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THE EMBEDDING PROBLEM

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To Vesselin Drensky for his 75th birthday

Abstract. We discuss a strategy to find possible counterexamples in the theory of Cayley Hamilton algebras in positive characteristic.

Foreword. *Given a positive integer n and an associative ring R give necessary end/or sufficient conditions that R may be embedded in the ring $M_n(A)$ of $n \times n$ matrices over a commutative ring A.*

This question has attracted a lot of research and has been pivotal in the development of the theory of rings with polynomial identities, A weaker formulation of the same question is:

Given an associative ring R give necessary end/or sufficient conditions that R may be embedded in the ring $M_n(A)$ of $n \times n$ matrices over a commutative ring A for some positive integer n.

Many papers have been written on this question and see [8] for a survey.
A complete answer as far as I can see is not available.

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0.0.1. Characteristic 0. In 1987 developing the idea of Cayley–Hamilton algebras I gave a complete answer in characteristic 0, for algebras with trace, and trace preserving representations, I call this the *strong embedding Theorem* see Theorem 1.9.

In 1988 Wiles [26] and 1991 Taylor [25], developing ideas of Mazur [10], were studying a similar problem, they wanted to understand under which conditions a class function $f : G \to A$ with G a group and A a commutative ring is the trace of a representation $\bar{f} : G \to GL(B)$ of G into $n \times n$ matrices over some commutative ring B containing A.

Remark 0.1. One needs in general a ring larger than A, for instance a real character of a finite group may be the character of a complex representation but not of a real one.

A necessary condition on f is to be a *pseudocharacter*, see Definition 1.10.

In positive characteristic one needs a rather different definition using axioms for the determinant rather than the trace.

As I will explain in §1.7 the notion of pseudocharacters and of Cayley–Hamilton algebras are very strongly connected.

In fact to a pseudocharacter f of dimension n with values in A one associates a homomorphism $\hat{f}$ of the group G into the invertible elements of a Cayley–Hamilton algebra $C_{A,n}(G)$ so that any homomorphism $\bar{f} : G \to GL(B)$ of G into $n \times n$ matrices over some commutative ring B containing A factors as $\bar{f} = g \circ \hat{f}$

$$\begin{array}{ccccc} \bar{f} : G & \xrightarrow{\hat{f}} & C_{A,n}(G) & \xrightarrow{g} & M_n(B) \ . \\ \downarrow f & & \downarrow t & & \downarrow tr \\ A & \longrightarrow & A \cdot 1 & \longrightarrow & B \end{array} \tag{1}$$

In characteristic 0, denote by $t(x)$ the formal trace of x, we have that $A \subset C_{A,n}(G), f(g) = t(\hat{f}(g))$.

The map g, compatible with trace, is injective by the strong embedding Theorem 1.9 for Cayley–Hamilton algebras. This implies that a pseudocharacter is in fact the character of a homomorphism $G \to GL(B)$.

0.0.2. Characteristic $p > 0$. A variation of this construction is defined in all characteristics. In positive characteristic it is well understood tat the trace is not enough to describe a representation. The representations of finite groups need not be semisimple, so there are many obstacles to extend the theory.

In positive characteristic, in order to define pseudocharactes or Cayley–Hamilton algebras, the axioms on a trace have to be replaced by axioms of a norm (or determinant) giving rise to a characteristic free Theory.

One has, for F a coefficient ring (usually $F = \mathbb{Z}$ or a field)

$$\begin{array}{ccccc} \bar{f} : F[G] & \xrightarrow{\hat{f}} & C_{A,n}(G) & \xrightarrow{g} & M_n(B)\ . \\ \downarrow f & & \downarrow d & & \downarrow \det \\ A & \xrightarrow{i} & A \cdot 1 & \xrightarrow{\bar{g}} & B \end{array} \tag{2}$$

Here $f, d, \det$ are multiplicative polynomial maps of degree n.

Two questions arise

(1) Is A contained in B? *that is a pseudocharacter is a character*

(2) Is g injective? *the embedding problem*

(3) Is g surjective to the invariants?

In this more general case the embedding problem seems to be still open.

In a recent paper [9], Jinyue Luo gave a example of a pseudocharacter of a group with 32 elements, that is not a character. So question (1) has a negative answer.

There is a simpler example, in characteristic 2, discussed later for the dihedral group $D4$. The two examples capture different phenomena.

In the last paragraph I will explain this example and show that in Formula (1) what happens in this case is that the map from the algebra A of values of the pseudocharacter to $C_{A,n}(G)$ is not injective while g is.

I will discuss the computer calculations which show that in this case the corresponding Cayley–Hamilton algebra still embeds into its universal representation. So Jinyue Luo gives some information but no example for the embedding problem in positive characteristic.

I start with a coincise history of the problem and of Cayley–Hamilton algebras. For details the reader can look at the ample literature.

In the first two sections I recall the known Theory and in the last section 3 I discuss Jinyue Luo's example.

1. A brief history of the problem and of Cayley–Hamilton algebras. 1966 In my Ph. D. thesis I started to stress a slightly different point of view for the embedding problem (cf. [12]).

1.1. n-dimensional representations. For a given $n \in \mathbb{N}$ and a commutative ring A by $M_n(A)$ we denote the ring of $n \times n$ matrices with coefficients in A. By a symbol $(a_{i,j})$ we denote a matrix with entries $a_{i,j} \in A$, for $i, j = 1, \dots, n$.

The construction $A \mapsto M_n(A)$ is a functor from the category $\mathcal{C}$ of commutative rings to the category $\mathcal{R}$ of associative rings.

To a map $f : A \to B$ is associated a map $M_n(f) : M_n(A) \to M_n(B)$ in the obvious way $M_n(f)((a_{i,j})) := (f(a_{i,j}))$.

Definition 1.1. *By an* n–dimensional representation *of a ring* R *we mean a homomorphism* $f : R \to M_n(A)$ *with* A *commutative.*

It is easily seen that the set valued functor $A \mapsto \hom_{\mathcal{R}}(R, M_n(A))$ is representable. That is:

Proposition 1.2. *There is a commutative ring* $T_n(R)$ *and a natural isomorphism* $\mathtt{j}_A$ *of functors:*

$$\hom_{\mathcal{R}}(R, M_n(A)) \overset{\mathtt{j}_A}{\cong} \hom_{\mathcal{C}}(T_n(R), A), \quad \mathtt{j}_A : f \mapsto \bar{f}$$

given by the commutative diagram $f = M_n(\bar{f}) \circ \mathtt{j}_R$*:*

(3)
$$\begin{array}{ccc} R & \xrightarrow{\mathtt{j}_R} & M_n(T_n(R)) \\ & \searrow^{f} & \downarrow{\scriptstyle M_n(\bar{f})} \\ & & M_n(A) \end{array} .$$

The map $\mathtt{j}_R : R \to M_n(T_n(R))$ is called the *universal* n*-dimensional representation of* R or *the universal map into* $n \times n$ *matrices.*

Of course it is possible that R has no n–dimensional representations, in which case $T_n(R) = \{0\}$.

Remark 1.3. A completely tautological statement is that R can be embedded in the ring $M_n(A)$ of $n \times n$ matrices over a commutative ring A if and only if j_R is an embedding.

The construction of $T_n(R)$ has an independent interest even if we know that R can be embedded in a ring $M_n(A)$. For instance it is easy to see that, if $R = M_n(F)$ with F a field the ring $T_n(R)$ is the coordinate ring of the projective linear group.

The same discussion can be performed when R is in the category $\mathcal{R}_F$ of algebras over a commutative ring F.

Then the functor $A \mapsto \hom_{\mathcal{R}_F}(R, M_n(A))$ is on commutative F algebras and $T_n(R)$ is an F algebra.

From now on F will be a fixed commutative ring, in particular we will restrict to $F = \mathbb{Z}$ or a field.

1.2. The projective group. The functor $A \mapsto \hom_{\mathcal{R}}(R, M_n(A))$ has a group of symmetries: the *projective linear group* $PGL(n)$.

It is best to define this as a representable group valued functor on the category $\mathcal{C}$ of commutative rings. The functor associates to a commutative ring A the group $\mathfrak{G}_n(A) := Aut_A(M_n(A))$ of A–linear automorphisms of the matrix algebra $M_n(A)$. To a morphism $f : A \to B$ one has an associated morphism $f_* : \mathfrak{G}_n(A) \to \mathfrak{G}_n(B)$ and the commutative diagram:

$$\begin{array}{ccc} M_n(A) & \xrightarrow{g} & M_n(A) \\ {\scriptstyle M_n(f)}\downarrow & & \downarrow{\scriptstyle M_n(f)} \\ M_n(B) & \xrightarrow{f_*(g)} & M_n(B) \end{array} \quad . \tag{4}$$

One has a natural homomorphism of the general linear group $GL(n, A)$ to $\mathfrak{G}_n(A)$ which associates to an invertible matrix X the inner automorphism $a \mapsto XaX^{-1}$. When A is a field this map is surjective.

The functor *general linear group* $A \mapsto GL(n, A)$ is represented by the Hopf algebra $\mathbb{H}_n := \mathbb{Z}[x_{i,j}][d^{-1}],\ i, j = 1, \dots, n$ with $d = \det(X)$, $X := (x_{i,j})$ with the usual structure given compactly by comultiplication δ, antipode S and counit ϵ:

$$\delta(X) = X \otimes X,\ S(X) := X^{-1},\ \epsilon : X \to 1_n.$$

The functor $A \mapsto \mathfrak{G}_n(A)$ is represented by the sub Hopf algebra $\mathbb{P}_n \subset \mathbb{Z}[x_{i,j}][d^{-1}]$ formed of elements homogeneous of degree 0.

