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LOCALLY THREE-WEIGHT PROPERTY FOR LINEAR CODES AND ITS APPLICATION*

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ABSTRACT. An $[n, k, d]_q$ code is a linear code of length n , dimension k and minimum weight d over $\mathbb{F}_q$. A fundamental problem in coding theory is to find $n_q(k, d)$, the minimum length n for which an $[n, k, d]_q$ code exists for given k, d and q . We introduce a new notion “ s -locally w -weight (mod q)” in the usual geometric method for linear codes over $\mathbb{F}_q$. We give some necessary conditions for $w = 3$ with $s = 1, 2$. As an application, we prove the non-existence of some linear codes attaining the Griesmer bound, which determines $n_8(4, d)$ and $n_9(4, d)$ for some d .

1. Introduction. Let $\mathbb{F}_q^n$ be the vector space of n -tuples over $\mathbb{F}_q$, the field of q elements. A linear code $\mathcal{C}$ of length n and dimension k , or an $[n, k]_q$ code is just a k -dimensional subspace of $\mathbb{F}_q^n$. $\mathcal{C}$ is called an $[n, k, d]_q$ code if the minimum Hamming weight $\min\{wt(c) \mid c \in \mathcal{C}, c \neq (0, \dots, 0)\}$ is d , where $wt(c)$

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is the number of non-zero entries in the vector c . We only consider linear codes having no coordinate which is identically zero.

A fundamental problem in coding theory is to find $n_q(k, d)$, the minimum length n for which an $[n, k, d]_q$ code exists. This problem is sometimes called the *optimal linear codes problem*, see [7, 9]. An $[n, k, d]_q$ code satisfies the inequality called the *Griesmer bound* [4, 24]:

$$n \geq g_q(k, d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x . The values of $n_q(k, d)$ are determined for all d only for some small values of q and k . For $k = 3$, $n_q(3, d)$ is known for all d for $q \leq 9$. See [3] for the updated linear codes bound. As for the updated tables of $n_q(k, d)$ for some small q and k , see [21]. We mainly deal with linear codes of dimension $k = 4$. The following results are known for $n_8(4, d)$ and $n_9(4, d)$.

Theorem 1.1 ([13, 17]).

- (a) $n_8(4, d) = g_8(4, d) + 1$ for $d = 279, 280, 399, 400, 407, 408$;
- (b) $n_8(4, d) = g_8(4, d)$ or $g_8(4, d) + 1$ for $d = 278, 398, 406$.

Theorem 1.2 ([14, 15]).

- (a) $n_9(4, d) = g_9(4, d) + 1$ for $d = 584, 585, 809, 810$;
- (b) $n_9(4, d) = g_9(4, d)$ or $g_9(4, d) + 1$ for $d = 583, 808$.
- (c) $n_9(4, d) \geq g_9(4, d) + 1$ for $d = 197, 198$;

See also [2] and [20] for $n_8(4, d)$. We prove the non-existence of $[g_8(4, d), 4, d]_8$ codes for $d = 278, 398, 406$ and of $[g_9(4, d), 4, d]_9$ codes for $d = 196, 583, 808$, giving the following.

Theorem 1.3.

- (a) $n_8(4, d) = g_8(4, d) + 1$ for $d = 278, 398, 406$;
- (b) $n_9(4, d) = g_9(4, d) + 1$ for $d = 583, 808$;
- (c) $n_9(4, d) \geq g_9(4, d) + 1$ for $d = 196$.

The weight distribution of $\mathcal{C}$ is the list of integers A_i , where A_i is the number of codewords of weight i , $0 \leq i \leq n$. The weight distribution with $(A_0, A_d, \dots) = (1, \alpha, \dots)$ is expressed as $0^1 d^\alpha \dots$.

Definition 1.1. A linear code $\mathcal{C}$ over $\mathbb{F}_q$ is w -weight $(\bmod q)$ if there exists a w -set $W \subseteq \mathbb{Z}_q = \{0, 1, \dots, q-1\}$ such that every non-zero codeword c of $\mathcal{C}$ satisfies $wt(c) \equiv v \pmod{q}$ for some $v \in W$.

The code obtained by deleting the same coordinate from each codeword of an $[n, k, d]_q$ code $\mathcal{C}$ is called a *punctured code* of $\mathcal{C}$. If there exists an $[n+1, k, d+1]_q$ code $\mathcal{C}'$ which gives $\mathcal{C}$ as a punctured code, $\mathcal{C}$ is called *extendable* and $\mathcal{C}'$ is an *extension* of $\mathcal{C}$. Hill and Lizak [10] proved that every $[n, k, d]_q$ code with $\gcd(d, q) = 1$ whose weights (i 's such that $A_i > 0$) are congruent to 0 or $d \pmod{q}$ is extendable, see also [8]. As a consequence, we get the following.

Theorem 1.4. Every $[n, k, d]_q$ code with $d \equiv -1 \pmod{q}$, whose weights of codewords are congruent to 0 or $-1 \pmod{q}$, is extendable.

As for 3-weight $(\bmod q)$ linear codes over $\mathbb{F}_q$, the following result is known.

Theorem 1.5 ([19, 27]). Every $[n, k, d]_q$ code with $q \geq 5$, $d \equiv -2 \pmod{q}$, whose weights of codewords are congruent to 0, -1 or $-2 \pmod{q}$, is extendable.

In this paper, we introduce a new notion “ s -locally w -weight $(\bmod q)$ ” in the usual geometric method for linear codes over $\mathbb{F}_q$. We give some necessary conditions for the s -locally 3-weight $(\bmod q)$ property for linear codes over $\mathbb{F}_q$ as the main theorems. The structure of this paper is as follows. In Section 2, we recall the usual geometric method through projective geometry and we state the main theorems (Theorems 2.3–2.6). In Section 3, we prove Theorem 1.3 using the results in Section 2. Finally in Section 4, we prove our main theorems by the dual version of the usual geometric method.

2. The geometric method and main theorems. In this section, we first recall the usual geometric method [1, 6, 7] to investigate linear codes over $\mathbb{F}_q$ through the projective geometry. Denote by $\text{PG}(r, q)$ the projective geometry of dimension r over $\mathbb{F}_q$. A j -flat is a projective subspace of dimension j in $\text{PG}(r, q)$. The 0-flats, 1-flats, 2-flats, $(r-2)$ -flats and $(r-1)$ -flats are called *points*, *lines*, *planes*, *secundums* and *hyperplanes*, respectively. We denote by θ_j the number of points in a j -flat, i.e., $\theta_j = (q^{j+1} - 1)/(q - 1)$.

Let $\mathcal{C}$ be an $[n, k, d]_q$ code with generator matrix G having no all-zero column. Then, the columns of G can be considered as a multiset of n points in $\Sigma = \text{PG}(k-1, q)$ denoted by $\mathcal{M}_{\mathcal{C}}$. A point P in Σ is called an i -point if it has multiplicity $m_{\mathcal{C}}(P) = i$ in $\mathcal{M}_{\mathcal{C}}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{M}_{\mathcal{C}}$ and let $\mathcal{P}_i$ be the set of i -points in Σ , $0 \leq i \leq \gamma_0$. For

any subset S of Σ , the *multiplicity of S with respect to $\mathcal{C}$* , denoted by $m_{\mathcal{C}}(S)$, is defined as

$$m_{\mathcal{C}}(S) = \sum_{P \in S} m_{\mathcal{C}}(P) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap \mathcal{P}_i|,$$

where $|T|$ denotes the number of elements in a set T . Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} \mathcal{P}_i$ such that $n = m_{\mathcal{C}}(\Sigma)$ and

$$n - d = \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\},$$

where $\mathcal{F}_j$ denotes the set of j -flats in Σ . Such a partition of Σ is called an $(n, n-d)$ -arc of Σ . Conversely an $(n, n-d)$ -arc of Σ gives an $[n, k, d]_q$ code in the natural manner. A line l with $t = m_{\mathcal{C}}(l)$ is called a t -line. A t -plane, a t -hyperplane and so on are defined similarly. For an m -flat Π in Σ , let

$$\gamma_j(\Pi) = \max\{m_{\mathcal{C}}(\Delta) \mid \Delta \subset \Pi, \Delta \in \mathcal{F}_j\}, \quad 0 \leq j \leq m.$$

We denote simply by γ_j instead of $\gamma_j(\Sigma)$. It holds that $\gamma_{k-2} = n - d$, $\gamma_{k-1} = n$. When $\mathcal{C}$ is Griesmer, the values $\gamma_0, \gamma_1, \dots, \gamma_{k-3}$ are also uniquely determined ([18]) as follows:

$$(2.1) \quad \gamma_j = \sum_{u=0}^j \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \quad \text{for } 0 \leq j \leq k-1.$$

Denote by $[h_1, \dots, h_k]$ the hyperplane of Σ defined by a non-zero vector $h = (h_1, \dots, h_k) \in \mathbb{F}_q^k$ as $\{\mathbf{P}(p_1, \dots, p_k) \in \Sigma \mid h_1 p_1 + \dots + h_k p_k = 0\}$, and by a_i the number of i -hyperplanes in Σ . Note that

$$(2.2) \quad a_i = A_{n-i}/(q-1) \quad \text{for } 0 \leq i \leq n-d.$$

The list of a_i 's is called the *spectrum* of $\mathcal{C}$. We usually use τ_j 's for the spectrum of a hyperplane of Σ to distinguish from the spectrum of $\mathcal{C}$. Simple counting arguments yield the following [16]:

$$(2.3) \quad \sum_{i=0}^{n-d} a_i = \theta_{k-1},$$

$$(2.4) \quad \sum_{i=1}^{n-d} i a_i = n \theta_{k-2},$$

$$(2.5) \quad \sum_{i=2}^{n-d} i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2} \sum_{s=2}^{\gamma_0} t(t-1)\lambda_t,$$

where λ_t denotes the number of t -points, i.e., $\lambda_t = |\mathcal{P}_t|$. When $\gamma_0 = 1$, one can get the following from (2.3)–(2.5):

$$(2.6) \quad \sum_{i=0}^{n-d-2} \binom{n-d-i}{2} a_i = \binom{n-d}{2} \theta_{k-1} - n(n-d-1)\theta_{k-2} + \binom{n}{2} \theta_{k-3}.$$

Lemma 2.1 ([25]). *Put $\epsilon = (n-d)q - n$ and $\mu_0 = \lfloor (w + \epsilon)/q \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x . Let Π be a w -hyperplane through a t -secundum δ . Then $t \leq (w + \epsilon)/q$ and the following hold.*

- (a) $a_w = 0$ if an $[w, k-1, d_0]_q$ code with $d_0 \geq w - \mu_0$ does not exist.
- (b) $\gamma_{k-3}(\Pi) = \mu_0$ if an $[w, k-1, d_1]_q$ code with $d_1 \geq w - \mu_0 + 1$ does not exist.
- (c) Let h_j be the number of j -hyperplanes through δ other than Π . Then $\sum_j h_j = q$ and

$$(2.7) \quad \sum_j (\gamma_{k-2} - j)h_j = w + \epsilon - qt.$$

The condition “ $A_j = 0$ for all $j \not\equiv 0, -1, -2 \pmod{q}$ ” in Theorem 1.5 can be written as “ $a_i = 0$ for all $i \not\equiv n, n+1, n+2 \pmod{q}$ ” by (2.2). We introduce a new notion “locally w -weight”, which is a little bit different from that in [14], as follows.

