerations as in Dixmier formulae and the shape of $\mathrm{supp}(f)$ show that $f_i = \xi_i^{a_i},\ g_i = c_i \xi_i^{b_i}$. It remains to check that $\dfrac{b_i}{a_i} = \lambda$.

It is clear for $i = 0$ and $i = 1$ because $x^M y^N = (x^m y^n)^\lambda$ and these are leading monomials of the corresponding forms. But then the smallest monomials of these forms satisfy the same relation and they are leading monomials for the forms supported by the next edges. Then we can apply this consideration to the next edges, etc. □

We will be interested in the right edges which have at least one vertex above the bisectrix of the first quadrant. We will call this edges *admissible*.

Lemmas on leading forms.

Lemma on forms. *For any $p \in \mathbb{C}[f,g]$ the form $p_i \in \mathbb{C}[\xi_i, \xi_i^{-1}, \tau_i]$ if $i > 0$.*

P r o o f. We will prove Lemma using induction on degree of $\partial_i = \mathrm{J}(f_i, -)$ which annihilates p_i.

Base: We know that if $\partial_i(p_i) = 0$ then ξ_i and p_i are algebraically dependent. Hence $p_i = c\xi_i^{d_p}$ because ξ_i is the maximal root of f_i.

Step. By induction $\partial_i(p_i) = \sum_{j,k} \xi_i^j \tau_i^k$; take $q_i = \sum_{j,k} \frac{1}{k+1} \xi_i^{j-a_i} \tau_i^{k+1}$. Then $\partial_i(p_i - q_i) = 0$ because $f_i = \mathrm{J}(f_i, \tau_i) = \mathrm{J}(\xi_i^{a_i}, \tau_i) = \xi_i^{a_i}$, and $p_i = q_i + c_i \xi_i^l$ where $l w_i(\xi_i) = w_i(p)$, provided l is an integer. □

For $p \in \mathbb{C}[x,y]$ denote by $\underline{p}$ the smallest monomial of p in the w lexicographic order.

Lemma on dependence for admissible edges. *If e_i is an admissible edge and $\overline{p_i}$ is dependent with $\overline{f_i}$ then $J(f_j, p_j) = 0$ for $0 < j < i$.*

P r o o f. We know that $p_{i-1} \in \mathbb{C}[\xi_{i-1}, \xi_{i-1}^{-1}, \tau_{i-1}]$. Hence $(f^d p)_{i-1} \in \mathbb{C}[\xi_{i-1}, \tau_{i-1}]$ if d is sufficiently large. Since e_i is admissible, the degree vector of $\underline{\xi_{i-1}}$ is above the bisectrix.

Next,

$$(f^d p)_{i-1} = \sum_{j w_{i-1}(\xi_{i-1}) + k w_{i-1}(\tau_{i-1}) = w_{i-1}(f^d p)} \xi_{i-1}^j \tau_{i-1}^k$$

and $\underline{(f^d p)_{i-1}} = \underline{\xi_{i-1}^j \tau_{i-1}^k}$ for the summand with maximal k because $\underline{\tau_{i-1}} = xy$ and the degree vector of $\underline{\xi_{i-1}}$ is above the bisectrix. Now, $\underline{p_{i-1}} = \overline{p_i}$ is dependent with $\overline{f_i}$. Hence the maximal k is zero, $(f^d p)_{i-1} = c\xi_{i-1}^{\mu}$, $\mu = \dfrac{w_{i-1}(f^d p)}{w_{i-1}(\xi_{i-1})}$ and $p_{i-1} = c\xi_{i-1}^{\mu - d a_{i-1}}$ because $f_{i-1} = \xi_{i-1}^{a_{i-1}}$. After that we can apply this considerations to e_{i-2} and so on until e_1. □

Remark 5. The form $p_{i-1} = c f_{i-1}^{\nu}$ where $\nu = \dfrac{w_i(p)}{w_i(f)}$. Just like in Lemma on similarity $p_j = c f_j^{\nu}$ for all $0 < j < i$.

By Lemma on independence we can find p for which p_i is algebraically independent with f_i. We can assume that $p_i = \tau_i \xi_i^{d_i}$, because we can decrease degree of τ_i by taking Jacobians with f if necessary.

Monomials $\overline{\tau_i\xi_i^{d_i}}$ and $\overline{f_i}$ are algebraically dependent. Hence by Lemma on dependence for admissible edges, p_j and f_j are algebraically dependent if $j<i$.

If $\mu_i=\dfrac{w_i(p)}{w_i(f)}$ then $p_j=c_if_j^{\mu_i}$ if $j<i$ and $f_j^{\mu_i}$ is a polynomial. We want to show that $f_j^{\lambda_i}$ is a polynomial where $\lambda_i=\dfrac{w_i(\tau_i)}{w_i(f)}$.

Since $\mathrm{J}(f,p_i)_i=\xi_i^{d_i+a_i}$ the form $f_j^{\nu_i}$ is a polynomial for $\nu_i=\dfrac{w_i(\xi_i^{d_i+a_i})}{w_i(\xi_i^{a_i})}$. Because $\nu_i=\dfrac{d_i}{a_i}+1$, $f_j^{\frac{d_i}{a_i}}$ is also a polynomial if $j\le i$.

Since $\mu_i=\dfrac{w_i(\tau_i\xi_i^{d_i})}{w_i(f)}=\lambda_i+\dfrac{d_i}{a_i}$ we can conclude that $f_j^{\lambda_i}$ is a polynomial if $j<i$.