It has a basis, over $\mathbb{Z}$, of elements ad^{-h} where a is a doubly standard tableaux with no rows of length n and of degree $h \cdot n$. For a proof see Theorem 3.4.21 in [1].

Finally we have a natural action of $\mathfrak{G}_n(A)$ on $\hom_{\mathcal{R}}(R, M_n(A))$ by composing a map f with an automorphism g.

As a consequence we have an action of the group $\mathtt{P}_n := Spec(\mathbb{P}_n)$ on $T_n(R)$ and on $M_n(T_n(R))$.

1) The map j_R has image into the subalgebra $M_n(T_n(R))^{\mathtt{P}_n}$ of invariants.

2) Finally the ring of invariants $T_n(R)^{\mathtt{P}_n}$ contains all the coefficients of the characteristic polynomials of the elements $\mathrm{j}_R(a),\ a \in R$.

One then has several new algebras and new questions appearing.

Question 1. Is the ring of invariants $T_n(R)^{\mathtt{P}_n}$ generated by the coefficients of the characteristic polynomials of the elements $\mathrm{j}_R(a),\ a \in R$?

Question 2. Is the ring of invariants $M_n(T_n(R))^{\mathtt{P}_n}$ generated by $T_n(R)^{\mathtt{P}_n}$ and $\mathrm{j}_R(a),\ a \in R$?

1.2.1. Step 1, the free algebra. The construction of j_R is in two steps.

One easily sees that when $R = F\langle x_i\rangle_{i\in\mathcal{I}}$ is a free algebra then:

Proposition 1.4. *$T_n(R) = F[\xi^{(i)}_{h,k}] = F[\Xi]$ is the polynomial algebra over F in the variables $\xi^{(i)}_{h,k},\ i \in \mathcal{I},\ h,k = 1,\dots,n$ and $\mathrm{j}_R(x_i) = \xi_i := (\xi^{(i)}_{h,k})$ is the* generic $n \times n$ matrix *with entries $\xi^{(i)}_{h,k}$.*

In this case both questions 1 and 2 have a positive answer. The invariant theory involved is presented in [14] when F is a field of characteristic 0, and may be considered as the *first fundamental theorem* of matrix invariants. For a characteristic free treatment the theorem, much more difficult to prove is due to Donkin [6]. A combinatorial proof of this and of further theorems in in [5].

For a general algebra R one presents it as the quotient of a free algebra $R = F\langle x_i\rangle_{i\in\mathcal{I}}/I$ and the the image of the ideal I generates in $M_n(F[\xi^{(i)}_{h,k}])$ an ideal of the form $M_n(J)$ and $T_n(R) = F[\xi^{(i)}_{h,k}]/J$.

1.3. The Cayley–Hamilton identity and its multilinear form. A basic fact for an $n \times n$ matrix a with entries in a commutative ring A is the construction of its characteristic polynomial

$$\chi_a(t) := \det(t - a) = t^n + \sum_{i=1}^{n} (-1)^i \sigma_i(a) t^{n-i}$$

and the Cayley Hamilton theorem $\chi_a(a) = 0$.

If F is an algebraically closed field then the elements $\sigma_i(a)$ are the elementary symmetric functions e_i in the eigenvalues $\lambda_1,\dots,\lambda_n$ of a.

We also have $tr(x^i) = \sum_{j=1}^{n} \lambda_j^i$ the power sum $\psi_j := \sum_j \lambda_j^i$.

The elementary symmetric functions are related to the power sums by the recursive formula:

$$(-1)^m \psi_{m+1} + \sum_{i=1}^{m} (-1)^{i-1} \psi_i e_{m+1-i} = (m+1) e_{m+1}. \tag{5}$$

Thus the power sums are polynomials with integer coefficients of the elementary symmetric functions while the elementary symmetric functions are polynomials with rational coefficients of the power sums.

We have the generating formula, for symmetric functions in n variables λ_j, obtained using the Taylor expansion for $\log(1+y)$.

$$\sum_{i=0}^{n}(-1)^i e_i u^i = \prod_{r=1}^{n}(1-\lambda_r u) = \exp(-\sum_{j=1}^{\infty}\frac{\psi_j}{j}u^j). \tag{6}$$

Finally for n variables ψ_{n+1} is a polynomial with rational coefficients of the power sums $\psi_i,\ i \leq n$.

The relationship with matrices is that for an $n \times n$ matrix x we have that $tr(x^{n+1})$ is a polynomial with rational coefficients of $tr(x^i),\ i \leq n$.

In fact this is strictly related to the Cayley Hamilton identity $CH_n(x)$ which can be viewed as a formal identity so that the previous relation follows from $tr(x \cdot CH_n(x)) = 0$, see §1.6.1.

1.4. 1974–76 The second fundamental Theorem of matrix invariants. *In this section we assume, unless otherwise specified, that the algebras are over* $\mathbb{Q}$.

Razmyslov [21] and Procesi [14] prove that in characteristic zero all matrix invariants are deduced from the Cayley–Hamilton identity. This is done by a formalization of the notion of trace.

A general theorem in a characteristic free form was later proved by Zubkov, it is much more difficult and see [31] and [5].

1.5. Axioms for a trace.

Definition 1.5. *An associative algebra with trace, over a field F, is an associative F algebra R with a 1-ary operation, denoted*

$$t : R \to R, \quad or \quad tr : R \to R,$$

which is assumed to satisfy the following axioms:

(1) *t is F–linear.*

(2) $t(a)b = b\,t(a), \quad \forall a, b \in R.$

(3) $t(ab) = t(ba), \quad \forall a, b \in R.$

(4) $t(t(a)b) = t(a)t(b), \quad \forall a, b \in R.$

1.6. The free trace algebra. Algebra with trace form a category with free algebras.

The free trace algebra over a set X of variables will be denoted by $\mathcal{F}_T\langle X\rangle$. By definition it is a trace algebra $\mathcal{F}_T\langle X\rangle$ containing X and such that for every trace algebra U the set of homomorphisms from $\mathcal{F}_T\langle X\rangle$ to U are in bijection with the maps $X \to U$. It can be described as follows.

Start from the usual free algebra $F\langle X\rangle$, then consider the classes of cyclic equivalence of monomials M, which we formally denote $tr(M)$.

The algebra $\mathcal{F}_T\langle X\rangle = F\langle x_i\rangle_{i\in\mathcal{I}}[tr(M)]$ is the polynomial ring in the infinitely many commuting variables $tr(M)$ over the free algebra $F\langle X\rangle$. Its trace algebra is the polynomial ring $F[tr(M)]$ in the infinitely many commuting variables $tr(M)$. The map $tr : M \mapsto tr(M)$ is the formal trace.

As for the usual theory of polynomial identities we have:

Definition 1.6. *Given a trace algebra U, an element $f \in \mathcal{F}_T\langle X\rangle$ is a* trace identity *for U if it vanishes under all evaluations $X \to U$.*

It is a pure trace identity *if $f \in F[tr(M)]$.*

Usually when speaking of trace identities we will assume that they have coefficients in $\mathbb{Q}$.

Then it is easy to see that the set of trace identities of a trace algebra U is a (trace) ideal of $\mathcal{F}_T\langle X\rangle$ which is closed under the endomorphisms (variable substitutions) of $\mathcal{F}_T\langle X\rangle$.

Such an ideal is called a *T–ideal*. Conversely any T–ideal I of $\mathcal{F}_T\langle X\rangle$ is the ideal of trace identities of an algebra, namely $\mathcal{F}_T\langle X\rangle/I$.

An algebra $\mathcal{F}_T\langle X\rangle/I$ with I a T–ideal, is called a *relatively free algebra on X*, since it is a free algebra in the *variety* of algebras satisfying the identities of I. Usually when discussing trace identities one assumes X to be countable. With some care one can also restrict to X finite.

1.6.1. The n formal Cayley Hamilton polynomial. In order to define this as an element in the free trace algebra in one variable x we *define* the elements $\sigma_i(x)$ in the free algebra, using Formula (6) by

$$\sum_{i=0}^{\infty}(-1)^i\sigma_i(x)u^i := \exp(-\sum_{j=1}^{\infty}\frac{tr(x^j)}{j}u^j). \tag{7}$$

We may define, for each n, in the free algebra with trace in a single variable x, the *n formal Cayley Hamilton polynomial*

$$CH_n(x) := x^n + \sum_{i=1}^{n} (-1)^i \sigma_i(x) x^{n-i}. \tag{8}$$

Example $\quad CH_1(x) = x - tr(x); \quad CH_2(x) = x^2 - tr(x)x + \frac{1}{2}(tr(x)^2 - tr(x^2)).$

$$CH_3(x) = x^3 - tr(x)x^2 + \frac{1}{2}(tr(x)^2 - tr(x^2))x - \frac{1}{3}tr(x^3) - \frac{1}{6}tr(x)^3 + \frac{1}{2}tr(x^2)tr(x).$$

It is convenient to use also the *multilinear* form of the Cayley–Hamilton identity and of the symmetric functions $\sigma_i(x)$, which can be obtained by the process of *full polarization*.