Definition 2.1. *Let $\mathcal{C}$ be an $[n, k]_q$ code with $k \geq 4$ and let w be an integer with $2 \leq w \leq q-2$. Let σ be a $(k-4)$ -flat and let $H_1, H_2, \dots, H_s$ be s distinct hyperplanes through σ in $\Sigma = \text{PG}(k-1, q)$ satisfying $m_{\mathcal{C}}(H_i) \equiv n + u_i \pmod{q}$ with $w \leq u_i \leq q-1$ for $1 \leq i \leq s$. We say that $\mathcal{C}$ is s -locally w -weight $\pmod{q}$ at σ if*

$$m_{\mathcal{C}}(\Pi) \equiv n, n+1, \dots, n+w-2 \text{ or } n+w-1 \pmod{q}$$

for any hyperplane $\Pi (\neq H_1, \dots, H_s)$ through σ .

The following theorem shows that s -locally 2-weight $\pmod{q}$ codes do not exist for $1 \leq s \leq q-1$ with $q \geq 3$.

Theorem 2.2 ([14]). Assume $q \geq 3$. Let $H_1, H_2, \dots, H_s$ be s distinct hyperplanes through a $(k-4)$ -flat σ in Σ . Assume $m_{\mathcal{C}}(\Pi) \equiv n$ or $n+1 \pmod{q}$ for any hyperplane Π ($\neq H_1, \dots, H_s$) containing σ . Then, $m_{\mathcal{C}}(H_i) \equiv n$ or $n+1 \pmod{q}$ for $1 \leq s \leq q-1$.

In this paper, we consider s -locally 3-weight $\pmod{q}$ codes for $s = 1, 2$. Assume $\mathcal{C}$ is an $[n, k]_q$ code with $q \geq 4$. Let Π_0 be a hyperplane of Σ with $m_{\mathcal{C}}(\Pi_0) \equiv n+u \pmod{q}$ with $3 \leq u \leq q-1$ and let Δ be a secundum of Σ contained in Π_0 . Let $\Pi_1, \dots, \Pi_q$ be the hyperplanes through Δ other than Π_0 . We say that Δ is of type $(\rho_0, \rho_1, \dots, \rho_{q-1})$ if the number of hyperplanes Π_j from $\Pi_0, \Pi_1, \dots, \Pi_q$ satisfying $m_{\mathcal{C}}(\Pi_j) \equiv n+i \pmod{q}$ is ρ_i . Δ is also called of type $(\rho_0, \rho_1, \dots, \rho_t)$ if $\rho_{t+1} = \dots = \rho_{q-1} = 0$. Let σ be a $(k-4)$ -flat σ in Π_0 . We first assume that $\mathcal{C}$ is 1-locally 3-weight $\pmod{q}$ at σ , that is, $m_{\mathcal{C}}(\Pi) \equiv n, n+1$ or $n+2 \pmod{q}$ for any hyperplane Π ($\neq \Pi_0$) containing σ .

Theorem 2.3. Let σ be a $(k-4)$ -flat in a hyperplane H with $m_{\mathcal{C}}(H) \equiv n+u \pmod{q}$ with $3 \leq u \leq q-1$ and let $\Delta_0, \Delta_1, \dots, \Delta_q$ be the $(k-3)$ -flats in H containing σ . Assume $\mathcal{C}$ is 1-locally 3-weight $\pmod{q}$ at σ and let c_{ρ_0, ρ_1} be the number of secundums from $\Delta_0, \dots, \Delta_q$ of type $(\rho_0, \rho_1, \dots, \rho_u)$ with $\rho_0 + \rho_1 + \rho_2 = q$ and $\rho_u = 1$. Then, one of the following holds:

- (a) u is even and $c_{i, u-2i} = 0$ for $0 \leq i \leq u/2-1$, $(c_{u/2, 0}, c_{u, q-u}) = (u-1, q+2-u)$;
- (b) $c_{i, u-2i} = 0$ for $1 \leq i \leq u/2$, $c_{0, u} = u$, $c_{0, 2} = q-u$;
- (c) $q+u$ is even with $5 \leq u \leq q-2$, $c_{0, 2} = c_{0, u} = 0$ and $(c_{1, u-2}, c_{(u+q)/2, 0}) = (u, \theta_1 - u)$.

Example 2.1. Let $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$, with $\alpha^2 = \alpha + 1$. Here, we denote α and α^2 by 2 and 3, respectively. Let $\mathcal{C}_2$ be the $[21, 4, 11]_4$ code with generator matrix

$$G_2 = \begin{bmatrix} 10001111111111111000 \\ 010000112233012311110 \\ 001010000000132122311 \\ 000111200000102313022 \end{bmatrix}.$$

Then, $\mathcal{C}_2$ has weight distribution $0^1 1^{11} 12^{12} 13^{12} 14^{18} 15^{48} 16^{66} 17^{36} 18^{36} 19^{12} 20^6$ with spectrum

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) = (2, 4, 12, 12, 22, 16, 6, 4, 3, 4).$$

Take a 0-point P and an 8-plane H in $\text{PG}(3, 4)$ as $P = (0, 0, 1, 3)$, $H = [1, 1, 0, 0]$. Then, every plane δ ($\neq H$) through P satisfies $m_{\mathcal{C}}(\delta) \equiv n, n+1$ or $n+2 \pmod{4}$.

Hence, $\mathcal{C}_1$ is 1-locally 3-weight (mod 4) at P . Let $\ell_0, \ell_1, \dots, \ell_4$ be the lines on H through P as

$$\ell_0 = [1, 0, 3, 1] \cap [0, 1, 3, 1], \ell_1 = [1, 0, 0, 0] \cap [0, 1, 0, 0], \ell_2 = [1, 0, 2, 3] \cap [0, 1, 2, 3],$$

$$\ell_3 = [1, 0, 1, 2] \cap [0, 1, 1, 2], \ell_4 = [1, 1, 1, 2] \cap [0, 0, 1, 2].$$

Since $m_{\mathcal{C}_2}([1, 0, 3, 1]) = m_{\mathcal{C}_2}([0, 1, 3, 1]) = 2 \equiv n + 1$, $m_{\mathcal{C}_2}([1, 2, 2, 3]) = 6 \equiv n + 1$, $m_{\mathcal{C}_2}([1, 3, 1, 2]) = 3 \equiv n + 2$, ℓ_0 has type $(0, 3, 1, 1)$. In this way, one can see that ℓ_1 and ℓ_2 also have type $(0, 3, 1, 1)$ and ℓ_3 and ℓ_4 have type $(3, 1, 0, 1)$, satisfying (b) in Theorem 2.3.

We also consider 2-locally 3-weight (mod q) codes. Let H_1, H_2 be hyperplanes with $m_{\mathcal{C}}(H_i) \equiv n + u_i \pmod{q}$ for $i = 1, 2$ with $3 \leq u_1 \leq u_2 \leq q - 1$ and let $\Delta_0 = H_1 \cap H_2$ be a $(k - 3)$ -flat of type $(\rho_0, \rho_1, \dots, \rho_{u_2})$. The index of Δ_0 is defined as $\alpha \in \{0, 1, 2\}$ satisfying $2\rho_0 + \rho_1 = u_1 + u_2 - 2 + (1 - \alpha)q$. For a $(k - 4)$ -flat σ contained in Δ_0 , the type of σ is defined as the pair of non-negative integers (ν_0, ν_1) , where ν_i is the number of hyperplanes H containing σ with $m_{\mathcal{C}}(H) \equiv n + i \pmod{q}$ for $i = 0, 1$.

Theorem 2.4. *Let H_1, H_2 be hyperplanes with $m_{\mathcal{C}}(H_i) \equiv n + u_i \pmod{q}$ for $i = 1, 2$ with $3 \leq u_1 \leq u_2 \leq q - 1$ and let σ be a $(k - 4)$ -flat of type (ν_0, ν_1) contained in the $(k - 3)$ -flat $\Delta_0 = H_1 \cap H_2$ with index α . Let $\Delta_1^{(i)}, \dots, \Delta_q^{(i)}$ be the $(k - 3)$ -flats ($\neq \Delta_0$) in H_i containing σ for $i = 1, 2$. Assume $u_1(u_1 - 2) + u_2(u_2 - 2) < q^2 - 4q$ and that $\mathcal{C}$ is 2-locally 3-weight (mod q) at σ . Let $d_{\rho_0, \rho_1}^{(i)}$ be the number of secundums from $\{\Delta_0^{(i)}, \Delta_1^{(i)}, \dots, \Delta_q^{(i)}\}$ of type $(\rho_0, \rho_1, \dots, \rho_{u_2})$. Let $\tau_{u_i} = \sum_{2\rho_0 + \rho_1 = u_i} d_{\rho_0, \rho_1}^{(i)}$ for $i = 1, 2$. Then, it holds that $u_i - 3 \leq \tau_{u_i} \leq u_i$ for $i = 1, 2$. either*

(a) $2\nu_0 + \nu_1 = \theta_1^2 + u + v - 3$, both of u_1 and u_2 are even and $\tau_{u_i} = u_i - 1 - \alpha$ for $i = 1, 2$;

(b) $2\nu_0 + \nu_1 = \theta_2 + u + v - 3 - eq$ and $\tau_{u_i} = e + u_i - \alpha$ for $i = 1, 2$ with $0 \leq e \leq q + \alpha - v$.

Theorem 2.5. *In the case (b) of Theorem 2.4, assume $u(u - 2) + v(v - 2) < q^2 - 4q$ for $\alpha = 0, 1$ and that $u(u - 2) + v(v - 2) < q^2 - 2q$ for $\alpha = 2$. Then, $e = 0$.*

Combining the above theorems, we get the following.

Theorem 2.6. *Let H_1, H_2 be hyperplanes with $m_{\mathcal{C}}(H_i) \equiv n + u_i \pmod{q}$ for $i = 1, 2$ with $3 \leq u_1 \leq u_2 \leq q - 1$ and let σ be a $(k - 4)$ -flat contained in the $(k - 3)$ -flat $\Delta_0 = H_1 \cap H_2$. Let $\Delta_1^{(i)}, \dots, \Delta_q^{(i)}$ be the $(k - 3)$ -flats ($\neq \Delta_0$) in H_i containing σ for $i = 1, 2$. Assume $u_1(u_1 - 2) + u_2(u_2 - 2) < q^2 - 4q$ and that $\mathcal{C}$ is 2-locally 3-weight $\pmod{q}$ at σ . Let $d_{\rho_0, \rho_1}^{(i)}$ be the number of secundums from $\{\Delta_0^{(i)}, \Delta_1^{(i)}, \dots, \Delta_q^{(i)}\}$ of type $(\rho_0, \rho_1, \dots, \rho_{u_2})$. Let $\tau_{u_i} = \sum_{2\rho_0 + \rho_1 = u_i} d_{\rho_0, \rho_1}^{(i)}$ for $i = 1, 2$. Then, it holds that $u_i - 3 \leq \tau_{u_i} \leq u_i$ for $i = 1, 2$.*

We prove Theorems 2.3, 2.4 and 2.5 in Section 4.

