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CHEIN AUTOMORPHISMS OF FREE METABELIAN ANTICOMMUTATIVE ALGEBRAS*

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Dedicated to the 75th birthday of Professor V. Drensky

ABSTRACT. We describe all automorphisms of a free metabelian anticommutative algebra of rank $n \geq 3$ over a field K that move only one variable while fixing the others. Such automorphisms are called Chein automorphisms in the cases of free metabelian groups and free metabelian Lie algebras. We show that all automorphisms of a free metabelian anticommutative algebra of rank $n = 2$ are linear, and that the simplest non elementary Chein automorphism of degree 3 is absolutely wild for all $n \geq 3$.

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1. Introduction. Let $\mathfrak{M}$ be an arbitrary variety of algebras over a field K and let $A = K_{\mathfrak{M}}\langle x_1, x_2, \dots, x_n \rangle$ be the free algebra of $\mathfrak{M}$ with a free set of generators $X = \{x_1, x_2, \dots, x_n\}$. As usual we denote by n -tuple $\phi = (f_1, \dots, f_n)$ the endomorphism ϕ of A such that $\phi(x_i) = f_i$, $1 \leq i \leq n$. In particular, $\text{Id} = (x_1, \dots, x_n)$ is the identity automorphism. Let $\text{Aut}(A)$ be the group of all automorphisms of A . An automorphism

$$(1) \quad (x_1, \dots, x_{i-1}, \alpha x_i + f, x_{i+1}, \dots, x_n),$$

where $0 \neq \alpha \in K$, $f \in K_{\mathfrak{M}}\langle X \setminus \{x_i\} \rangle$, is called *elementary*. The subgroup $\text{TAut}(A)$ of $\text{Aut}(A)$ generated by all elementary automorphisms is called the *tame automorphism group*, and the elements of this subgroup are called *tame automorphisms* of A . Nontame automorphisms of A are called *wild*.

An automorphism ϕ is called *approximately tame* [21] if it can be approximated by a sequence of tame automorphisms $\{\psi_k\}_{k \geq 0}$ with respect to the *formal power series topology* (see exact definitions in Section 7). An automorphism ϕ is called *absolutely wild* if it is not approximately tame.

In 1968 O. Chein [5] introduced a specific type of automorphisms of free metabelian groups of finite rank $n \geq 3$, and proved that they are not tame when $n = 3$. Nevertheless they are tame when $n \geq 4$ since all automorphisms of free metabelian groups of rank $n \geq 4$ are tame by the well known Bachmuth-Mochizuki-Roman'kov Theorem [2, 16].

Analogues of these automorphisms for free metabelian algebras were studied by V. Roman'kov and his coauthors [9, 18, 19] as well as by R. Nauryzbaev [12, 14]. Let M be a free metabelian Lie algebra in the variables $x_1, \dots, x_n$. As usual, the group of all linear automorphisms of M will be identified with the group of all invertible matrices $GL_n(K)$. In 2015, Nauryzbaev [15] proved that $\text{TAut}(M)$ is generated by all linear automorphisms and the quadratic automorphism

$$(2) \quad (x_1 + [x_2, x_3], x_2, \dots, x_n)$$

if either $n \geq 4$ and K is a field of characteristic $\neq 3$, or $n \geq 5$ and K is a field of arbitrary characteristic.

A peculiar feature of metabelian Lie algebras is that they admit automorphisms of the form (1) that are not necessarily elementary; that is, f may also involve x_i . We call them *Chein automorphisms* as in the case of groups [8, 17]. Notice that, by our definition, all elementary automorphisms form a subset of Chein automorphisms. These automorphisms move only one variable and fix the

others. The subgroup $\text{ATAut}(M)$ of $\text{Aut}(M)$ generated by all Chein automorphisms will be called the *almost tame automorphism group*, and the elements of this subgroup will be called *almost tame automorphisms* of M [12]. Thus