For this, given a permutation $\sigma \in S_m$, we decompose

$$\sigma = (i_1 i_2 \dots i_h) \dots (j_1 j_2 \dots j_\ell)(s_1 s_2 \dots s_t)$$

in cycles and we set:

$$T_\sigma(x_1, x_2, \dots, x_m) \tag{9}$$

$$= tr(x_{i_1} x_{i_2} \dots x_{i_h}) \dots tr(x_{j_1} x_{j_2} \dots x_{j_\ell}) tr(x_{s_1} x_{s_2} \dots x_{s_t}). \tag{10}$$

This is justified by the classical Formula due to Kostant

$$tr(\sigma^{-1} \circ x_1 \otimes x_2 \otimes \dots \otimes x_n) = T_\sigma(x_1, \dots, x_n).$$

From the basic elements T_σ of Formula (9) take $m = k + 1$.

We may assume that the last cycle ends with $s_t = k + 1$ so the last factor is of the form $tr((x_{s_1} x_{s_2} \dots x_{s_{t-1}}) x_{k+1})$, hence we have that

$$T_\sigma(x_1, x_2, \dots, x_{k+1}) = tr(\psi_\sigma(x_1, x_2, \dots, x_k) x_{k+1}) \tag{11}$$

where $\psi_\sigma(x_1, x_2, \dots, x_k)$ is the element of $\mathcal{F}_T\langle X \rangle$ given by the formula

$$\psi_\sigma(x_1, x_2, \dots, x_k) \tag{12}$$

$$= tr(x_{i_1} x_{i_2} \dots x_{i_h}) \dots tr(x_{j_1} x_{j_2} \dots x_{j_\ell}) x_{s_1} x_{s_2} \dots x_{s_{t-1}}.$$

Then we have the multilinear form,

Proposition 1.7.

(1) *For each k the polarized form of $\sigma_k(x)$ is the expression*

$$T_k(x_1, \dots, x_k) = \sum_{\sigma \in S_k} \epsilon_\sigma T_\sigma(x_1, \dots, x_k). \tag{13}$$

(2) *The polarized form of* $CH_n(x)$ *is*

$$CH(x_1,\dots,x_n)=(-1)^n\sum_{\sigma\in S_{n+1}}\epsilon_\sigma\psi_\sigma(x_1,x_2,\dots,x_n). \tag{14}$$

Here ϵ_σ *denotes the sign of* σ.

(3)

$$tr(CH(x_1,\dots,x_n)x_{n+1})=(-1)^nT_{n+1}(x_1,\dots,x_{n+1}). \tag{15}$$

For the proof see [4] or Proposition 12.1.12 in [1].

Example $n=2$ ($CH_2(x)$ and its polarized multilinear form).

$$CH_2(x)=x^2-tr(x)x+det(x)=x^2-tr(x)x+\frac{1}{2}(tr(x)^2-tr(x^2)). \tag{16}$$

$$\begin{aligned}x_1x_2+x_2x_1-tr(x_1)x_2-tr(x_2)x_1-tr(x_1x_2)+tr(x_1)tr(x_2)\\ =CH_2(x_1+x_2)-CH_2(x_1)-CH_2(x_2).\end{aligned}$$

Also from the decomposition into cosets $S_{n+1}=S_n\bigcup_{i=1}^{n}S_n(i,n+1)$, of the symmetric group, one has the recursive formula

$$T_{n+1}(x_1,\dots,x_{n+1})=T_n(x_1,\dots,x_n)tr(x_{n+1})-\sum_{i=1}^{n}T_n(x_1,\dots,x_ix_{n+1},\dots,x_n). \tag{17}$$

1.7. 1987 Cayley–Hamilton algebras. The notion of Cayley–Hamilton algebra (CH algebras for short), see Definition 1.8, was introduced in 1987 by Procesi [16], in characteristic 0, as an axiomatic treatment of the Cayley–Hamilton theorem. This was done in order to clarify the Theory of n–dimensional representations, [15], cf. Definition 1.1, of an associative and in general noncommutative algebra R with 1 (from now on just called *algebra*).

The definition given in [16] is the following.

Definition 1.8. *An* n*-Cayley–Hamilton algebra is an algebra over* $\mathbb{Q}$ *with trace, such that* $tr(1)=n$ *and the formal* n*-Cayley-Hamilton polynomial* (8) *is a trace identity for* R.

The Theory developed in §1.1 can be extended to n-Cayley–Hamilton algebras by just imposing that the representations be compatible with trace. We then have the universal map into $n\times n$ matrices of an n-Cayley–Hamilton algebra.

In this case questions 1 and 2 have a positive answer. The main Theorem of [16] is that

Theorem 1.9. *If* R *is an* n*-Cayley–Hamilton algebra, then the universal map* $\mathtt{j}_R:R\to M_n(T_n(R))^{\mathtt{P}_n}$ *is an isomorphism and both questions* 1) *and* 2) *have a positive answer.*

This gives a rather satisfactory answer to the embedding problem and we call it the *Strong embedding Theorem cf.* [16].

1.7.1. The case of group algebras. Motivated by the Theory of deformations of Galois representations Wiles [26] and Taylor [25], introduced the following notion:

Definition 1.10. *A pseudocharacter (or pseudorepresentation) of a group G, of degree n with coefficients in a commutative ring A, is a map $t : G \to A$ satisfying the following three properties:*

(1) $t(1) = n$.

(2) $t(ab) = t(ba),\ \forall a, b \in G$.

(3) $T_{n+1}(g_1, \dots, g_{n+1}) = 0,\ \forall g_i \in G$ *(with $T_k(g_1, \dots, g_k)$ defined from t for all k by Formula* (13)*).*

Frobenius [7] discovered already that, for A an algebraically closed field, this is a property of an n–dimensional character.

Then the problem is:

Problem 1.11. Prove that there exists a commutative A algebra B and a homomorphism ρ of G into $GL(B)$ such that the $T(g) = tr(\rho(g))$.

The ring A is assumed to be a $\mathbb{Q}$ algebra otherwise it is usually not true that the trace determines the representation. Condition 3) is necessary in view of the previous identity (15).

The fact that one needs to extend A to some B appears already for finite groups and $A = \mathbb{R}$. For instance the quaternion group with 8 elements has a 2–dimensional representation over $\mathbb{C}$ but not $\mathbb{R}$ with the trace real.

From the main Theorem 1.9 of Cayley Hamilton algebras the answer to this question is positive.

The connection between the previous pseudocharacters and Cayley Hamilton algebras is the following.

For an algebra with trace R one considers the *Kernel* of the trace, that is

$$K_t := \{a \in R \mid t(ab) = 0,\ \forall b \in R]\}.$$

One sees that K_t is a proper ideal of R and the trace induces a trace in R/K_t. Moreover if $a \neq 0$ is in the trace algebra we have that $t(a \cdot 1) = n \cdot a$ so $a \notin K_t$ and the trace algebra of R/K_t is the same as that of R.

For a pseudocharacter of a group G with values in A, one considers the group algebra $A[G]$ and then extends the map t to a trace with trace algebra A. Next one considers the *Kernel* of the trace.

It is then an easy fact to see (cf. Formula (15)) that, if t is a pseudocharacter of G of degree n, then $A[G]/K_t$ is a n–Cayley Hamilton algebra.

In particular when $A \supset \mathbb{Q}$ one can apply Theorem 1.9. In general we have:

Proposition 1.12. *If R is an algebra with trace t and the trace satisfies*

$$t(1) = n, \quad T_{n+1}(x_1, \dots, x_{n+1}) = \sum_{\sigma \in S_{n+1}} \epsilon_\sigma T_\sigma(x_1, \dots, x_{n+1}) = 0,$$

then $CH_n(x_1, \dots, x_n) \in K_t$ and R/K_t is an n– Cayley Hamilton algebra containing the trace algebra of R.

Observe that, from Formula (17) it follows that, if R satisfies the multilinear identity T_{n+1}, then it satisfies T_j, $\forall j \geq n+1$.

One can then apply the strong embedding theorem to R/K_t.

2. Characteristic free theory.

2.1. Norms. The theory was developed only in characteristic 0, for two reasons; the first being that at that time the characteristic free results of Donkin [6] and Zubkov [31] were not available.

The second reason was mostly because it looked not likely that the *main theorem* 1.9 could possibly hold in general due to the fact that its proof is based on the fact that the projective group is linearly reductive in characteristic zero but not in positive characteristic.

The first concern can now be considered to have a positive solution, due to the contributions of several people, and we may take the book [5] as reference. As for the second, that is the main theorem in positive characteristic, the issue remains unsettled.

The present author feels that it should not be true in general that the map $\mathtt{j}_R$ of Formula (3) is injective or surjective to the invariants, but has no counterexamples. A recent paper [9] gives a counterexample to the strictly related problem of pseudocharacters, see §3, so it gave me new interest in the question.

A partial theory in general characteristics, replacing the trace with the determinant, appears already in Procesi [13] and [17].

For the general definition we follow Chenevier [3]:

Definition 2.1. *Given an algebra R over a commutative ring A and $n \in \mathbb{N}$, an A* valued *n–*norm *is a multiplicative polynomial law, as A–modules, $N : R \to A$ homogeneous of degree n (see Definition 2.2).*

An algebra R over a commutative ring A with an n–norm $N : R \to A$ is a Cayley Hamilton algebra if, for every commutative A algebra B, each $a \in B \otimes_A R$ satisfies its characteristic polynomial *$\chi_a(t) := N(t - a)$, that is $\chi_a(a) = 0$.*

It is not hard to see that, if A contains $\mathbb{Q}$ then this definition is equivalent to Definition 1.8. On the other hand there are several problems with this definition. One can construct the universal map into matrices compatible with the norm (for matrices the determinant) but then one has to prove several non trivial Theorems even to start to develop some Theory.