3. Proof of Theorem 1.3. As applications of Theorems 2.3–2.6, we prove the non-existence of $[g_8(4, d), 4, d]_8$ codes for $d = 278, 398, 406$ and the non-existence of $[g_9(4, d), 4, d]_9$ codes for $d = 196, 583, 808$. The following two lemmas are obtained from Lemma 4.9 in [13].

Lemma 3.1. *The spectrum of a $[28, 3, 24]_8$ code is $(a_0, a_4) = (10, 63)$ and the set of 0-lines form a 10-arc of lines no three of which are concurrent.*

Lemma 3.2. *Every $[27, 3, 23]_8$ code is extendable and its spectrum is $(a_0, a_3, a_4) = (10, 9, 54)$.*

Lemma 3.3. *There exists no $[319, 4, 278]_8$ code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[319, 4, 278]_8$ code. Then, $(\gamma_0, \gamma_1, \gamma_2) = (1, 6, 41)$ from (2.1), and an i -plane through a fixed t -line in $\Sigma = \text{PG}(3, 8)$ satisfies

$$(3.1) \quad t \leq \frac{i + 9}{8}$$

by Lemma 2.1. If there exists a 2-plane δ_2 , it follows from (3.1) that there exists no 2-line on δ_2 , a contradiction. If there exists a 11-plane, it corresponds to a $[11, 3, d_0]_8$ code with $d_0 \geq 11 - \lfloor 20/8 \rfloor = 9$, which does not exist, see [3]. Thus, $a_{11} = 0$. In this way, one can get

$$a_i = 0 \text{ for all } i \notin \{0, 1, 7-10, 15, 23-28, 31-33, 39-41\}.$$

This procedure to rule out some possible multiplicities of hyperplanes using the known results on $n_q(k - 1, d)$ and Lemma 2.1 and the possible spectra for an

$(n - d)$ -hyperplane is called the **first sieve** [15]. The equality (2.7) gives

$$(3.2) \quad \sum_j (41 - j)h_j = w + 9 - 8t$$

with $\sum_j h_j = 8$. Suppose $a_i > 0$ and let δ_i be an i -plane. Note that $\gamma_1(\delta_i) = 4$ for $i = 26, 27, 28$ from (3.1). Suppose $a_{28} > 0$ and let P be a 1-point on δ_{28} . Then, all the lines on δ_{28} through P are 4-lines. The possible solutions of (3.2) with $(w, t) = (28, 4)$ are

$$(3.3) \quad (h_{39}, h_{40}, h_{41}) = (2, 1, 5), (1, 3, 4), (0, 5, 3),$$

giving $c_{2,1} + c_{1,3} + c_{0,5} = 9$, and $\mathcal{C}$ is locally 3-weight (mod 8) at P . This contradicts that $c_{2,1} = c_{1,3} = 0$ and $c_{0,5} = 5$ from Theorem 2.3(b) with $u = 5$. Hence, $a_{28} = 0$. Similarly, applying Theorem 2.3 with $u = 4$, one can prove $a_{27} = 0$.

Suppose $a_{26} > 0$ and let P be a 1-point on δ_{26} . Then, the lines on δ_{26} through P consist of either seven 4-lines and two 3-lines or eight 4-lines and one 2-line. For the former case, one can apply Theorem 2.3 with $u = 3$ to get a contradiction since the RHS of (3.2) with $(w, t) = (26, 3)$ is 11. For the latter case, we apply Theorem 2.6 to get a contradiction when possible solutions of (3.2) with $(w, t) = (26, 2)$ (resp. $(26, 4)$) are $(h_{26}, h_{39}, h_{40}, h_{41}) = (1, 2, 0, 5), (1, 1, 2, 4)$ or $(1, 0, 4, 3)$ (resp. $(h_{39}, h_{40}, h_{41}) = (1, 1, 6), (0, 3, 5)$). Thus, $a_{26} = 0$. One can also prove $a_{10} = 0$ applying Theorem 2.3 with $u = 3$ since every line through a 1-point on a 10-plane is a 2-line.

Now, we have $A_i = 0$ for all $i \not\equiv 0, -1, -2 \pmod{8}$ from (2.2). Applying Theorem 1.5, $\mathcal{C}$ is extendable, which contradicts Theorem 1.1 (a). This completes the proof. $\square$

Lemma 3.4. *There exists no $[456, 4, 398]_8$ code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[456, 4, 398]_8$ code. Note that $(\gamma_0, \gamma_1, \gamma_2) = (1, 8, 58)$ from (2.1). An i -plane through a fixed t -line in $\Sigma = \text{PG}(3, 8)$ satisfies

$$(3.4) \quad t \leq \frac{i + 8}{8}$$

by Lemma 2.1. Then, one can get

$$a_i = 0 \text{ for all } i \notin \{0, 1, 8, 9, 10, 24-28, 32, 33, 40, 41, 42, 48, 49, 56, 57, 58\}$$

by the first sieve. The equality (2.7) gives

$$(3.5) \quad \sum_j (58 - j)h_j = w + 8 - 8t$$

with $\sum_j h_j = 8$.

Suppose $a_{28} > 0$ and let δ_{28} be a 28-plane. Then, $\gamma_1(\delta_{28}) = 4$ from (3.4) and δ_{28} has spectrum $(\tau_0, \tau_4) = (10, 63)$ by Lemma 3.1. Hence, for any 1-point P on δ_{28} , all the lines on δ_{28} through P are 4-lines. Since the possible solutions of (3.5) with $(w, t) = (28, 4)$ are

$$(h_{56}, h_{57}, h_{58}) = (0, 4, 4), (1, 2, 5), (2, 0, 6),$$

giving $c_{0,4} + c_{1,2} + c_{2,0} = 9$, where $c_{i,j}$ denotes the number of lines through P of type $(i, j, 8 - i - j, 0, 1)$. Thus, $\mathcal{C}$ is locally 3-weight (mod 8) at P . On the other hand, Theorem 2.3 with $u = 4$ yields that $c_{0,4} + c_{1,2} + c_{2,0} = 3$ or 4, a contradiction.

Next, suppose $a_{27} > 0$ and let δ_{27} be a 27-plane. Then, δ_{27} has spectrum $(\tau_0, \tau_3, \tau_4) = (10, 9, 54)$ by Lemma 3.2. For any 1-point P on δ_{27} , the lines on δ_{27} through P consist of eight 4-lines and one 3-line. Since the possible solutions of (3.5) with $(w, t) = (27, 4)$ are $(h_{56}, h_{57}, h_{58}) = (1, 1, 6)$ or $(0, 3, 5)$ and since the RHS of (3.5) with $(w, t) = (27, 3)$ is 11, we have $c_{1,1} + c_{0,3} \geq 8$, and $\mathcal{C}$ is locally 3-weight (mod 8) at P . This contradicts Theorem 2.3 with $u = 3$. Hence, $a_{27} = 0$.

Now, we have $A_i = 0$ for all $i \not\equiv 0, -1, -2 \pmod{8}$ from (2.2). Applying Theorem 1.5, $\mathcal{C}$ is extendable, which contradicts Theorem 1.1 (a). This completes the proof. $\square$

Lemma 3.5. *There exists no $[465, 4, 406]_8$ code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[465, 4, 406]_8$ code. Then, $(\gamma_0, \gamma_1, \gamma_2) = (1, 8, 59)$ from (2.1). An i -plane through a fixed t -line satisfies $t \leq (i + 7)/8$ by Lemma 2.1, and one can get

$$a_i = 0 \text{ for all } i \notin \{0, 1, 9, 10, 25-28, 33, 41, 42, 49, 57, 58, 59\}$$

by the first sieve. The equality (2.7) gives

$$(3.6) \quad \sum_j (59 - j)h_j = w + 7 - 8t$$

with $\sum_j h_j = 8$. From (2.6), we have

$$(3.7) \quad \sum_{i \leq 57} \binom{59-i}{2} a_i = 3045.$$

Suppose $a_0 > 0$ and let δ_0 be a 0-plane. Setting $w = t = 0$, the maximum possible contribution of h_j 's in (3.6) to the LHS of (3.7) is $(h_{57}, h_{58}, h_{59}) = (3, 1, 4)$. Hence we get

$$(\text{LHS of (3.7)}) \leq 3\theta_2 + \binom{59}{2} = 1930,$$

a contradiction. Hence $a_0 = 0$.

Suppose $a_{28} > 0$ and let δ_{28} be a 28-plane. For a 1-point P on δ_{28} , all the lines on δ_{28} through P are 4-lines since $\gamma_1(\delta_{28}) = 4$. The possible solutions of (3.6) with $(w, t) = (28, 4)$ are

$$(h_{57}, h_{58}, h_{59}) = (1, 1, 6) \text{ or } (0, 3, 5),$$

giving $c_{1,1} + c_{0,3} = 9$, and $\mathcal{C}$ is locally 3-weight (mod 8) at P . This contradicts that $c_{1,1} = 0$ and $c_{1,3} = 3$ from Theorem 2.3 with $u = 3$. Hence, $a_{28} = 0$.

Now, we can apply Theorem 1.5 and Theorem 1.1 (a) to get a contradiction. $\square$

As an application of Theorems 2.3 and 2.6, we prove the non-existence of $[g_9(4, d), 4, d]_9$ codes for $d = 196, 583, 808$.

Lemma 3.6. *There exists no $[222, 4, 196]_9$ code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[222, 4, 196]_9$ code. Then, an i -plane through a fixed t -line in $\Sigma = \text{PG}(3, 9)$ satisfies

$$(3.8) \quad t \leq \frac{i+12}{9}$$

by Lemma 2.1. Then, by the first sieve, we get

$$a_i = 0 \text{ for all } i \notin \{0, 1, 6-10, 15, 16, 17, 24, 25, 26\}.$$

The equality (2.7) gives

$$(3.9) \quad \sum_j (26-j)h_j = w + 12 - 9t$$

with $\sum_j h_j = 9$. From (2.6), we have

$$(3.10) \quad \sum_{i \leq 17} \binom{26-i}{2} a_i = 6760.$$

Suppose $a_0 > 0$ and let δ_0 be a 0-plane. Setting $w = t = 0$, the maximum possible contribution of h_j 's in (3.9) to the LHS of (3.10) is $(h_{15}, h_{25}, h_{26}) = (1, 1, 7)$. Hence we get

$$(\text{LHS of (3.10)}) \leq \binom{11}{2} \theta_2 + \binom{26}{2} = 5330,$$

a contradiction. Hence $a_0 = 0$. One can show $a_1 = 0$ similarly.