$$\text{TAut}(M) \subseteq \text{ATAut}(M) \subseteq \text{Aut}(M).$$

The automorphisms of the algebra M of rank $n = 3$ have been extensively investigated by numerous authors. In 1992, V. Drensky [7] (see also [3]) proved that the exponential automorphism

$$\exp(\text{ad}[x_1, x_2]) = (x_1 + [[x_1, x_2], x_1], x_2 + [[x_1, x_2], x_2], x_3 + [[x_1, x_2], x_3])$$

of M is wild. Moreover, this automorphism is not almost tame [12, 18]. Every non-elementary Chein automorphism is also shown to be wild [12, 18]. The defining relations for elementary automorphisms [13] and for Chein automorphisms [14] have also been determined. It was recently observed [21] that the examples of wild automorphisms mentioned here are actually absolutely wild.

The first example of a Chein automorphism is

$$(3) \quad (x_1 + [[x_2, x_3], x_1], x_2, \dots, x_n),$$

which is absolutely wild when $n = 3$ and plays an important role in the study of $\text{Aut}(M)$ for $n \geq 4$. In 1993 Bryant and Drensky [4] (see also [11]) proved that every automorphism of M of rank $n \geq 4$ is approximately tame in characteristic zero. Thus (3) is approximately tame in characteristic zero. Recently it was shown in [28] that $\text{Aut}(M_n)$ is generated modulo $GL_n(K)$ by (2) and (3) when either $n = 4$ and the characteristic of K is not 3, or $n \geq 5$ and K is an arbitrary field. It remains an open question whether (3) is tame or wild for $n \geq 4$.

This paper was motivated by the behavior of the Chein automorphism (3) in free metabelian Lie algebras. Here we study automorphisms of free metabelian anticommutative algebras. These algebras also admit Chein automorphisms. First we describe all Chein automorphisms of free metabelian anticommutative algebras. We show that every automorphism of a free metabelian anticommutative algebra of rank 2 is tame; that is, all such automorphisms are linear. We also show that the analogue of the Chein automorphism (3) for free metabelian anticommutative algebras of rank $n \geq 3$ is absolutely wild.

2. Free metabelian anticommutative algebras. Let K be an arbitrary field. An algebra A over K is called *anti-commutative* if the identity

$$(4) \quad x^2 = 0$$

is satisfied. This identity implies

$$(5) \quad xy = -yx.$$

An algebra A is called *metabelian* if it satisfies the identity

$$(6) \quad (xy)(zt) = 0.$$

Let A be a free metabelian anticommutative algebra in the variables $x_1, \dots, x_n$. The identities (4)–(6) imply that every element of $A^2 = AA$ is a linear combination of the elements of the form

$$(7) \quad (((x_i x_j) x_{k_1}) \cdots) x_{k_s},$$

where $i > j$, $1 \leq k_r \leq n$ for all $1 \leq r \leq s$, and $s \geq 0$.

Let $Y = A/A^2$ and denote by y_i the images of x_i in Y . Then Y is an algebra with trivial multiplication and with a linear basis $y_1, \dots, y_n$. Denote by U the free associative algebra with identity and with free generators $z_i = R_{y_i}$, where $1 \leq i \leq n$. In fact, U is the universal (multiplicative) enveloping algebra of Y in the variety of all anticommutative algebras [25, 27] and $z_i = R_{y_i}$ is the universal operator of right multiplication by y_i . Notice that $L_{y_i} = -R_{y_i}$ by (5).

Let T be a free right U -module with free generators $t_1, \dots, t_n$. Turn the space $Y \oplus T$ into an algebra by

$$(a_1 + m_1)(a_2 + m_2) = m_1 R_{a_2} - m_2 R_{a_1}, \quad a_1, a_2 \in Y, \quad m_1, m_2 \in T.$$

Obviously, $Y \oplus T$ is a metabelian anticommutative algebra.