The starting point is to interpret the characteristic free results of Donkin [6] and Zubkov [31] describing the invariants of matrices by generators and relations as a description of the free Cayley–Hamilton algebra.

The next step which is sketched in the next paragraph is to formulate this description in terms of a theory of determinants of Ziplies in [30] and Vaccarino in [27, 28].

Thus the theory can be developed through the axiomatization of a *trace* or of *determinant*. In the first case it is closer to the Theory of *pseudocharacters* of groups, see [18]. The definition via trace is also closer to the language of Universal algebra, while the one using norms is more categorical in nature.

Recently there has been a renewed interest in these topics, we point out two recent preprints. The first [11] by Mohamed Moakher, Julian Quast *Symplectic determinant laws and invariant theory* arXiv:2310.15822 in which a symplectic analogue is developed, this could be considered as a study of Cayley–Hamilton algebras with symplectic involution, and then [9] by Jinyue Luo, *Pseudorepresentations not arising from genuine representations*, `https://arxiv.org/pdf/2310.16953.pdf` containing the proposed counterexample.

2.2. Polynomial laws and determinants. Since the notion of polynomial law appears throughout this paper we start by recalling the definitions.

Given a commutative ring F, an F module M, and a commutative F algebra B one has the *base change* functor from F–modules to B–modules:

$$ {}_BM := B \otimes_F M. \tag{18}$$

Recall that, in [23] and [24], Roby defines:

Definition 2.2. *A* polynomial law *between two F modules M, N is a natural transformation of the two set valued functors on the category $\mathcal{C}_F$ of commutative F algebras:*

$$ f_B :_B M \to_B N, \quad B \in \mathcal{C}_F. \tag{19}$$

Such a law is *homogeneous of degree n* if:

$$ f_B(ba) = b^n f_B(a),\ \forall b \in B,\ \forall a \in_B M,\ \forall B \in \mathcal{C}_F.$$

Given M let us denote, for N any module, by $\mathcal{P}_n(M, N)$ the polynomial laws homogeneous of degree n from M to N. This, by Roby's Theory, is a set valued functor on modules N and it is representable.

This is done by constructing the divided powers $\Gamma_n(M) = \Gamma_{n,F}(M)$ (over the base F) together with the map $\mu_M : m \mapsto m^{[n]}$ of M to $\Gamma_n(M)$.

The divided powers $\Gamma_{n,F}(M)$ are constructed by generators and relations. The construction is compatible with base change, that is given a commutative F algebra B we have:

$$ \Gamma_{n,B}({}_BM) = {}_B\Gamma_{n,F}(M) = B \otimes_F \Gamma_{n,F}(M).$$

When there is no ambiguity we drop the symbol F which is assumed to be fixed and just write $\Gamma_n(M) := \Gamma_{n,F}(M)$.

In fact in most applications there is a more concrete description of $\Gamma_n(M)$. For instance if M is a free (or just projective) F module one describes the divided power as symmetric tensors:

$$\Gamma_n(M) \simeq (M^{\otimes n})^{S_n}, \quad M^{\otimes n} = M \otimes_F M \otimes_F M \otimes \cdots \otimes_F M, \tag{20}$$

and $m^{[n]} = m \otimes m \otimes \ldots \otimes m = m^{\otimes n}$.

From now on we will assume to be in this case. Notice that, by change of coefficient rings, if $F \to A$ is a map of commutative rings we have ${}_AM := A \otimes_F M$ and

$${}_AM^{\otimes n} = (A \otimes_F M)^{\otimes n}, \quad {}_A\Gamma_n(M) = \Gamma_n({}_AM).$$

Remark 2.3. In general a polynomial law is not determined by the map $f : M \to N$. Under further hypothesis this is the case, the simplest being that the base commutative ring F contains an infinite field.

2.2.1. Multiplicative polynomial laws. If we have two F–algebras R, S we have the notion of *multiplicative polynomial law* $d : R \to S$ that is a polynomial law satisfying:

$$d(1) = 1, \quad d(ab) = d(a)d(b), \ \forall a, b \in_B R, \ \forall B.$$

2.2.2. Schur algebras. One proves that, if R is an algebra, then $\Gamma_n(R)$ is also an algebra, which we call *the n^{th}–Schur algebra of R*, see [5], and μ_R is a universal multiplicative polynomial map, homogeneous of degree n.

That is any multiplicative polynomial law, homogeneous of degree n from R to an algebra S factors through μ_R and a homomorphism of $\Gamma_n(R)$ to S.

When we have $\Gamma_n(R) = (R^{\otimes n})^{S_n}$, Formula (20), the algebra structure on $\Gamma_n(R)$ is induced by the tensor product of algebras.

Denoting by $\mathcal{M}_n(R, S)$ the set of multiplicative polynomial laws homogeneous of degree n from R to S we have the isomorphism:

$$\phi : d \mapsto d \circ \mu_R, \quad \hom_{\mathcal{R}}(\Gamma_n(R), S) \overset{\phi}{\simeq} \mathcal{M}_n(R, S), \quad S \in \mathcal{R}. \tag{21}$$

If S is any algebra we denote by S_{ab} its *abelianization*, that is S modulo the ideal C_S generated by all commutators $[x, y] = x \cdot y - y \cdot x$.

When we restrict, in Formula (21), to S commutative, this functor, on commutative algebras, is represented by $\pi : \Gamma_n(R) \to \Gamma_n(R)_{ab}$ that is the abelianization of $\Gamma_n(R)$.

$$\mathcal{M}_n(R, A) \simeq \hom_{\mathcal{C}}(\Gamma_n(R)_{ab}, A), \quad A \in \mathcal{C}. \tag{22}$$

Definition 2.4. *We call the commutative algebra $\Gamma_n(R)_{ab}$ the n^{th}* norm algebra of R *and the induced polynomial map $r \to r^{\otimes n}$ the n^{th}* universal norm of R.

For a group G n^{th} universal norm of its group algebra $F[G]$ is called the n^{th} universal pseudocharacter of G. *Notice that now such a pseudocharacter should be considered as a function on G^n and not on G.*

This is an object studied by Roby in [24], and by Ziplies in [30] and Vaccarino in [27], [28] and discussed in some detail in my book with De Concini [5].

2.3. Determinants. Given an algebra R the composition of a homomorphism $j : R \to M_n(A)$ with the determinant

$$\det \circ j :_B R \xrightarrow{j} M_n({}_BA) \xrightarrow{\det}_B A \tag{23}$$

is a multiplicative polynomial law, homogeneous of degree n. One then obtains a natural transformation of functors:

$$\hom_{\mathcal{R}_F}(R, M_n(A)) \to \mathcal{M}_n(R, A), \tag{24}$$

and a commutative diagram.

$$\begin{array}{ccccc} R & \xrightarrow{\mathrm{j}_R} & M_n(T_n(R)) & \xrightarrow{\det} & T_n(R) \\ & \searrow^{f} & \downarrow{\scriptstyle M_n(\bar{f})} & & \downarrow{\scriptstyle \bar{f}} \\ & & M_n(A) & \xrightarrow{\det} & A. \end{array} \tag{25}$$

Since $T_n(R)$ is commutative, $\det \circ j_R$ factors as

$$R \xrightarrow{i_R} \Gamma_n(R) \xrightarrow{\pi} \Gamma_n(R)_{ab} \xrightarrow{D} T_n(R),\ D \circ \pi \circ i_R = \det \circ j_R. \tag{26}$$

The homomorphism $D : \Gamma_n(R)_{ab} \longrightarrow T_n(R)$ is clearly invariant under the group of automorphisms $\mathfrak{G}_n$ so it factors through $T_n(R)^{\mathfrak{G}_n}$.

Remark 2.5. [Problem] One of the main problems of the theory is to understand when is that $D : \Gamma_n(R)_{ab} \to T_n(R)^{\mathfrak{G}_n}$ is an isomorphism.

This happens in several cases but not always as the examples in the last section show.

In this example D is not injective, but we have no clue about the surjectivity of the map.

We start with an important example.

If F is an infinite field and V a vector space of some finite dimension m over F one has, for $R = End_F(V)$, that

$$(End_F(V)^{\otimes n})^{S_n} = End_F(V^{\otimes n})^{S_n} = End_{F[S_n]}(V^{\otimes n}).$$

The algebra $End_{F[S_n]}(V^{\otimes n})$ is spanned by the elements $g^{\otimes n}$ where $g \in GL(V)$, this in characteristic 0 is the classical Schur–Weyl duality proved in [4] in a characteristic free form.