Suppose $a_{10} > 0$ and let P be a 1-point on a 10-plane δ_{10} . Then, there are nine 2-lines and one 1-line on δ_{10} through P . Since the possible solutions of (3.9) with $w = 10$ satisfy $h_9 = h_{10} = 0$ for $t = 1, 2$, one can apply Theorem 2.3 with $u = 4$ to get either $c_{0,4} = c_{1,2} = 0$ and $c_{2,0} = 3$ from (a) or $c_{1,2} = c_{2,0} = 0$ and $c_{0,4} = 4$ from (b). On the other hand, the possible solutions of (3.9) with $w = 9$ is $(h_{24}, h_{25}, h_{26}) = (1, 2, 6)$ or $(h_{24}, h_{25}, h_{26}) = (0, 4, 5)$ for $t = 2$, which implies that $c_{1,2} + c_{0,4} \geq 9$, a contradiction. Thus, $a_{10} = 0$. One can similarly prove $a_9 = 0$.

Now, applying Theorem 1.5, $\mathcal{C}$ is extendable, which contradicts Theorem 1.2 (c). This completes the proof. $\square$

Lemma 3.7 ([22]). *The spectrum of a $[48, 3, 42]_9$ code is $(a_0, a_3, a_6) = (3, 16, 72)$.*

Lemma 3.8. *There exists no $[657, 4, 583]_9$ code.*

Proof. We give a sketch of the proof. Let $\mathcal{C}$ be a putative Griesmer $[657, 4, 583]_9$ code. Then, an i -plane through a fixed t -line in $\Sigma = \text{PG}(3, 9)$ satisfies $t \leq (i + 9)/9$ and we have

$$a_i = 0 \text{ for all } i \notin \{0, 1, 9, 10, 27, 28, 36, 37, 45-48, 54, 55, 63-65, 72-74\}$$

by the first sieve. The equality (2.7) gives

$$(3.11) \quad \sum_j (74 - j) h_j = w + 9 - 9t$$

with $\sum_j h_j = 9$. Suppose $a_{48} > 0$ and let P be a 1-point on a 48-plane δ_{48} . Then, as we see in the proof of the previous lemma, there are at most one 48-plane ($\neq \delta_{48}$) through P . Since the possible solutions of (3.11) with $(w, t) = (48, 6)$ is $(h_{72}, h_{73}, h_{74}) = (1, 1, 7)$ only, we have $c_{1,1} \geq 9$, which contradicts Theorems 2.3 and 2.6 with $u = 3$. Hence, $a_{48} = 0$.

Applying Theorem 1.5, $\mathcal{C}$ is extendable, which contradicts Theorem 1.2 (a). $\square$

Lemma 3.9 ([26]). *Let $\mathcal{C}$ be a Griesmer $[n, k, d]_q$ code, where q is a power of a prime p . If q divides d , then $\mathcal{C}$ is p -divisible.*

Let $\mathcal{C}$ be a Griesmer $[102, 3, 90]_9$ code. Then, the multiset $\mathcal{F} = 2\text{PG}(2, 9) - \mathcal{M}_{\mathcal{C}}$ forms a $(80, 8)_9$ -minihyper. Hence, we get the following from the classification of $(80, 8)_9$ -minihypers by Hill and Ward (Theorem 43 in [11]).

Lemma 3.10. *The possible spectra of a $[102, 3, 90]_9$ code are $(a_3, a_6, a_9, a_{12}) = (8, 0, 0, 83), (0, 9, 6, 76), (0, 8, 8, 75), (0, 1, 22, 68), (0, 0, 24, 67)$.*

Lemma 3.11. *There exists no $[910, 4, 808]_9$ code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[910, 4, 808]_9$ code. Then, $(\gamma_0, \gamma_1, \gamma_2) = (2, 12, 102)$ from (2.1), and an i -plane through a fixed t -line in $\Sigma = \text{PG}(3, 9)$ satisfies

$$(3.12) \quad t \leq \frac{i+8}{9}$$

by Lemma 2.1. Hence, $i \geq 9 \cdot 3 - 8 = 19$ by Lemma 3.10, and we have

$$a_i = 0 \text{ for all } i \notin \{46, 47, 48, 73-81, 91, 100, 101, 102\}$$

by the first sieve. The equality (2.7) gives

$$(3.13) \quad \sum_j (102 - j)h_j = w + 8 - 9t$$

with $\sum_j h_j = 9$. From (2.6), we have

$$(3.14) \quad \sum_{i \leq 100} \binom{102-i}{2} a_i = 81\lambda_2 - 4030.$$

Suppose $a_{81} > 0$ and let δ_{81} be an 81-plane. Then, the spectrum of δ_{81} is $(\tau_0, \tau_9) = (1, 90)$ [5]. Setting $w = 81$, the maximum possible contribution of h_j 's in (3.13) to the LHS of (3.14) is $(h_{46}, h_{73}, h_{100}, h_{107}) = (1, 1, 2, 5)$ for $t = 0$ and $(h_{100}, h_{102}) = (4, 5)$ for $t = 9$. Hence we get

$$(\text{LHS of (3.14)}) \leq \binom{56}{2} + \binom{29}{2} + 2 + 4\tau_9 + \binom{21}{2} = 2518,$$

whence $\lambda_2 \leq 80$, which contradicts that $\lambda_2 = 90 + \lambda_0 \geq 90$. Hence $a_{81} = 0$. One can similarly prove $a_{80} = a_{79} = 0$. We also have $a_{91} = 0$ since a putative 91-plane has only 10-lines, which do not appear in a 102-plane.

Next, suppose $a_{78} > 0$ and let δ_{78} be a 78-plane. Let P be a 1-point on δ_{78} and let b_j be the number of j -lines through P in δ_{78} . Since (3.13) has no solution for $(w, t) = (78, 6)$, the possible b_j 's are $(b_8, b_9) = (3, 7)$ or $(b_6, b_9) = (1, 9)$. Note that the possible solutions of (3.13) with $(w, t) = (78, 9)$ are $(h_{100}, h_{101}, h_{102}) = (2, 1, 6), (1, 3, 5), (0, 5, 4)$. Assume $(b_8, b_9) = (3, 7)$. Then, there is no 76-, 77- or 78-plane through P other than δ_{78} and we have $c_{2,1} + c_{1,3} + c_{0,5} \geq 7$, which contradicts that $c_{2,1} + c_{1,3} + c_{0,5} = 5$ by Theorem 2.3.

Assume $(b_6, b_9) = (1, 9)$. Then, $c_{2,1} + c_{1,3} + c_{0,5} \geq 9$. If there is no 76-, 77- or 78-plane through P other than δ_{78} , then we get a contradiction again. So, we may assume that the solutions of (3.13) with $(w, t) = (78, 6)$ satisfy $h_{76} + h_{77} + h_{78} = 1$. Assume there is a 78-plane ($\neq \delta_{78}$) through P . Then, $c_{2,1} + c_{1,3} + c_{0,5} \leq 5$ by Theorem 2.6, a contradiction. Next, assume there is a 76- or 77-plane ($\neq \delta_{78}$) through P . Setting $(u, v) = (3, 5)$ or $(4, 5)$, Theorem 2.6 yields that $c_{2,1} + c_{1,3} + c_{0,5} \leq 5$, a contradiction again. Thus, $a_{78} = 0$.

Similarly, one can prove $a_{77} = a_{76} = 0$. Applying Theorem 1.5, $\mathcal{C}$ is extendable, which contradicts Theorem 1.2 (a). This completes the proof. $\square$

Now, Theorem 1.3 follows from Theorems 1.1, 1.2 and Lemmas 3.3–3.6, 3.8, 3.11.

4. Proof of Theorems 2.3–2.5. Let $\mathcal{C}$ be an $[n, k, d]_q$ code with a generator matrix $G = [g_{ij}]$ whose i -th row is g_i , $1 \leq i \leq k$. In this section, we prove Theorems 2.3–2.5 by the dual version of the geometric method in Section 2, that is, the columns of G are considered as hyperplanes of $\Sigma = \text{PG}(k-1, q)$. For $P = \mathbf{P}(p_1, \dots, p_k) \in \Sigma$, the weight of P with respect to G , denoted by $w_G(P)$, is defined [27] as

$$w_G(P) = |\{j \mid \sum_{i=1}^k g_{ij}p_i \neq 0\}| = wt(\sum_{i=1}^k p_i g_i).$$

For $i = 0, 1, 2, \dots, q-1$, let

$$N_i = \{P \in \Sigma \mid w_G(P) \equiv -i \pmod{q}\}.$$

For a line $\ell = \{P_0, P_1, \dots, P_q\}$ in Σ , it holds that

$$(4.1) \quad \sum_{i=0}^q w_G(P_i) \equiv 0 \pmod{q},$$

see [27]. ℓ is called of type $(\rho_0, \rho_1, \rho_2, \dots, \rho_{q-1})$ if $\rho_i = |\ell \cap N_i|$ for $0 \leq i \leq q-1$ in what follows. Let δ be a plane with $\nu_i = |N_i \cap \delta|$ for $0 \leq i \leq q-1$. It is known that $\sum_{i \geq 2} \nu_i \geq q$ if $\nu_0 + \nu_1 < \theta_2$ as follows.

Lemma 4.1 ([14]). *Assume $q \geq 3$ and $\nu_0 + \nu_1 \geq q^2 + 2$. Then, it holds that $\nu_0 + \nu_1 = \theta_2$.*

Next, assume $q \geq 4$ and $\nu_0 + \nu_1 + \nu_2 \geq \theta_2 - 2$. We first consider the possible lines of type $(\rho_0, \rho_1, \dots, \rho_{q-1})$ in δ . Let $\mathcal{S}$ be the set of possible (ρ_0, ρ_1) 's when $\rho_0 + \rho_1 + \rho_2 = \theta_1$. Then,

$$(4.2) \quad \mathcal{S} = \{(i, q+2-2i) \mid 1 \leq i \leq \lfloor q/2 \rfloor + 1\} \cup \{(1, 0), (0, 2), (\theta_1, 0)\}$$

from (4.1). Let $\mathcal{S}_u$ be the set of possible (ρ_0, ρ_1) 's when $\rho_0 + \rho_1 + \rho_2 = q$ and $\rho_u = 1$ for some $3 \leq u \leq q-1$. Then, from (4.1), we get $\mathcal{S}_u = \mathcal{T}_u \cup \mathcal{T}_u^+$ with

$$(4.3) \quad \begin{aligned} \mathcal{T}_u &= \{(i, u-2i) \mid 0 \leq i \leq \lfloor u/2 \rfloor\}, \\ \mathcal{T}_u^+ &= \{(i, u+q-2i) \mid u \leq i \leq \lfloor (q+u)/2 \rfloor\}. \end{aligned}$$

Indeed, for $(i, j = u+q-2i) \in \mathcal{T}_u^+$, we have $i = (q-i-j) + u \geq u$, giving $u \leq i \leq (q+u)/2$ since $j \geq 0$. Note that $2\rho_0 + \rho_1 \equiv u \pmod{q}$ if $(\rho_0, \rho_1) \in \mathcal{S}_u$, which yields the following.