Remark 6. By Lemma on divisibility $f_i^{\lambda_i}$ is not a polynomial. Hence $f_i^{\mu_i}$ is not a polynomial though $p_j=c_if_j^{\mu_i}$ for $0<j<i$.

Left leading edge e_0. Consider now the edge e_0, the corresponding weight w_0, and the leading form f_0. Recall that $\alpha<0,\ \beta>0,\ \alpha+\beta<0$. By Lemma on independence there are polynomials in $\mathbb{C}[f,g]$ with the leading w_0 form algebraically independent with f_0. We can choose among these forms a form χ such that $\mathrm{J}(f_0,\chi)=f_0^a$ where a is minimal possible. For the admissible right edges $\tau_i=\dfrac{\chi_i}{f_i^{a_i}}$ is a polynomial. This is not the case for the left edges. If the corresponding q would be a polynomial then the rational function $q^{w(\rho)}\exp\left(\int-\dfrac{\alpha}{q}dz\right)$ (see (3)) will have a root because $-\dfrac{\alpha}{q}=\sum\limits_i\dfrac{d_i}{z-\nu_i}$ and $\sum\limits_i d_i=0$. But then p would have a pole in this point.

Nevertheless we can define the leading monomial of τ as the ratio of leading monomials of its numerator and denominator and the support of τ which may be infinite by presenting τ as a series: $\tau=\overline{\tau}s(z)$ where $z=x^{-\beta}y^{\alpha}$.

Lemma on left edge. *The leading monomials of f_0 and τ are algebraically independent.*

Proof. The leading monomial of f_0 is x^my^n. Assume that $\overline{\tau}=x^iy^j$ is algebraically dependent with x^my^n. Then $x^iy^j=(x^my^n)^{\mu}$ where $\mu<0$ because $\alpha m+\beta n>0$ while $\alpha i+\beta j=\alpha+\beta<0$, and $\mathrm{supp}(\tau)$ belongs to the ray $\mu(m,n)+t(-\beta,\alpha),\ t\in\mathbb{R},\ t\ge0$. But then $\overline{\mathrm{J}(f_0,\tau)}<x^my^n$. □

Remark 7. We can conclude that $\overline{\tau} = xy$. Therefore the same proof as in Lemma on dependence for admissible edges shows that if $\overline{p_0}$ and $\overline{f_0}$ are algebraically dependent then p_0 and f_0 are algebraically dependent. Hence $f_0^{\lambda_i}$ is a polynomial for all λ_i corresponding to the admissible edges including e_1.

Conclusion. For a monomial μ denote by $|\mu|$ its degree vector and by v_0 the vertex of $\mathcal{N}(f)$ corresponding to $\overline{f}$.

To finish we prove two Theorems.

Theorem 1. *For a reduced f there are no counterexamples with* $\deg_y(f) = 2\deg_x(f)$ *and vertical edge v_1.*

P r o o f. Suppose $|\overline{f_1}| = k\langle 1, 2\rangle$. Since edges supporting f_1 and τ_1 are parallel $|\overline{\tau_1}| = \langle 1, 2\rangle$ and $\lambda_1 = \dfrac{1}{k}$. Hence $f_0 = \xi_0^k$ and $\xi_0 = (xy^2 + cy^i)^k$ where $i = 0$ because $i < 2$ and $i \neq 1$, as we mentioned above. Hence $i = 0$. But then $f(0, y) = c_1$. If $f = c_1 + xf_{(1)}(y) + \ldots$, $g = g_{(1)}(y) + \ldots$ then $f_{(1)}g'_{(1)} = 1$, $g_{(1)} = g_{0,0} + g_{0,1}y$ which is impossible (see Remark 4). □

Theorem 2. *If f is reduced and $v_0 = \left(\prod_{i=1}^{k} p_i^{\delta_i}\right)(a, b)$ where all p_i are prime numbers and* $\gcd(a, b) = 1$ *then $\mathcal{N}(f)$ has at most* $\sum_i \delta_i - 1$ *admissible edges.*

P r o o f. Recall that $f_i = \xi_i^{a_i}$ where ξ_i is the maximal root of f_i. For some polynomials in $\mathbb{C}[f, g]$ the form $h_i = c_{h,i}\xi_i^{b_i(h)}, c_{h,i} \in \mathbb{C}$. We can write $h_i = c_{h,i}f_i^{\frac{b_i(h)}{a_i}}$ and choose among fractions $\dfrac{b_i(h)}{a_i}$ one with the largest denominator d_i after these fractions are written in the reduced form. It is a nice exercise to show that $d_i\dfrac{b_i(h)}{a_i} \in \mathbb{Z}$. If we take a polynomial H for which $\dfrac{b_i(H)}{a_i}$ has denominator d_i then by Remark 5 $H_j = c_{H,i}f_j^{\frac{b_i(H)}{a_i}}$ for j, $i \geq j > 0$. Therefore $d_{i-1} \geq d_i$. Actually, it follows from Remark 6 that $d_{i-1} > d_i$. Also, $\dfrac{d_{i-1}}{d_i} \in \mathbb{Z}$

Now we can finish the proof. For the last admissible edge d_s is at least the denominator of $\lambda = \dfrac{\deg_y(g)}{\deg_y(f)}$ which is a fraction. So d_s is at least one of the

p_i's. Since d_1 obtained from d_s by multiplying it by $s-1$ factors it has at least s factors and by Lemma on divisibility d_1 has at most $\sum_i \delta_i - 1$ factors, thus $s \leq \sum_i \delta_i - 1$. □

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