If the characteristic of F is 0 one has the decomposition

$$V^{\otimes n} = \oplus_{\lambda \vdash n | ht(\lambda) \leq m} M_\lambda \otimes S_\lambda(V),$$

where M_λ is the irreducible representation of S_n corresponding to λ and $S_\lambda(V)$, a Schur functor, the corresponding irreducible representation of $GL(V)$. One proves that

$$End_F(V^{\otimes n})^{S_n} = \oplus_{\lambda \vdash n | ht(\lambda) \leq m} End(S_\lambda(V)). \tag{27}$$

Each summand of this decomposition is a simple algebra and abelian only when $S_\lambda(V)$ is 1–dimensional. This happens only if $n = i \cdot m$ and $S_\lambda(V) = \bigwedge^m (V)^{\otimes i}$. In this case

$$(End_F(V)^{\otimes n})^{S_n}_{ab} = End(\bigwedge^m (V)^{\otimes i}),\ N(a) = \det(a)^i,\ a \in End_F(V). \tag{28}$$

By the remarkable theory of *good filtrations* or combinatorially of standard double tableaux, in positive characteristic or over the integers the previous formula can be replaced by a canonical filtration of which Formula (27) is the associated graded representation. From this one could deduce a general form of Formula (28).

We have a fundamental result, Theorem 20.24 of [5].

Theorem 2.6. *Consider the free algebra $A\langle X\rangle$ in some set of variables $X = \{x_i\}_{i\in I}$ with A a field or the integers.*
Then $D : \Gamma_n(A\langle X\rangle)_{ab} \to T_n(A\langle X\rangle)^{\mathfrak{S}_n}$ is an isomorphism.

Notice that $T_n(A\langle X\rangle)^{\mathfrak{S}_n}$ is *the ring of invariants of X-tuples of $n \times n$ matrices.*

Theorem 2.6 is based on a Theorem of Procesi [14] and Razmyslov [21] characterizing trace identities of matrices and proved by Zieplies [29], when $A = \mathbb{Q}$ and Vaccarino [27], when $A = \mathbb{Z}$.

The general case of Vaccarino is based on the characteristic free results of Donkin [6] and Zubkov [31] on the invariants of matrices and a careful combinatorial study of $\Gamma_n(A\langle X\rangle)$ inspired by the work of Zieplies, (see also [5]).

In fact since $A\langle X\rangle$ is a free A module, its divided power is more conveniently described as the symmetric tensors:

$$\Gamma_n(A\langle X\rangle) \simeq (A\langle X\rangle^{\otimes n})^{S_n}. \tag{29}$$

Since, using the basis of monomials, the space $A\langle X\rangle^{\otimes n}$ is a permutation representation of S_n, one has a combinatorial description of $(A\langle X\rangle^{\otimes n})^{S_n}$. The abelian quotient, isomorphic to the ring of invariants of matrices, does not have a combinatorial description and it is a rather hard object to study.

Example 2.7. If $X = \{x\}$ is a single variable we have that $A\langle X\rangle = A[x]$ is the commutative ring of polynomials, $A[x]^{\otimes n} = A[x_1, \dots, x_n]$ so $\Gamma_n(A[x])$ is the algebra of symmetric polynomials in n–variables, commutative.

Consider the elementary symmetric function e_i defined by:

$$(t - x)^{\otimes n} = \prod_{i=1}^{n} (t - x_i) = t^n - e_1 t^{n-1} + e_2 t^{n-2} - \ldots + (-1)^n e_n. \tag{30}$$

The map D maps $(t-x)^{\otimes n}$ to $\det(t-\xi) = t^n + \sum_{i=1}^{n}(-1)^i\sigma_i(\xi)t^{n-i}$, the characteristic polynomial of a generic matrix $\xi = (\xi_{i,j})$.

So the elementary symmetric function e_i maps to $\sigma_i(\xi)$, the generators of invariants of a single matrix.

This is a very special case of Theorem 2.6.

2.4. Generalized symmetric functions. Let us discuss the case of the free algebra (29) over $A = \mathbb{Z}$.

The algebra $\mathbb{Z}\langle X\rangle^{\otimes n}$ has a basis over $\mathbb{Z}$ formed by tensor products on monomials which is permuted by S_n, therefore $\mathbb{Z}\langle X\rangle^{\otimes n})^{S_n}$ has a basis indexed by the orbits of S_n on this tensor basis.

In the isomorphism between its abelianization and the invariants of matrices a tensor $u^{\otimes n}$, $u = u(x_1, \cdots, x_m)$ is identified to the determinant $\det(u(x_1, \cdots, x_m))$ where the x_i are evaluated into matrices.

As a consequence one sees that the orbit of a tensor $m^{\otimes h} \otimes 1^{\otimes n-h}$ with m a monomial evaluates to the h coefficient $\sigma_h(m)$ of the characteristic polynomial of m evaluated into matrices. As in the theory of symmetric functions one can pass to the limit for n to ∞ and think of the $\sigma_h(m)$ as stable polynomial laws.

Then there are universal formulas ruling these stable laws [5] Theorem 4.15 p. 37. This is due to Amitsur, [2], (who stated it for matrix invariants), and later independently by Reutenauer and Schützenberger [22].

Theorem 2.8 (Amitsur)**.** *Given an integer $n \in \mathbb{N}$, non commutative variables x_i and commutative parameters t_i:*

$$\sigma_n\left(\sum_i t_i x_i\right) = \sum_{\substack{(p_1<\ldots<p_k)\subset W_0,\\ j_1,\ldots,j_k\in\mathbb{N},\ \sum j_i\ell(p_i)=n}} (-1)^{n-\sum j_i} t^{\sum_{i=1}^{k} j_i\nu(p_i)}\sigma_{j_1}(p_1)\ldots\sigma_{j_k}(p_k) \tag{31}$$

Here W_0 denotes the set of Lyndon words ordered by the degree lexicographic order. For a word p, $\nu(p)$ is the vector $(a_1, \ldots, a_n)$ with a_i counting how many times the variable x_i appears in p. Finally $t^{(a_1,\ldots,a_n)} := \prod_{i=1}^{n} t_i^{a_i}$.

Recall that a *primitive word* is a word which is not a proper power of a shorter word and a Lyndon word is a primitive word which is minimal among its cyclic equivalence under lexicographic order.

2.5. The group case. Given a group G we want to understand what could be a pseudocharacter in positive characteristic.

One can start to describe the universal norm of degree n on the group algebra $\mathbb{Z}[G]$ or $\mathbb{Z}/(p)[G]$.

The algebra $\mathbb{Z}[G]^{\otimes n}$ has as basis the elements $g_1 \otimes g_2 \otimes \cdots \otimes g_n$ and its symmetric part $[\mathbb{Z}[G]^{\otimes n}]^{S_n}$ the S_n orbits $O(g_1, g_2, \cdots, g_n)$ of these elements. One has then the n^{th} universal norm algebra

$$A_n(G) := [\mathbb{Z}[G]^{\otimes n}]^{S_n}_{ab}. \tag{32}$$

The multiplication of the orbits is given by universal formulas which resemble Amitsur's Theorem.

Then a *pseudocharacter* of G with values in a commutative ring A is a homomorphism of $A_n(G)$ to A. It may be given as a function on the classes of the orbits $O(g_1, g_2, s, \cdots, g_n)$.

The condition that such a function corresponds to a homomorphism is given by a set of complicated identities the and one can refer to [3].

In characteristic 0 these functions are deduced all by the *trace* σ_1 using the identities of symmetric functions, hence the name.

In this case the main question is to understand if a pseudocharacter comes from a representation into the $n \times n$ matrices over some commutative A algebra.

The complicated formulas are discussed in detail in the case $n = 2$, see Section 3.

Remark 2.9. In particular one has the universal norm with values in $A_n(G)$ and then this comes from a representation into the $n \times n$ matrices over some commutative A algebra if and only if the universal map D of Formula (26) is injective.

This is not always true as the example of Jinyue Luo discussed in §3, shows.

2.6. A universal Cayley–Hamilton algebra. In the recent paper [19] I have introduced the following construction.

Given any associative F–algebra R with 1, we have seen that $\Gamma_n(R)_{ab}$ can be viewed as a *universal determinant*, Formula (26). This gives rise to a norm on $R \otimes \Gamma_n(R)_{ab}$

$$R \otimes_F \Gamma_n(R)_{ab} \to \Gamma_n(R)_{ab} \otimes_F \Gamma_n(R)_{ab} \xrightarrow{\mu} \Gamma_n(R)_{ab},\ \mu(a \otimes b) := a \cdot b. \tag{33}$$

We can then impose as relations all evaluations of the characteristic polynomials (as polynomial laws).

The resulting quotient algebra will be denoted by $C_n(R)$. It is universal in the sense that, given any algebra homomorphism $f : R \to S$ with S a Cayley–Hamilton algebra, the map factors through a map $\bar{f} : C_n(R) \to S$ of Cayley–Hamilton algebras, and we refer to [19] for some general properties.

In particular if $f : R \to M_n(A)$ is the universal map into $n \times n$ matrices over a commutative ring we have a factorization

$$\begin{array}{ccccccc} f : R & \xrightarrow{i} & C_n(R) & \xrightarrow{\bar{f}} & M_n(A)^{\mathfrak{G}_n} & \longrightarrow & M_n(A)\ . \\ \downarrow{\scriptstyle r^{\otimes n}} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\ A_n(R) & \xrightarrow{\bar{i}} & A_n(R) \cdot 1 & \xrightarrow{j} & A^{\mathfrak{G}_n} & \longrightarrow & A \end{array} \tag{34}$$

If R is an algebra over $\mathbb{Q}$ then by the strong embedding Theorem $\bar{f}$, $\bar{i}$, j are all isomorpisms.