Lemma 4.2. $\mathcal{S} \cap \mathcal{S}_u = \emptyset$ for $3 \leq u \leq q-1$.

For two integers u, v with $3 \leq u \leq v \leq q-1$, let $\mathcal{D}_{u,v}$ be the set of all possible (ρ_0, ρ_1) 's when $\rho_0 + \rho_1 + \rho_2 = q-1$ with either $\rho_u = 2$ if $u = v$ or $\rho_u = \rho_v = 1$ if $u < v$. Then, from (4.1), we get $\mathcal{D}_{u,v} = \mathcal{T}_{u,v} \cup \mathcal{T}_{u,v}^+ \cup \mathcal{T}_{u,v}^-$ with

$$(4.4) \quad \begin{aligned} \mathcal{T}_{u,v} &= \{(i, u+v-2-2i) \mid \max\{0, u+v-\theta_1\} \leq i \leq \lfloor (u+v)/2 \rfloor - 1\}, \\ \mathcal{T}_{u,v}^+ &= \{(i, u+v-2+q-2i) \mid u+v-1 \leq i \leq \lfloor (u+v+q)/2 \rfloor - 1\}, \\ \mathcal{T}_{u,v}^- &= \{(i, u+v-2-q-2i) \mid 0 \leq i \leq \lfloor (u+v-q)/2 \rfloor - 1\}. \end{aligned}$$

For a line m of type $(\rho_0, \rho_1, \dots, \rho_v)$ with $(\rho_0, \rho_1) \in \mathcal{D}_{u,v}$, the *index* of m is defined as the integer α satisfying $2\rho_0 + \rho_1 = u + v - 2 + (1 - \alpha)q$, i.e., $\alpha = 0, 1, 2$ if (ρ_0, ρ_1) is in $\mathcal{T}_{u,v}^+$, $\mathcal{T}_{u,v}$, $\mathcal{T}_{u,v}^-$, respectively.

For $(\rho_0, \rho_1) = (i, u + v - 2 - 2i) \in \mathcal{T}_{u,v}$, we have $\rho_0 + \rho_1 + 2 \leq \theta_1$, giving $i \geq u + v - \theta_1$. We get $i \geq u + v - 1$ for $(i, u + v - 2 + q - 2i) \in \mathcal{T}_{u,v}^+$ similarly.

Lemma 4.3. $\mathcal{D}_{u,v} = \mathcal{T}_{u,v} \cup \mathcal{T}_{u,v}^+ \cup \mathcal{T}_{u,v}^-$ satisfies $|\mathcal{D}_{u,v}| = 1$ and

- (a) $|\mathcal{T}_{u,v}^+| = 0$ if $u + v \geq \theta_1$;
- (b) $|\mathcal{T}_{u,v}^-| = 0$ if $u + v \leq \theta_1$;
- (c) $\mathcal{T}_{u,v}^- \cap \mathcal{S} = \emptyset$ if $q \neq u + v - 4$ and $\mathcal{T}_{u,v}^- \cap \mathcal{S} = \{(0, 2), (1, 0)\}$ if $q = u + v - 4$.

Note that $(\rho_0, \rho_1) \in \mathcal{D}_{u,v}$ satisfies $2\rho_0 + \rho_1 \equiv u + v - 2 \pmod{q}$. We consider the following three cases.

- Case 1: $\nu_0 + \nu_1 + \nu_2 = q^2 + q$ and $\nu_u = 1$ with $3 \leq u \leq q - 1$;
- Case 2: $\nu_0 + \nu_1 + \nu_2 = q^2 + q - 1$ and $\nu_u = \nu_v = 1$ with $3 \leq u < v \leq q - 1$;
- Case 3: $\nu_0 + \nu_1 + \nu_2 = q^2 + q - 1$ and $\nu_u = 2$ with $3 \leq u \leq q - 1$.

We note that no two lines of different types in $\mathcal{S} \cup \mathcal{S}_u \cup \mathcal{S}_v \cup \mathcal{D}_{u,v}$ have the same (ρ_0, ρ_1) except for the case $q = u + v - 4$ in Lemma 4.3(c). Let $\Lambda_1 = \mathcal{S} \cup \mathcal{S}_u$ for Case 1; $\Lambda_1 = \mathcal{S} \cup \mathcal{S}_u \cup \mathcal{S}_v \cup \mathcal{D}_{u,v}$ for Case 2; $\Lambda_1 = \mathcal{S} \cup \mathcal{S}_u \cup \mathcal{D}_{u,u}$ for Case 3. We denote by $c_{i,j}$ the number of lines of type $(\rho_0, \rho_1, \dots, \rho_{q-1})$ with $(i, j) = (\rho_0, \rho_1)$ in δ . Such lines are called (i, j) -lines. Then we have the following equalities which are similar to (2.3)–(2.5):

$$(4.5) \quad \sum_{(i,j) \in \Lambda_1} c_{i,j} = \theta_2,$$

$$(4.6) \quad \sum_{(i,j) \in \Lambda_1} i c_{i,j} = \theta_1 \nu_0,$$

$$(4.7) \quad \sum_{(i,j) \in \Lambda_1} j c_{i,j} = \theta_1 \nu_1,$$

$$(4.8) \quad \sum_{(i,j) \in \Lambda_1} \binom{i}{2} c_{i,j} = \binom{\nu_0}{2},$$

$$(4.9) \quad \sum_{(i,j) \in \Lambda_1} \binom{j}{2} c_{i,j} = \binom{\nu_1}{2},$$

$$(4.10) \quad \sum_{(i,j) \in \Lambda_1} \binom{i+j}{2} c_{i,j} = \binom{\nu_0 + \nu_1}{2}.$$

From (4.8)–(4.10), we also get

$$(4.11) \quad \sum_{(i,j) \in \Lambda_1} ij c_{i,j} = \binom{\nu_0 + \nu_1}{2} - \binom{\nu_0}{2} - \binom{\nu_1}{2} = \nu_0 \nu_1.$$

4.1. Case 1. We first consider the case when there is only one exceptional point.

Lemma 4.4. Assume $q \geq 4$, $\nu_0 + \nu_1 + \nu_2 = q^2 + q$ and $\nu_u = 1$ for some $3 \leq u \leq q-1$. Let $c := \sum_{(i,j) \in \mathcal{T}_u} c_{i,j} = \sum_i c_{i,u-2i}$. Then, $2\nu_0 + \nu_1 - u + 1 = \theta_2 + q$ or θ_2 . Moreover,

(a) u is even and $(\nu_0, \nu_1) = (u\theta_1 - \binom{u}{2}, (q-u)(q-u+2))$, $c = c_{u/2,0} = u-1$, $c_{1,0} = c_{0,2} = c_{i,u-2i} = 0$ for $i < u/2$, $c_{\theta_1,0} = u$, $c_{u,q-u} = q-u+2$ when $2\nu_0 + \nu_1 - u + 1 = \theta_2 + q$;

(b) When $2\nu_0 + \nu_1 - u + 1 = \theta_2$, it holds either

(b-1) $(\nu_0, \nu_1) = (\binom{q}{2} - \binom{u}{2} + u, 2q + u(u-2))$, $c_{i,u-2i} = 0$ for $i > 0$, $c = c_{0,u} = u$, $c_{1,0} = c_{\theta_1,0} = 0$, $c_{0,2} = q-u$, or

(b-2) $q+u$ is even with $5 \leq u \leq q-2$ and $(\nu_0, \nu_1) = (u + (\theta_1 - u)(q+u)/2, u(u-2))$, $c_{\theta_1,0} = 1$, $c_{0,2} = c_{0,u} = 0$, $c_{1,u-2} = u$, $c_{1,0} = c_{(u+q)/2,0} = \theta_1 - u$.

Proof. For $q = 4$, the only possible plane δ with $\nu_3 = 1$ satisfies $(\nu_0, \nu_1, \nu_2) = (6, 11, 3)$ with $(c_{1,4}, c_{2,2}, c_{0,2}, c_{0,3}, c_{3,1}) = (6, 9, 1, 3, 2)$ from Table 3.3 in [23], which yields (b-1).

Assume $q \geq 5$. Let R be the unique point of N_u in Σ . Note that $\sum_{(i,j) \in \mathcal{S}_u} c_{i,j} = \theta_1$ and that $(q+2)\theta_2 = \theta_1^3 + 1$. Calculating $\sum_{(i,j) \in \Lambda_1} (2i+j-q-2)c_{i,j}$, $\sum_{(i,j) \in \Lambda_1} i(2i+j-q-2)c_{i,j}$ and $\sum_{(i,j) \in \Lambda_1} j(2i+j-q-2)c_{i,j}$ by way of (4.5)–(4.11), we get

$$(4.12) \quad \theta_1(2\nu_0 + \nu_1 - \theta_1^2 - u + 2) - 1 = -q(c_{1,0} + c_{0,2} - c_{\theta_1,0} + c),$$

$$(4.13) \quad \nu_0(2\nu_0 + \nu_1 - \theta_2 - u + 1) = -q(c_{1,0} - \theta_1 c_{\theta_1,0} + \sum_i i c_{i,u-2i}),$$

$$(4.14) \quad \nu_1(2\nu_0 + \nu_1 - \theta_1^2 - u + 1) = -q(2c_{0,2} + \sum_i (u - 2i)c_{i,u-2i}),$$

respectively. It follows from (4.12) that $2\nu_0 + \nu_1 - \theta_1^2 - u + 2 \equiv 1 \pmod{q}$. Since the RHS of (4.14) is non-positive, we have $2\nu_0 + \nu_1 \leq \theta_1^2 + u - 1$.

Assume $2\nu_0 + \nu_1 = \theta_1^2 + u - 1$. Then, $c_{0,2} = c_{i,u-2i} = 0$ for $0 \leq i < u/2$ from (4.14). Suppose u is odd. Then, we have $c = 0$. Considering the lines through R , we have

$$2\nu_0 + \nu_1 = \sum_{(i,j) \in \mathcal{T}_u^+} (2i + j)c_{i,j} = (u + q)\theta_1 = \theta_1^2 + u - 1,$$

giving $u = 1$, a contradiction. Hence u is even. It follows from (4.12) that

$$c_{1,0} - c_{\theta_1,0} + c = -1$$

with $c = c_{u/2,0}$, whence $c_{\theta_1,0} > 0$. Considering the lines through R , we have

$$2\nu_0 + \nu_1 = uc + (u + q)(\theta_1 - c) = \theta_1^2 + u - 1,$$

giving $c = u - 1$. Suppose $c_{1,0} > 0$ and let ℓ_0 be a $(1, 0)$ -line. Then $c_{\theta_1,0} = c + 1 + c_{1,0} \geq u + 1$, and the $(\theta_1, 0)$ -lines meet ℓ_0 in the point of N_0 , which contradicts that there are $u - 1$ $(u/2, 0)$ -lines through R . Thus, $c_{1,0} = 0$, $c_{\theta_1,0} = c + 1 = u$. It follows from (4.13) that

$$\nu_0 = \theta_1 c_{\theta_1,0} - (u/2)c_{u/2,0} = u\theta_1 - u(u - 1)/2,$$

giving $\nu_1 = \theta_1^2 + u - 1 - 2\nu_0 = (q - u)(q - u + 2)$. Since $\nu_1 = (q - u)(\theta_1 - c)$, we get $c_{u,q-u} = q - u + 2$. Thus, we obtain (a) when $2\nu_0 + \nu_1 = \theta_1^2 + u - 1$.