Im positive characteristic for the free algebra these maps are isomorphisms, this is the interpretation by Ziplies and Vaccarino, of the statement of the first and second fundamental theorem of invariant theory.

For a general algebra R, for the moment there is only the counterexample of Jinyue Luo, discussed in §3, showing that $\bar{i}$ need not be injective.

In [19] we have given a very explicit description of $C_n(R)$ when $R = \oplus_i M_{n_i}(F)$.

What I want to do is to discuss the case in which $R = F[G]$ the group algebra of a finite group and to try to understand the role of the characteristic of F. The idea, following Jinyue Luo, is to use some very simple finite 2-groups and $F = \mathbb{Z}/(2)$, $n = 2$ so that the complicated Amitsur's identities and the construction of $C_n(R)$ can be almost done by hand, hoping to find in this simple cases a counterexample of the embedding problem.

For the moment I tried to use Luo's example but in this case the algebra embeds ito the universal representation!

2.7. Semisimple case.

Example 2.10. Before doing this let me recall the example where F is algebraically closed and $R = \oplus_{i=1}^{k} M_{n_i}(F)$, is a direct sum of matrix algebras over the base field F.

In [19] given n, I consider the set

$$X_R := (h_1, \cdots, h_k) \mid h_i \in \mathbb{N},\ \sum_i h_i n_i = n \tag{35}$$

Observe that, while for the n_i we require $n_i \neq 0$ for the h_i we allow also the choice $h_i = 0$.

Given $\underline{h} \in X_R$ set $R_{\underline{h}} := \oplus_{i|h_i \neq 0} M_{n_i}(F)$ set $(j_1, \ldots, j_t)$ the indices with $h_i \neq 0$. We then can equip this algebra with a structure of Cayley–Hamilton algebra by setting

$$N(x_{j_1}, \ldots, x_{j_t}) := \prod_i \det(x_{j_i})^{h_i}.$$

This Cayley–Hamilton algebra is simple in the category of Cayley–Hamilton algebras, and two different ones of these are clearly not isomorphic, with an isomorphism which is the identity on R.

Observe that this is the usual determinant in the semisimple representation in which the irreducibles appear each h_i times. We now can form the direct sum of all these Cayley–Hamilton algebras.

$$C_R := \oplus_{\underline{h} \in X_R} R_{\underline{h}}. \tag{36}$$

This is a Cayley–Hamilton algebra with norm algebra $B := \oplus_{\underline{h} \in X_R} F$.

Theorem 2.11 (Theorem 2.25 of [19]). *We have*

$$\Gamma_n(R)_{ab} = \oplus_{\underline{h}\in X_R} F, \quad \textit{and} \quad C_n(R) = C_R := \oplus_{\underline{h}\in X_R} R_{\underline{h}}. \tag{37}$$

In particular

$$\dim_F \Gamma_n(R)_{ab} = \|X_R\|, \quad \dim_F C_n(R) = \sum_{\underline{h}\in X_R} \sum_{i|h_i\neq 0} n_i^2. \tag{38}$$

Remark 2.12. The construction of $C_n(R)$ commutes with the change of base field, that is if $F \subset G$ is a field extension

$$G \otimes_F C_n(R) = C_n(G \otimes_F R) \tag{39}$$

In particular the theory developed for direct sums of matrix algebras extends to finite dimensional separable algebras.

3. Possible counterexamples.

3.1. Some general considerations.

Remark 3.1. Both the algebra $C_n(F[G]), A_n(G)$ (Definition 32) are defined over $\mathbb{Z}$. This implies that for any prime p

$$\dim_{\mathbb{Q}} C_n(\mathbb{Q}[G]) \leq \dim_{\mathbb{Z}/(p)} C_n(\mathbb{Z}/(p)[G]),$$

$$\dim_{\mathbb{Q}} A_n(\mathbb{Q}[G]) \leq \dim_{\mathbb{Z}/(p)} A_n(\mathbb{Z}/(p)[G])$$

Having a precise description of $C_n(F[G])$ one then needs a description of the universal map in $n \times n$ matrices. One works first over $\mathbb{Z}$.

One presents G by generators g_i and relations, for each i one has that g_i maps to some $n \times n$ matrix

$$\xi_i := \begin{vmatrix} a^i_{1,1} & a^i_{1,2} & \cdots & a^i_{1,n} \\ \cdots & & & \\ a^i_{n,1} & a^i_{n,2} & \cdots & a^i_{n,n} \end{vmatrix}$$

where the elements $a_{i,j}$ lie in the polynomial ring over $\mathbb{Z}$ given by these generators and the relations obtained from the relations among the elements g_i.

In this case since we have that all constructions can be done over $\mathbb{Z}$, an analogue of Remark 3.1 holds.

In fact the relations among the g_i give rise to polynomial relations with integer coefficients, so one should first consider the ring $U_n(G)$ quotient of the polynomial ring over $\mathbb{Z}$ in the $a_{i,j}$ modulo the relations.

This ring need not be torsion free but it may have some torsion.

$\mathbb{Q} \otimes_{\mathbb{Z}} U_n(G)$ is the universal ring in characteristic 0 while $\mathbb{Z}/(p) \otimes_{\mathbb{Z}} U_n(G)$ is the universal ring in characteristic p.

3.2. n = 2. Let us discuss the case $n = 2$, if $F = \mathbb{Q}$ then assume that G has a, 1-dimensional irreducible representations and b, 2-dimensional irreducible representations. Then it has a two dimensional representations sum of two isomorphic irreducible representations, for each the dimension of $R_{\underline{h}}$ is 1. Then $\binom{a}{2}$ two dimensional representations sum of two non isomorphic irreducible representations, for each the dimension of $R_{\underline{h}}$ is 2. Finally for each of the 2–dimensional irreducible representations the dimension of $R_{\underline{h}}$ is 4. A total dimension

$$\dim_{\mathbb{Q}} C_2(\mathbb{Q}[G]) = a^2 + 4b, \quad \dim_{\mathbb{Q}} A_2(\mathbb{Q}[G]) = \binom{a+1}{2} + b. \tag{40}$$

The algebra $(R \otimes R)^{S_2}$ can be defined over $\mathbb{Z}$ and has as possible basis the elements

$$d(g) := g \otimes g,\ t(g,h) = t(h,g) := g \otimes h + h \otimes g,\ h \neq g \in G. \tag{41}$$

$$\dim_{\mathbb{Z}}(R \otimes R)^{S_2} = \binom{|G|+1}{2}. \tag{42}$$

The multiplication rules are

$$d(g)d(h) = d(gh),\ d(g)t(a,b) = t(ga,gb),\ t(a,b)d(g) = t(ag,bg) \tag{43}$$

$$t(a,b)t(c,d) = t(ac,bd) + t(ad,bc).$$

If $ac = bd$ then $t(a,b)t(c,d) = t(ad,bc)$ similarly if $ad = bc$ then $t(a,b)t(c,d) = t(ac,bd)$.

Set $t(h) := t(h,1)$ so $d(g) = g \otimes g$ and $t(h) = h \otimes 1 + 1 \otimes h$ generate

$$t(a,b) = t(a)t(b) - t(ab) = d(a)t(a^{-1}b) = d(b)t(b^{-1}a) = t(ba^{-1})d(a) = t(ab^{-1})d(b)$$

$$\begin{aligned} t(a)t(b)t(c) &= t(abc) + t(ab,c) + t(ac,b) + t(a,bc) \\ &= t(abc) + t(ab)t(c) - t(abc) + t(ac)t(b) - t(acb) + t(a)t(bc) - t(abc) \\ &= t(ab)t(c) + t(ab)t(c) + t(ac)t(b) - t(acb) + t(a)t(bc) - t(abc). \end{aligned}$$

For the commutator ideal one can construct a basis using

$$[d(g), d(h)] = d[gh] - d[hg],\ [d(g), t(h,k)] = t(gh,gk) - t(hg,kg),$$

$$[t(f,g), t(h,k)] = t(fh,gk) + t(gh,fk) - t(hf,kg) + t(hg,kf).$$

Let us denote the classes of these elements in $A_2(G) := (R \otimes R)^{S_2}_{ab}$

$$D(g) := g \otimes g, \quad T(g) := g \otimes 1 + 1 \otimes g,\ T(g,h) = T(h,g) := g \otimes h + h \otimes g,\ h \neq g \in G. \tag{44}$$

We have the following identities in $A_2(G)$

$$D(gh) = D(g)D(h), \quad T(gh) = T(hg),\ T(1) = 2$$

$$D(g)T(g^{-1}h) - T(g)T(h) + T(gh) = 0$$

$$\iff D(g)T(g^{-1}h) + T(gh) = T(g)T(h) \tag{45}$$

$$\implies D(g)T(g^{-1}h) = D(h)T(h^{-1}g)$$

or

$$T(k) = D(k)T(k^{-1}).$$

If $2 = 0$ we have $T(h^2) = T(h)^2$. On the other hand the full relations depend upon G since even in characteristic 0 the dimension of $A_2(G)$ depends on G and in fact upon the number of irreducible representations of G of dimensions 1 and 2, see the general formula (35) and Theorem 2.11.

Once we have computed $A_2(G)$ the construction of the Cayley–Hamilton algebra is by considering first $A_2(G)[G]$ the group algebra of G with coefficients $A_2(G)$. Then we want to impose the Cayley–Hamilton identity as polynomial map and this consists in imposing in $A_2(G)[G]$ the relations

$$g^2 - T(g)g + D(g) = 0,\ hg + gh - T(h)g - T(g)h + T(h)T(g) - T(hg) \tag{46}$$

for all $h, g \in G$.