Next, assume $2\nu_0 + \nu_1 = \theta_2 + u - 1 - qe$ for some non-negative integer e . Then, the equalities (4.12)-(4.14) are replaced by

$$(4.15) \quad \theta_1 e + q = c_{1,0} + c_{0,2} - c_{\theta_1,0} + c,$$

$$(4.16) \quad \nu_0 e = c_{1,0} - \theta_1 c_{\theta_1,0} + \sum_i i c_{i,u-2i},$$

$$(4.17) \quad \nu_1(e + 1) = 2c_{0,2} + \sum_i (u - 2i)c_{i,u-2i}.$$

It follows from (4.15) and our assumption that $2 \times (4.16) + (4.17)$ gives

$$(4.18) \quad \nu_1 - qe^2 + (q^2 - q + u - 2)e = (u - 2)c + 2q(1 - c_{\theta_1,0}).$$

Counting the number of points in the lines through R , we get

$$2\nu_0 + \nu_1 = uc + (q + u)(\theta_1 - c) = \theta_2 + u - 1 - qe,$$

giving $c = e + u$, and $e \leq q - u + 1$ since $c \leq \theta_1$. Hence, (4.18) gives

$$(4.19) \quad \nu_1 + q(q - 1 - e)e = 2q + u(u - 2) - 2qc_{\theta_1,0}.$$

We shall prove $e = 0$. For $1 \leq e \leq q - u + 1$, we have $q(q - 1 - e)e > 2q + u(u - 2)$ for $3 \leq u \leq q - 2$ and

$$\nu_1 + q(q - 1 - e)e \geq \theta_1 + q(q - 1 - e)e > 2q + u(u - 2)$$

for $u = q - 1$ with even q since every line through R contains a point of N_1 when u is odd, both of which contradict (4.19). Suppose $u = q - 1$ with odd q . For $e = 1$, (4.19) gives $\nu_1 + 2qc_{\theta_1,0} = 3$, whence $c_{\theta_1,0} = 0$ and $\nu_1 = 3$. Then, one can take a line in δ containing at least two points of N_0 but none of $N_1 \cup N_u$. This contradicts (4.2). For $e = 2$, we have $q(q - 1 - e)e > 2q + u(u - 2)$ for $q \geq 5$, which contradicts (4.19). Hence, we obtain $e = 0$, $c = u$, and (4.19) yields

$$\nu_1 = 2q + u(u - 2) - 2qc_{\theta_1,0}.$$

Assume $c_{\theta_1,0} > 0$. Then, $c_{0,2} = c_{0,u} = 0$ since every line contains a point of N_0 . From (4.17), we have $\nu_1 = \sum_i (u - 2i)c_{i,u-2i}$, which implies that $c_{i,u+q-2i} = 0$ for $u \leq i < (q + u)/2$, and $c_{(q+u)/2,0} = \theta_1 - u$. Hence, $q + u$ is even, and $u \leq q - 2$. Suppose $c_{\theta_1,0} \geq 2$. Then, (4.15) yields that $c_{1,0} = c_{\theta_1,0} + q - u \geq 3$. Let ℓ_0 be a $(1, 0)$ -line and let P be the point of N_0 on ℓ_0 . Then, every $(\theta_1, 0)$ -line meets ℓ_0 in P . Since $c_{\theta_1,0} \geq 2$, all of $(1, 0)$ -lines and $(\theta_1, 0)$ -lines in δ are on the point P . Take a $((q + u)/2, 0)$ -line meeting ℓ_0 in a point of N_2 , say Q . Then, any (i, j) -line through Q other than $\langle P, Q \rangle$ and $\langle Q, R \rangle$ satisfies $2i + j = 2 + q$. So, counting the number of points in the lines through Q , we get a contradiction as

$$2\nu_0 + \nu_1 = (2 + q)(q - 1) + 2 + (q + u) = q^2 + 2q + u.$$

Thus, $c_{\theta_1,0} = 1$ and $\nu_1 = u(u - 2)$. Then, (4.15) gives $c_{1,0} = \theta_1 - u$ and (4.16) gives $\sum_i ic_{i,u-2i} = u = c = \sum_i c_{i,u-2i}$, which yields $c_{1,u-2} = u$, and $\nu_0 =$

$u + (\theta_1 - u)(q + u)/2$. Let $\mathcal{L}$ be the set of $q + 4 - u$ lines consisting of the $(\theta_1, 0)$ -line, the $(1, 0)$ -lines and any two of the $(1, u - 2)$ -lines. Counting the number of points in the lines through each point as above, one can show that $\mathcal{L}$ forms a $(q + 4 - u)$ -arc of lines. For $u = 3$ with odd $q \geq 5$, $\mathcal{L}$ forms a $(q + 1)$ -arc of lines. This contradicts the fact that any q -arc is contained in the unique $(q + 1)$ -arc (see Theorem 10.28 in [12]) since there are three $(1, u - 2)$ -lines. For $u = 4$ with $q = 2^h$, $h \geq 3$, one can take a line ℓ with $|\ell \cap N_1| = 1$, for $\nu_1 = 8$, which is impossible from (4.2). Hence (b-2) holds.

Finally, assume $c_{\theta_1, 0} = 0$. Then, $\nu_1 = 2q + u(u - 2)$, $\nu_0 = (q^2 - q - u^2 + 3u)/2$. We also get $c_{1, 0} = c_{i, u-2i} = 0$ for $1 \leq i \leq u/2$ from (4.16), $c_{0, u} = c = u$, and (4.15) gives $c_{0, 2} = q - u$, giving (b-1). $\square$

Theorem 2.3 follows from Lemma 4.4. We continue the investigation for the Cases 2 and 3 in the following subsections.

4.2. Case 2. Now, we consider the Case 2 with $N_u = \{R_u\}$, $N_v = \{R_v\}$, $3 \leq u < v \leq q - 1$. Assume $\langle R_u, R_v \rangle$ is a (ρ_0, ρ_1) -line with index α , i.e., $\alpha = 0, 1, 2$ when (ρ_0, ρ_1) is in $\mathcal{T}_{u, v}^+$, $\mathcal{T}_{u, v}$, $\mathcal{T}_{u, v}^-$, respectively. Recall from Lemma 4.3 that the condition $u + v < q + 1$ (resp. $u + v > q + 1$) is needed when $\alpha = 0$ (resp. $\alpha = 2$) and that $\alpha = 1$ if $u + v = q + 1$. Calculating $\sum_{(i, j) \in \Lambda_1} (2i + j - q - 2)c_{i, j}$,

$\sum_{(i, j) \in \Lambda_1} i(2i + j - q - 2)c_{i, j}$ and $\sum_{(i, j) \in \Lambda_1} j(2i + j - q - 2)c_{i, j}$ using (4.5)-(4.11) again, we get the following equalities:

$$(4.20) \quad \theta_1(2\nu_0 + \nu_1 - \theta_1^2 - u - v + 4) - 1 = -q(c_{1, 0} + c_{0, 2} - c_{\theta_1, 0} + t_u + t_v + \alpha),$$

$$(4.21) \quad \nu_0(2\nu_0 + \nu_1 - \theta_2 - u - v + 3) = -q(c_{1, 0} - \theta_1 c_{\theta_1, 0} + t_0 + \alpha \rho_0),$$

$$(4.22) \quad \nu_1(2\nu_0 + \nu_1 - \theta_1^2 - u - v + 3) = -q(2c_{0, 2} + t_1 + \alpha \rho_1),$$

where $t_u = \sum_{(i, j) \in \mathcal{T}_u} c_{i, j}$, $t_v = \sum_{(i, j) \in \mathcal{T}_v} c_{i, j}$, $t_0 = \sum_{(i, j) \in \mathcal{T}_u \cup \mathcal{T}_v} i c_{i, j}$ and $t_1 = \sum_{(i, j) \in \mathcal{T}_u \cup \mathcal{T}_v} j c_{i, j}$.

It holds that

$$(4.23) \quad 2\nu_0 + \nu_1 - \theta_1^2 - u - v + 4 \equiv 1 \pmod{q}$$

and

$$(4.24) \quad 2\nu_0 + \nu_1 \leq \theta_1^2 + u + v - 3$$

from (4.20) and (4.22), respectively. Counting the number of points through $R_u \in N_u$, we obtain

$$(4.25) \quad \begin{aligned} 2\nu_0 + \nu_1 &= ut_u + (q + u)(q - t_u) + u + v - 2 + (1 - \alpha)q \\ &= u + v - 2 + q(q + u + 1 - \alpha - t_u). \end{aligned}$$

Lemma 4.5. *Assume $2\nu_0 + \nu_1 = \theta_1^2 + u + v - 3$. Then, u and v are even, $c_{0,2} = c_{1,0} = 0$, $t_u = c_{u/2,0}$, $t_v = c_{v/2,0}$, $(t_u, t_v, c_{\theta_1,0}) = (u - 1 - \alpha, v - 1 - \alpha, u + v - 1 - \alpha)$ and $\alpha \leq 1$.*

Proof. Assume $2\nu_0 + \nu_1 = \theta_1^2 + u + v - 3$. Then, it follows from (4.25) that $t_u = u - 1 - \alpha$ and we get $c_{0,2} = \alpha\rho_1 = c_{i,j} = 0$ for all $(i, j) \in \mathcal{T}_u \cup \mathcal{T}_v \cup \mathcal{T}_{u,v} \cup \mathcal{T}_{u,v}^-$ with $j > 0$ from (4.22). Substituting $2\nu_0 + \nu_1 = \theta_1^2 + u + v - 3$ to (4.20), we also obtain

$$(4.26) \quad c_{\theta_1,0} = c_{1,0} + t_u + t_v + \alpha + 1 = c_{1,0} + u + v - \alpha - 1 \geq v,$$

since $u \geq 3$ and $\alpha \leq 2$. If v is odd, then $t_v = 0$, and we get $v = \alpha + 1 \leq 3$ from (4.25), giving a contradiction. If u is odd and v is even, then $t_u = 0$, and $u = \alpha + 1 \leq 3$, giving $u = 3$ and $\alpha = 2$ with $u + v > q + 1$, $(\rho_0, \rho_1) \in \mathcal{T}_{u,v}^-$. So, $v = q - 1$ since $v \leq q - 1$. Then, $(\rho_0, \rho_1) = (0, 0)$, which contradicts the fact that there exists a $(\theta_1, 0)$ -line. Hence, u and v are even, $v \geq 6$ and $t_u = c_{u/2,0}$, $t_v = c_{v/2,0}$. Suppose $c_{1,0} > 0$ and let P be a point of N_0 on a $(1, 0)$ -line. Since $(\theta_1, 0)$ -lines are on P , a $(v/2, 0)$ -line must contain P and R_v , which contradicts (4.26) as $t_v = v - 1 - \alpha \geq 3$. Thus, $c_{1,0} = 0$.

Now, assume $\alpha = 2$. Then, $\rho_1 = 0$, and the existence of a $(\theta_1, 0)$ -line from (4.26) implies that $\rho_0 > 0$, whence

$$(4.27) \quad u + v \geq q + 4.$$

From (4.21), we get

$$\begin{aligned} \nu_0 &= \theta_1 c_{\theta_1,0} - t_0 - 2\rho_0 \\ &= q(u + v - 2) - 1 - u(u - 3)/2 - v(v - 3)/2 \end{aligned}$$

since $t_u = u - 3$, $\rho_0 = (u + v - 2 - q)/2$ and $c_{\theta_1,0} = u + v - 3$. Then, from the equality $2\nu_0 + \nu_1 = \theta_1^2 + u + v - 3$, one can get

$$(4.28) \quad \begin{aligned} \nu_1 &= q^2 - 2q(u + v - 3) + u(u - 2) + v(v - 2) \\ &= (q + 1 - u)^2 + (q + 1 - v)^2 - q^2 + 2q - 2. \end{aligned}$$

From (4.27) and (4.28), ν_1 takes the largest value when $(u, v) = (6, q - 2)$ or $(q - 2, 6)$ with q even since u and v are even. But $\nu_1 = -8q + 32 < 0$ when $(u, v) = (6, q - 2)$ with $q \geq 8$, a contradiction. Hence $\alpha \neq 2$. $\square$

From (4.23) and (4.24), we may assume that $2\nu_0 + \nu_1 = \theta_2 + u + v - 3 - eq$ with non-negative integer e when $2\nu_0 + \nu_1 < \theta_1^2 + u + v - 3$.

Lemma 4.6. *Assume $2\nu_0 + \nu_1 = \theta_2 + u + v - 3 - eq$ with $e \geq 0$. Then,*

- (a) $t_u = e + u - \alpha$, $t_v = e + v - \alpha$ with $e \leq q + \alpha - v$.
- (b) $e = 0$ if $u(u - 2) + v(v - 2) < q^2 - 4q$ when $\alpha = 0$ or 1.
- (c) $e = 0$ if $u(u - 2) + v(v - 2) < q^2 - 2q$ when $\alpha = 2$.
- (d) If $e = c_{\theta_1,0} = 0$ and $\alpha = 0$ or 1, then $c_{1,0} = 0$, $\nu_1 = u(u - 2) + v(v - 2) + 2q$ and $(c_{0,u}, c_{0,v}, c_{0,2}) = (u - \alpha, v - \alpha, q - u - v + \alpha)$. In addition, $\rho_0 = 0$ when $\alpha = 1$.
- (e) If $e = c_{\theta_1,0} = 0$ and $\alpha = 2$, then $c_{1,0} = c_{0,2} = 0$, $\nu_1 = u(u - 2) + v(v - 2)$, $q = u + v - 2$ and $(c_{0,u}, c_{0,v}, \rho_0) = (u - 2, v - 2, 0)$.

Proof. We get (a) from (4.25) since $t_v \leq q$. Since $2\nu_0 + \nu_1 = \theta_2 + u + v - 3 - eq$, the equalities (4.20)-(4.22) can be written as

$$(4.29) \quad \theta_1 e + q = c_{1,0} + c_{0,2} - c_{\theta_1,0} + t_u + t_v + \alpha,$$

$$(4.30) \quad \nu_0 e = c_{1,0} - \theta_1 c_{\theta_1,0} + t_0 + \alpha \rho_0,$$

$$(4.31) \quad \nu_1(e + 1) = 2c_{0,2} + t_1 + \alpha \rho_1,$$

respectively. From (4.29), $2 \times (4.30) + (4.31)$ gives

$$\begin{aligned} \nu_1 - e^2 q + e(q^2 - q + u + v - 4) &= (u - 2)t_u + (v - 2)t_v + 2q(1 - c_{\theta_1,0}) \\ &\quad + \alpha(u + v - 4 + (1 - \alpha)q). \end{aligned}$$

Hence, using the equalities in (a), we obtain

$$(4.32) \quad \nu_1 + eq(q - 1 - e) = u(u - 2) + v(v - 2) + 2q(1 - c_{\theta_1,0})$$

for $\alpha = 0, 1$ and

$$(4.33) \quad \nu_1 + eq(q - 1 - e) = u(u - 2) + v(v - 2) - 2qc_{\theta_1,0}.$$

for $\alpha = 2$. Assume $\alpha = 0$ or 1 and $u(u-2) + v(v-2) < q^2 - 4q$. If $e > 0$, then $1 \leq e \leq q-2$ from (a), and $u(u-2) + v(v-2) + 2q < q(q-2) \leq eq(q-1-e)$, which contradicts (4.32). Hence, (b) holds. One can prove (c) similarly.

Finally, assume $e = c_{\theta_1,0} = 0$. Then, from (4.30), we have $c_{1,0} = c_{i,j} = 0$ for all $(i, j) \in \mathcal{T}_u \cup \mathcal{T}_v$ with $i > 0$, giving $t_u = c_{0,u}$, $t_v = c_{0,v}$, $\alpha\rho_0 = 0$. We also get $c_{0,2} = q - t_u - t_v - \alpha = q - u - v + \alpha$ from (4.29) and (a). When $\alpha = 2$, the condition $u + v \geq q + 2$ is needed, giving $c_{0,2} = 0$ and $q = u + v - 2$. ν_1 is obtained from (4.32) and (4.33). Hence, our assertion (d) and (e) hold. $\square$

It could be possible that $2\nu_0 + \nu_1 = \theta_2 + u + v - 3$ only when both of u and v are even by Lemma 4.5. Now, we consider the case when $2\nu_0 + \nu_1 = \theta_2 + u + v - 3 - eq$ with integer $e \geq 0$. We determine the value e for all $q \geq v + 1$ with $3 \leq u < v \leq 8$ so that one can apply our result to the optimal linear codes problem for $v < q \leq 9$.

Lemma 4.7. *Let $(u, v) = (3, 4)$. Then,*

- (a) *For $q \geq 7$, $2\nu_0 + \nu_1 = \theta_2 + 4$ and $t_3 = 3 - \alpha$, $t_4 = 4 - \alpha$.*
- (b) *For $q = 5$, $2\nu_0 + \nu_1 = 30$ or 35 and $t_3 \leq 4$, $t_4 \leq 5$.*

Proof. Assume $q \geq 7$ (or $q = 5$ with $(\rho_0, \rho_1) \in \mathcal{T}_{3,4}^-$). Then, $e = 0$, $t_3 = 3 - \alpha$, $t_4 = 4 - \alpha$ and $2\nu_0 + \nu_1 = \theta_2 + 4$ by Lemma 4.6. Assume $q = 5$ with $(\rho_0, \rho_1) \in \mathcal{T}_{3,4} \cup \mathcal{T}_{3,4}^+$. Then, (4.32) gives $\nu_1 + 5e(4-e) = 21 - 10c_{\theta_1,0}$. From Lemma 4.6 (a), we have $t_3 + \alpha = e + 3$ with $e \leq 2$. If $e = 2$, then $c_{\theta_1,0} = 0$ and $\nu_1 = 1$, which contradicts that $\nu_1 \geq t_3 \geq e + 2$. Hence $e \leq 1$. $\square$

From Lemma 4.6, one can prove the following similarly.

Lemma 4.8. *It holds that $t_u = u - \alpha$, $t_v = v - \alpha$ and $2\nu_0 + \nu_1 = \theta_2 + u + v - 3$ for*

- (a) *$(u, v) = (3, 5), (4, 5), (3, 7)$ for all $q \geq v + 1$;*
- (b) *$(u, v) = (3, 6)$ for $q \geq 8$;*
- (c) *$(u, v) = (4, 6), (5, 6), (4, 7), (5, 7)$ for $q \geq 9$;*
- (d) *$(u, v) = (6, 7), (3, 8), (4, 8), (5, 8), (6, 8), (7, 8)$ for $q \geq 11$.*

Lemma 4.9. *It holds that $2\nu_0 + \nu_1 = \theta_2 + u + v - 3 - eq$ with $e = 0$ or 1 for*

- (a) *$(u, v) = (3, 6), (4, 6), (5, 6)$ for $q = 7$;*

(b) $(u, v) = (4, 6), (5, 6), (4, 7), (5, 7), (6, 7)$ for $q = 8$;

(c) $(u, v) = (6, 7), (3, 8), (4, 8), (5, 8), (6, 8), (7, 8)$ for $q = 9$.

4.3. Case 3. Here, we consider the Case 3: $\nu_0 + \nu_1 + \nu_2 = q^2 + q - 1$ and $\nu_u = 2$ with $3 \leq u \leq q - 1$. Let $N_u = \{R, R'\}$. Assume $\langle R, R' \rangle$ is a (ρ_0, ρ_1) -line with index α . Let t_u^R (resp. $t_u^{R'}$) be the number of (i, j) -lines through R (resp R') with $(i, j) \in \mathcal{T}_u$. Note that $t_u = \sum_{(i,j) \in \mathcal{T}_u} c_{i,j} = t_u^R + t_u^{R'}$. From the

equalities (4.20)-(4.22) with $u = v$, we have $2\nu_0 + \nu_1 \equiv \theta_1^2 + 2u - 3 \pmod{q}$ and $2\nu_0 + \nu_1 \leq \theta_1^2 + 2u - 3$. Then, Lemmas 4.5 and 4.6 can be rewritten as follows.

Lemma 4.10. *Assume $2\nu_0 + \nu_1 = \theta_1^2 + 2u - 3$. Then, u is even, $c_{0,2} = c_{1,0} = 0$, $t_u = c_{u/2,0}$, $(t_u^R, c_{\theta_1,0}) = (u - 1 - \alpha, 2u - 1 - \alpha)$ and $\alpha \leq 1$.*

Proof. Assume $2\nu_0 + \nu_1 = \theta_1^2 + 2u - 3$ with $(\rho_0, \rho_1) \in \mathcal{D}_{u,u}$. Then, as in the discussion for Case 2, we have that $t_u^R = u - 1 - \alpha$ and that $c_{0,2} = \alpha\rho_1 = c_{i,j} = 0$ for all $(i, j) \in \mathcal{T}_u \cup \mathcal{T}_v \cup \mathcal{T}_{u,v} \cup \mathcal{T}_{u,v}^-$ with $j > 0$. Also, (4.26) can be rewritten as

$$(4.34) \quad c_{\theta_1,0} = c_{1,0} + t_u^R + t_u^{R'} + \alpha + 1 = c_{1,0} + 2u - \alpha - 1 \geq u,$$

since $u \geq 3$ and $\alpha \leq 2$. If u is odd, then $t_u = 0$, and we get $3 \leq u = \alpha + 1 \leq 3$, giving $\alpha = 2$ and $u = 3$. Then, $\rho_1 = 0$, $q = 4$ and $\rho_0 = 0$, which contradicts (4.34). Hence, u is even and $t_u = c_{u/2,0}$. Suppose $c_{1,0} > 0$ and let P be a point of N_0 on a $(1, 0)$ -line. Since $(\theta_1, 0)$ -lines are on P , a $(u/2, 0)$ -line must be $\langle P, R \rangle$ or $\langle P, R' \rangle$, giving $t_u^R = 1$, and $\rho_0 \geq 2u - \alpha$ from (4.34). Hence, $\alpha = 0$ and $u = 2$, a contradiction. Thus, $c_{1,0} = 0$. One can prove $\alpha \neq 2$ similarly to the proof of Lemma 4.5. $\square$

Lemma 4.11. *Assume $2\nu_0 + \nu_1 = \theta_2 + 2u - 3 - eq$ with $e \geq 0$. Then,*

(a) $t_u^R = t_u^{R'}$, $t_u^R = e + u - \alpha$ with $e \leq q + \alpha - u$.