That is one has to describe the quotient $C_2(F[G]) =: S_2(G) := A_2(G)[G]/I$ of $A_2(G)[G]$ modulo the ideal I generated by the elements (46). In characteristic 0 this is Theorem 2.11.

The simplest example is $G := \mathbb{Z}/(2) = \{1, \tau\}$. Then $A_2(G)$ has a basis over $\mathbb{Z}/(2)$ by $1, T(\tau),\ D(\tau)$ with

$$D(\tau)^2 = 1,\ T(\tau)^2 = 0,\ D(\tau)T(\tau) = T(\tau)$$

The relations (46) reduce to

$$1 + T(\tau)\tau = D(\tau)$$

so $S_2(G)$ has a basis over $\mathbb{Z}/(2)$ by $1,\ \tau,\ T(\tau),\ T(\tau)\tau$.

Multiplication given by $\tau^2 = 1,\ T(\tau)^2 = 0$.

The universal ring is generated by elements a, b, c, d with

$$\tau^2 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^2 = 1 \iff a^2 + bc = d^2 + bc = 1,\ ab + bd = ac + cd = 0$$

One has $T(\tau) = a + d$ and one easily sees that $S_2(G)$ embeds into its universal ring. In fact it is easier to use $t := a + d$ as one of the variables and

$$\tau = \begin{vmatrix} t - d & b \\ c & d \end{vmatrix},\ t^2 = tb = tc = 0.$$

The universal ring is

$$\mathbb{Z}/(2)[b,c] \oplus \mathbb{Z}/(2)t.$$

In general one can use Grobner bases in order to verify if the mapping is an embedding, by extracting from the finite monomials appearing in the formulas a linearly independent list and then just using linear algebra.

Conjecture. If G is abelian then $S_2(G)$ embeds into its universal ring.

I think this follows from an argument by Jinyue Luo

So for a counterexample one may try for G a non abelian 2–group with $F = \mathbb{Z}/(2)$.

The simplest are D_4 the dihedral group and H_4 the quaternion group.

Their group algebras are not isomorphic over $\mathbb{Q}$ but actually also not isomorphic over $\mathbb{Z}/(2)$ since one shows that the two commutative algebras $A_2(G)$ are not isomorphic over $\mathbb{Z}/(2)$ by computing the dimensions of the powers of the two radicals.

$$D_4 = \{1, a, b, b^2, b^3, ab, ab^2, ab^3\}, \quad a^2 = 1,\ b^4 = 1,\ ab = b^3a$$

the center is $1, b^2$, the elements of order 4 are b, b^3 and are conjugate by a the ones of order 2 are a, b^2, ab, ab^2, ab^3 divided in 4 conjugacy classes $b^2, \{a, ab^2\}, \{ab, ab^3\}$. Moreover b^2 is a commutator so $D(b^2) = 1$ and by Cayley–Hamilton $T(b^2)b^2 = 0$. We have generators for $S_2(D_4)$

$$D(a)^2 = D(b)^2 = 1,\ T(a),\ T(b),\ T(ab)$$

relations

$$T(a)^2 = T(ab)^2 = T(b)^4 = 0.$$

The program I describe later shows that in this case the algebra $C_{2,\mathbb{Z}/(2)}(D_4)$ embeds, but the pseudocharacter has dimension 11 and in the map to $C_{2,\mathbb{Z}/(2)}(D_4)$ has a 1–dimensional kernel. Same results for H_4.

3.2.1. A candidate. In a recent preprint, [9] Jinyue Luo presents a counterexample for a finite group with 32 elements based on computer computations.

3.3. The group. The group is the following, let D_4 be the dihedral group with 8 elements given by 2 generators a, b and and relations

$$a^2 = 1, b^4 = 1, aba = b^3.$$

Its center is $1, b^2$ of order 2.

The group is $G := D_4 \times D_4/H$ modulo $H = \{(1,1), (b^2, b^2)\}$ the identification of the two centers.

G has center $C = \{1, (1, b^2) = (b^2, 1)\}$ and modulo its center is isomorphic to $\mathbb{Z}/(2)^4$.

It has 12 elements of order 4 whose square is $(1, b^2) = (b^2, 1)$. The other elements different from 1 have square 1.

She shows that the element $t(1, b^2) \neq 0$ in the universal pseudocharacter maps to 0 in the universal representation over $\mathbb{Z}$.

She also claims, Theorem 4.2, that D_4 does not provide a counterexample but as I will explain this depends on the fact that she works over $\mathbb{Z}$ while I seek examples directly in $\mathbb{Z}/(2)$. Also for $D4$ the element $t(b^2) \neq 0$ over $\mathbb{Z}$ there is no torsion but reducing modulo 2 this maps to 0 in the universal representation.

The fact that $t(x) = 0$ already in $C_2(\mathbb{Z}/(2)[G])$ has a simple general explanation.

Proposition 3.2. *Given a group G let $x \in G$ be an element of order 2, i.e. $x^2 = 1$ and also a commutator. Then $t(x)$ maps to 0 in the associated Cayley–Hamilton algebra over $\mathbb{Z}/(2)$.*

P r o o f. Since x is a commutator its determinant is 1, then the Cayley–Hamilton identity

$$0 = x^2 + t(x)x + d(x) = 1 + t(x)x + 1 = t(x)x \implies t(x) = 0. \qquad \square$$

Now for $x = b^2$ we have $x^2 = 1$. The fact that $d(x) = 1$ depends upon the fact that x is a commutator since $b^2 = aba^{-1}b^{-1} = abab^3 = b^3b^3$.

For the fact that $t(x) \neq 0$ in A, I have no theoretical explanation.

In fact one can compute that the universal pseudocharacter for D_4 has dimension 11 (as in Characteristic 0) over $\mathbb{Z}/(2)$ but its image in the universal Cayley–Hamilton algebra (and in the universal representation) has dimension 10.

In fact its universal Cayley–Hamilton algebra embeds in the universal representation.

The algebra $C_2(\mathbb{Z}/(2)[D_4])$ has dimension 20 over $\mathbb{Z}/(2)$ which agrees with the dimension over $\mathbb{Q}$.

Finally the canonical map $C_2(\mathbb{Z}/(2)[D_4]) \to C_2(\mathbb{Z}/(2)[G])$ has a 1–dimensional kernel.

Remarks. It is easily seen that over $\mathbb{C}$ the group G has 16 irreducible 1–dimensional representations and one 4–dimensional irreducible representation.

Let us apply Example 2.10 to $C_2(\mathbb{C}[G]) = C_2(\mathbb{C}^{16} \oplus M_4(\mathbb{C})) = C_2(\mathbb{C}^{16})$. We have thus the subgroup C maps to 1, $G/H = \mathbb{Z}/(2)^4$ has 16, 1–dimensional irreducible representations so it has $16 + \binom{16}{2} = 136$ non isomorphic 2–dimensional representations and $A_2(\mathbb{C}[G]) = \mathbb{C}^{136}$ while the dimension of $C_2(\mathbb{C}[G])$ is $16 + 2\binom{16}{2} = 256$.

On the other hand by the computations of Jinyue Luo one has that $\Gamma_2(\mathbb{Z}[G])_{ab}$ has a non 0 two torsion and giving an example of a Pseudorepresentation not arising from genuine representation.

In particular the dimension of $A := A_2(\mathbb{Z}/(2)[G])$ over $\mathbb{Z}/(2)$ is 137 just one more the characteristic zero dimension 136.

Finally I computed $\dim C_2(\mathbb{Z}/(2)[G]) = 272$ while $\dim C_2(\mathbb{Q}[G]) = 256$.

It turns out that this extra element is the class of $1 \otimes (b^2, 1) + (b^2, 1) \otimes 1$, in the universal representation this is the trace $tr(x)$ of the central element $x = (b^2, 1)$.

Some information on A. A is local with radical J and $\dim_{\mathbb{Z}/(2)} J = 136$ we have

$$\dim_{\mathbb{Z}/(2)} J^2 = 117,\ \dim_{\mathbb{Z}/(2)} J^3 = 65,\ \dim_{\mathbb{Z}/(2)} J^4 = 16,\ J^5 = 0.$$

On the other hand the group algebra over $A_2(\mathbb{Z}/(2)[G])$ has dimension 4384, over $\mathbb{Z}/(2)$, and by computer the ideal generated by the Cayley–Hamilton identities has dimension 4112=4384-272.

The computation is done as follows, let us denote for simplicity by

$$A := A_2(\mathbb{Z}/(2)[G])$$

the norm algebra.

One has to define the trace $t(i)$ of a group element i as the class in A of $\{1, i\}$ and the determinant $d(i)$ of a group element i as the class in $A_2(\mathbb{Z}/(2)[G])$ of $\{1, i\}$. Therefore the symbolic Cayley–Hamilton algebra $C_2(\mathbb{Z}/(2)[G])$ has dimension 272 while $C_2(\mathbb{Q}[G])$ has dimension 256. Moreover the group G maps injectively in $C_2(\mathbb{Z}/(2)[G])$ while it has as Kernel the center in $C_2(\mathbb{Q}[G])$.

Finally in $C_2(\mathbb{Z}/(2)[G])$ the determinant of the 32 elements of G is 1 only for the center of G. Under the map to $C_2(\mathbb{Z}/(2)[G])$ the commutative algebra A has a 1–dimensional kernel generated by $t(b^2, 1)$.