(b) $e = 0$ if $u(u - 2) < q^2/2 - 2q$ when $\alpha = 0$ or 1.

(c) $e = 0$ or $(q, u, e) = (4, 3, 3)$ if $u(u - 2) < q^2/2 - q$ when $\alpha = 2$.

(d) If $e = c_{\theta_1,0} = 0$ and $\alpha = 0$ or 1, then $c_{1,0} = 0$, $t_u = c_{0,u}$, $\nu_1 = 2u(u - 2) + 2q$ and
 $(t_u^R, c_{0,2}) = (u, q - 2u)$ when $\alpha = 0$; $(t_u^R, c_{0,2}, \rho_0) = (u - 1, q - 2u + 1, 0)$ when $\alpha = 1$.

(e) If $e = c_{\theta_1,0} = 0$ and $\alpha = 2$, then q is even, $u = q/2 + 1$, $\nu_1 = q^2/2 - 2$, $\nu_0 = q^2/4 + q + 1$, $t_u = c_{0,u}$, $c_{1,0} = c_{0,2} = 0$ and $(\rho_0, \rho_1) = (0, 0)$.

Proof. One can see (4.29)-(4.31) as $t_u = t_u^R$ and $t_v = t_u^{R'}$ for Case 3, whence (4.32) and (4.33) can be replaced by

$$(4.35) \quad \nu_1 + eq(q - 1 - e) = 2u(u - 2) + 2q(1 - c_{\theta_1,0})$$

for $\alpha = 0, 1$ and

$$(4.36) \quad \nu_1 + eq(q - 1 - e) = 2u(u - 2) - 2qc_{\theta_1,0}.$$

for $\alpha = 2$. So, (a) and (b) are straightforward. Assume $\alpha = 2$ and $u(u - 2) < q^2/2 - q$. If $1 \leq e \leq q - 2$, then $2u(u - 2) < q(q - 2) \leq eq(q - 1 - e)$, which contradicts (4.36). If $e = q - 1$, we have $u = 3$ from (a), and $q = 4$ since $2u \geq q + 2$ is needed for $\alpha = 2$. Hence, (c) holds.

Finally, assume $e = c_{\theta_1,0} = 0$. Then, from (4.30), we have $c_{1,0} = \alpha\rho_0 = c_{i,j} = 0$ for all $(i, j) \in \mathcal{T}_u \cup \mathcal{T}_v$ with $i > 0$, giving $t_u^R = t_u^{R'} = c_{0,u}/2$. We also get $c_{0,2} = q - 2t_u^R - \alpha = q - 2u + \alpha$ from (4.29) and (a). ν_1 is obtained from (4.35). Hence we get (d) for $\alpha = 0, 1$. When $\alpha = 2$, the condition $2u \geq q + 2$ yields that $c_{0,2} = 0$ and $q = 2u - 2$. Hence, q is even, $u = q/2 + 1$, $\nu_1 = q^2/2 - 2$ from (4.36), giving (e). $\square$

Now, Theorem 2.4 follows from Lemmas 4.5, 4.10, and Theorem 2.5 follows from Lemmas 4.6, 4.11. We investigate more detail for all $q \geq u + 1$ with $3 \leq u \leq 8$.

Table 1: (ν_0, ν_1) and $c_{i,j}$'s for $q = 4$

ν_0	ν_1	$c_{1,4}$	$c_{2,2}$	$c_{3,0}$	$c_{1,0}$	$c_{0,2}$	$c_{5,0}$	$c_{0,3}$	$c_{1,1}$	$c_{3,1}$	$c_{0,0}$	$c_{2,0}$	$c_{1,2}$
3	6		3		3	6		2	6		1		
5	6	1	4	2	1	4			8				1
5	6		6	1	2	3		2	6			1	
7	6	1	6	3		2			6	2			1
7	6		8	2	1	1		2	4	2		1	
9	6		9	3				2		6	1		
9	6	1	8	2			1		4	4			1

Lemma 4.12. Assume $q \geq 4$, $\nu_0 + \nu_1 + \nu_2 = q^2 + q - 1$ and $\nu_3 = 2$. Let $N_3 = \{R, R'\}$ and let m be the line $\langle R, R' \rangle$ of index α . Then, one of the following holds.

(a) For $q \geq 7$, $(\nu_0, \nu_1) = \left(\binom{q}{2} - 1, 2q + 6 \right)$, $c_{1,0} = c_{2,0} = c_{1,1} = c_{1,2} = c_{\theta_1,0} = 0$ and either

(a-1) $(c_{0,2}, c_{0,3}) = (q - 5, 4)$, $t_3^R = 2$ and m is a $(0, 4)$ -line if $\alpha = 1$ or

(a-2) $(c_{0,2}, c_{0,3}) = (q - 6, 6)$ and $t_3^R = 3$ if $\alpha = 0$.

(b) For $q = 5$, it holds either

(b-1) $(\nu_0, \nu_1) = (14, 6)$, $c_{6,0} = c_{1,0} = c_{1,2} = 1$, $(c_{1,1}, c_{4,0}, c_{3,2}) = (4, 4, 2)$ and the lines through each point of N_3 consist of two $(1, 1)$ -lines, two $(4, 0)$ -lines, one $(3, 2)$ -line and the $(1, 2)$ -line m , or

(b-2) $(\nu_0, \nu_1) = (9, 16)$, $c_{0,2} = c_{1,0} = c_{2,0} = c_{1,1} = c_{1,2} = c_{\theta_1,0} = 0$, $(c_{0,3}, c_{3,2}, c_{0,4}) = (4, 6, 1)$ and the lines through each point of N_3 consist of two $(0, 3)$ -lines, three $(3, 2)$ -lines and the $(0, 4)$ -line m .

(c) For $q = 4$, the possible (ν_0, ν_1) and $c_{i,j}$'s are as in Table 1.

Proof. When $\alpha = 2$, the condition $q + 2 \leq 2u = 6$ is needed, which implies $q = 4$. For $q = 4$, all the possible (ν_0, ν_1, ν_2) and $c_{i,j}$'s are known, see Tables 3.1-3.4 in [23], giving (c). So, we assume $q \geq 5$ with $\alpha = 0$ or 1.

We may assume that $2\nu_0 + \nu_1 = \theta_2 + 3 - eq$ with integer $e \geq 0$. Assume $q \geq 7$. Then, $e = 0$, $t_3^R = 3 - \alpha$ and $2\nu_0 + \nu_1 = \theta_2 + 3$ by Lemma 4.11. Assume $q = 5$ with $(\rho_0, \rho_1) \in \mathcal{T}_{3,4} \cup \mathcal{T}_{3,4}^+$. Then, (4.35) gives

$$(4.37) \quad \nu_1 + 5e(4 - e) = 16 - 10c_{\theta_1,0}.$$

From Lemma 4.11 (a), we have $t_3^R + \alpha = e + 3$ with $e \leq 3$. If $e = 2$, then $5e(4 - e) = 20 > 16$, which contradicts (4.37). If $e = 1$ or $e = 3$, then $c_{\theta_1,0} = 0$ and $\nu_1 = 1$ by (4.37), which contradicts that $\nu_1 \geq t_3^R = e + 3 - \alpha \geq 3$. Thus, $e = 0$ for all $q \geq 5$.

Now, we have $\nu_1 = 6 + 2q - 2qc_{\theta_1,0}$ from (4.35). For even q , the lines through R other than m have a point of N_1 , and we have $\nu_1 \geq q$, which yields that $c_{\theta_1,0} = 0$. Assume q is odd and that $c_{\theta_1,0} = 1$. Then, $\nu_1 = 6$ and $\nu_0 = (q^2 + q - 2)/2$, $c_{0,2} = c_{0,3} = c_{0,4} = 0$. Suppose $\alpha = 0$. Then, one can take a point $P \in N_0$ of m which is not on the $(\theta_1, 0)$ -line. Since any line through P contains a point of N_1 , we have $\nu_1 = 6 \geq q + 1$, giving $q = 5$, which contradicts that $\mathcal{T}_{3,3}^+ = \emptyset$ for $q = 5$. Thus, $\alpha = 1$, $t_3^R = 2$ and $(\rho_0, \rho_1) \in \mathcal{T}_{3,3} = \{(1, 2), (2, 0)\}$. Then, one can take an

(i, j) -line ℓ_0 through R with $(i, j) \in \mathcal{T}_3^+$, $j \geq 2$. Considering the lines through a fixed point of N_0 on ℓ_0 not on the $(\theta_1, 0)$ -line again, we have $\nu_1 \geq q - 1 + 2$, giving $q = 5$. We also obtain $c_{1,0} = \rho_0 = 1$ from (4.29) and (4.30). Hence, $c_{1,2} = 1$, and (b-1) holds. Finally, assume $c_{\theta_1,0} = 0$. Then, we obtain $c_{1,0} = c_{1,1} = \alpha\rho_0 = 0$, $t_3^R = 3 - \alpha$ and $c_{0,2} = q - 6 + \alpha$ by Lemma 4.11 (d). Hence, m is a $(0, 4)$ -line when $\alpha = 1$, giving (a) and (b-2). $\square$

From Lemma 4.11, one can prove the following similarly.

Lemma 4.13. *It holds that $t_u^R = u - \alpha$ and $2\nu_0 + \nu_1 = \theta_2 + 2u - 3$ for*

- (a) $u = 4, 5$ for all $q \geq u + 3$;
- (b) $u = 7$ for $q \geq 11$;
- (c) $u = 6, 8$ for all $q \geq u + 5$.

Lemma 4.14. *It holds that $2\nu_0 + \nu_1 = \theta_2 + 2u - 3 - eq$ with $e = 0$ or 1 for*

- (a) $u = 5$ for $q = 7$;
- (b) $u = 6, 7$ for $q = 8, 9$;
- (c) $u = 8$ for $q = 11$.

Lemma 4.15. *It holds that $2\nu_0 + \nu_1 = \theta_2 + 2u - 3 - eq$ with $e = 0, 1$ or 2 for*

- (a) $u = 4$ for $q = 5$;
- (b) $u = 6$ for $q = 7$;
- (c) $u = 8$ for $q = 9$.

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