Using Grobner basis I verified by a Mathematica program, that the algebra $C_2(\mathbb{Z}/(2)[G])$ maps injectively into its universal representations in 2×2 matrices over $\mathbb{Z}/(2)$. In particular while the map of G into its universal representations in 2×2 matrices over $\mathbb{Q}$ has kernel its center C and $C_2(\mathbb{Q}[G])$ is commutative, over $\mathbb{Z}/(2)$ the group G maps injectively and $C_2(\mathbb{Z}/(2)[G])$ is non commutative.

I have also checked with my program her claims that in the universal map to 2×2 matrices the corresponding trace algebra has dimension 136 so the universal map has a 1 dimensional kernel.

All other groups with 8, 16, 32 elements give always embeddings, I started with the 267 groups with 64 elements which take a lot of computer time and for the moment I have checked more than 150 cases and have found no counterexample.

The necessary data to work with these groups have been obtained from the GAP library of small groups, using a program provided to me by Thomas Breuer whom I thank heartfully.

3.4. The Mathematica program. Some words about the Mathematica program, available on request, I describe it for G, for D_4 it is much simpler.

3.4.1. The pseudocharacter. One indexes the group G with the numbers $1, \ldots,$ 32 and first has to encode the basis of Formula (41), for the algebra $\mathtt{T} := (R \otimes R)^{S_2}$ with $R = \mathbb{Z}/(2)[G]$, with indices $\{i\}\ \{i, j\}, i < j$ and develop the multiplication table using the Formulas (43). This basis has 528 elements.

Using this one constructs a basis of the commutator ideal I which has dimension 391, so the quotient $A := \mathtt{T}/I$ has dimension 137.

This is A and one computes a set S of 137 elements from the basis of Formula (41) which form a basis for the quotient A.

3.4.2. The Cayley–Hamilton algebra. The group algebra $A[G]$ of dimension $32 \times 137 = 4384$ over $\mathbb{Z}/(2)$ has the natural basis $s \cdot i,\ s \in S,\ i \in G$.

For a group element i or for two group elements i, j one has the identities

$$i^2 + t(i)i + d(i), \quad i \cdot j + j \cdot i + t(j)i + t(i)j + t(i)t(j) + t(i \cdot j). \tag{47}$$

giving thus 496 elements of the group algebra $A[G]$. The universal Cayley–Hamilton algebra is the group algebra $A[G]$ modulo the ideal J generated by these elements. By multiplying the 496 basic identities by the natural basis of $A[G]$ formed by the 4384 elements $s \cdot i$ one obtains a list of 2174464 elements linear generators of J, many of these elements coincide so one has a total of 6688 linear generators for J.

At this point I was forced to write a program to extract from these a basis of J, I used the fact that one can identify n dimensional vectors over $\mathbb{Z}/(2)$ with subsets of $[1, \cdots, n]$ where the sum is the symmetric difference of sets.

In this way I obtained a basis with 4112 elements for J so that the universal 2-Cayley–Hamilton algebra of G has dimension 272.

At this point I had to describe the universal map in 2×2 matrices over the universal commutative $\mathbb{Z}/(2)$ algebra $A := A_2(\mathbb{Z}/(2)[G])$.

The group G has 4 generators, so I picked 16 variables to describe the corresponding 2×2 matrices.

There are then 32 group relations which give rise to 128 polynomial relations in the 16 variables. The quotient of the polynomial ring modulo these relations is the universal commutative ring A.

Using a Groebner basis and polynomial reduction I finally checked that the images of the 272 basis elements for the universal Cayley–Hamilton algebra map into 272 linearly independent matrices, so the universal map is injective.

So what to do next to try to find an example where the universal map is not injective?

One possibility is to try other finite groups.

For S_4 the algebra $A = A_2(\mathbb{Z}/(2)[S_4])$ has dimension 5 and the associateed Cayley–Hamilton algebra $C_2 = C_2(\mathbb{Z}/(2)[S_4])$ has dimension 12 but the image of A in C_2 has dimension 4. C_2 embeds into matrices.

In characteristic 0 the dimensions are 4 and 8.

The program can be easily adapted to other groups, I am running the program for groups of order 64, there are 267 such groups and I have tested for the moment half of them always getting that the CH algebra associated embeds.

Another possibility would be to consider quotients of the algebra of this or similar examples.

Observe that if R is a 2-Cayley–Hamilton algebra and I an ideal of R in order that the norm $d(x)$ factors to R/I it is necessary and sufficient that for all $r \in R$ and $x \in I$ we have $d(r) \simeq d(r+x),\ mod\ I$.

For $n = 2$ we have

$$d(r+x) = d(r) + d(x) + t(x)t(r) + t(r \cdot x). \tag{48}$$

So the condition is $d(x) + t(x)t(r) + t(r \cdot x) \in I$, $\forall r \in R$, $x \in I$.

This is equivalent to $d(x), t(x) \in I$, $\forall x \in I$, if this is satisfied we say that I is a Cayley–Hamilton-ideal.

If I does not contain all these elements we claim that the ideal generated by I and all these elements is a Cayley–Hamilton–ideal. This follows from Formula (48).

Thus it is possible to implement the study of quotient algebras of the algebra $C_2(\mathbb{Z}/(2)[G])$.

I have tested a few hundred ideals picking randomly generators for these ideals and, using the same method used for $C_2(\mathbb{Z}/(2)[G])$, I checked the dimension of the quotient and of its image in its universal map into 2×2 matrices always getting that the two numbers coincide.

I have no theoretical explanation for this and so no proof that this holds for all quotients of $C_2(\mathbb{Z}/(2)[G])$.

So for the moment I have been unable to see if the embedding problem in positive characteristic has a negative solution for Cayley–Hamilton algebras. But I also have no strategy to try to prove that the embedding problem in positive characteristic has a positive solution.

In fact the proof in characteristic 0 goes as follows. One presents a Cayley–Hamilton algebra R as the quotient $R = F\langle X\rangle/I$ of the free algebra $F\langle X\rangle$ modulo a Cayley–Hamilton ideal I.

Then one has the universal embedding for the free Cayley–Hamilton algebra as invariant algebra, with $G = PGL(n, F)$ or $\mathfrak{G}_n$ as group scheme. With the notations of Proposition 1.4 we have the isomorphism i:

$$F\langle X\rangle \to C_n(F\langle X\rangle) \xrightarrow{i} M_n(F[\Xi])^G \subset M_n(F[\Xi]).$$

Let J be the two sided ideal of $M_n(F[\Xi])$ generated by I, by a general fact $J = M_n(\bar{J})$ with $\bar{J} \subset F[\Xi]$ an ideal.

Then one has a commutative diagram:

$$\begin{array}{ccccc} C_n(F\langle X\rangle) & \xrightarrow{i} & M_n(F[\Xi])^{\mathfrak{G}_n} & \longrightarrow & M_n(F[\Xi]) \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ C_n(R) & \xrightarrow{j} & M_n(F[\Xi]/\bar{J})^{\mathfrak{G}_n} & \longrightarrow & M_n(F[\Xi]/\bar{J})\,. \end{array} \tag{49}$$

We have that $C_n(R)$ embeds into $n \times n$ matrices if and only if j is an embedding. This means that, since i is an isomorphism

$$J^{\mathfrak{G}_n} = J \cap C_n(F\langle X\rangle) = I.$$

In characteristic 0 this is proved by a non commutative version of the usual method of the Reynolds operator, which depends upon the fact that G is linearly reductive.

The main question is thus to see if the injective map $i : I \to J^{\mathfrak{S}_n}$ is an isomorphism.

Let us recall the argument in characteristic 0. In order to simplify notations let us denote $B := M_n(F[\Xi])$ we may also assume that the set of variables in the free algebra is infinite.

First $J = \Big\{ a = \sum_j a_j u_j b_j \Big\} \mid \ a_j, b_j \in B, \ u_j \in I.$

In [16] we then use a variable x not appearing in a_j, b_j and consider, for $a \in J^{\mathfrak{S}_n}$ the element $tr(ax) = tr\Big(\sum_j b_j x a_j u_j\Big) \in F[\Xi]^{\mathfrak{S}_n}$. In characteristic 0 we have the Reynolds operator $\mathfrak{R}$ which projects to the invariants and commutes with trace and satisfies the identity $\mathfrak{R}(ru) = \mathfrak{R}(r)u, \ \forall r \in B, \ u \in B^{\mathfrak{S}_n}$ so

$$tr(ax) = tr\left(\sum_j \mathfrak{R}(b_j x a_j) u_j \right).$$

If one can prove a similar statement in positive characteristic then the remaining part of the argument in [16] would complete the proof.

The existence of the Reynolds operator depends upon the fact that in characteristic 0 the space of invariants is a direct summand as G module. This is certainly not true in our case. For instance take the coordinate ring of a $p \times p$ matrix X in characteristic p. The space of linear functions contains the invariant $tr(X)$ but as G module is indecomposable.

What one knows is that the ring $F[\Xi]$ has a good filtration, see [5], but I do not see how to exploit this for the problem. In fact the experimental fact that the dimension of the Cayley–Hamilton algebra in characteristic 2 is sometimes bigger that the dimension of the Cayley–Hamilton algebra in characteristic 0, is not an encouraging sign for this approach.

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