Lemma 1. (1) *The homomorphism $\rho : A \rightarrow Y \oplus T$ defined by $\rho(x_i) = y_i + t_i$ for all i is an embedding;*

(2) *The elements of the form (7) is a basis of A^2 .*

Proof. It is sufficient to check that the images of elements of the form (7) in $Y \oplus T$ under ρ are linearly independent. In fact,

$$\rho(x_i x_j) = t_i z_j - t_j z_i$$

and

$$\rho((((x_i x_j) x_{k_1}) \dots) x_{k_s}) = (t_i z_j - t_j z_i) z_{k_1} \dots z_{k_s}.$$

Let us introduce a deglex order on the set of all basis elements

$$t_i z_{i_1} \dots z_{i_k}$$

of T by setting $t_i < t_j$ and $z_i < z_j$ when $i < j$. Then the leading monomial of $\rho(((x_i x_j) x_{k_1}) \dots x_{k_s})$ is $t_i z_j z_{k_1} \dots z_{k_s}$. Thus, the leading monomials of the images of all distinct elements of the form (7) are different and therefore linearly independent. $\square$

By this lemma, we will identify A with the subalgebra of $Y \oplus T$ generated by $x_i = y_i + t_i$, where $1 \leq i \leq n$. Then every element $a \in A$ can be uniquely represented in the form

$$a = y + t_1 u_1 + \dots + t_n u_n,$$

where $y \in Y$ and $u_1, \dots, u_n \in U$.

Set

$$\frac{\partial a}{\partial x_i} = u_i$$

for all $1 \leq i \leq n$. Then $\frac{\partial a}{\partial x_i}$ are called the Fox derivatives of a [23, 24, 26]. Set also

$$\partial(a) = (u_1, \dots, u_n)^t = \left(\frac{\partial a}{\partial x_1}, \dots, \frac{\partial a}{\partial x_n} \right)^t,$$

where t denotes the transposition. We have $\partial(x_i) = e_i^t$, where e_i is the row vector with a 1 in the i th place and zeros elsewhere.

Lemma 2. (1) If $g \in A^2$ then the column $\partial g = (r_1, r_2, \dots, r_n)^t$ satisfies the conditions

$$(8) \quad r_i = \sum_{j=1}^n z_j w_j^{(i)}, \quad w_j^{(i)} = -w_i^{(j)}, \quad w_i^{(i)} = 0,$$

for all i, j .

(2) If the column $(r_1, r_2, \dots, r_n)^t \in U^n$ satisfies the conditions (8) then there exists $g \in A^2$ such that $\partial g = (r_1, r_2, \dots, r_n)^t$.

Proof. (1) Since ∂ is linear, it is sufficient to check the conditions (8) for an arbitrary element of the form $g = (x_s x_k)u$, where $s > k$ and $u \in U$. We have

$$(x_s x_k)u = (t_s z_k - t_k z_s)u = t_s z_k u - t_k z_s u = t_s r_s + t_k r_k,$$

where $r_s = z_k u = z_k w_k^{(s)}$, $r_k = z_s(-u) = z_s w_s^{(k)}$. Consequently, $w_k^{(s)} = -w_s^{(k)}$ and $w_i^{(j)} = 0$ when $\{i, j\} \neq \{s, k\}$. This proves (1).

(2) Suppose that $r_1, r_2, \dots, r_n \in U$ satisfy the conditions (8). Set $g = \sum_{1 \leq j < i \leq n} (x_i x_j) w_j^{(i)}$. Then

$$\begin{aligned} \partial(g) &= \partial \left(\sum_{1 \leq j < i \leq n} (x_i x_j) w_j^{(i)} \right) = \partial \left(\sum_{1 \leq j < i \leq n} (t_i z_j - t_j z_i) w_j^{(i)} \right) \\ &= \partial \left(\sum_{1 \leq j < i \leq n} t_i z_j w_j^{(i)} - \sum_{1 \leq j < i \leq n} t_j z_i w_j^{(i)} \right) = \partial \left(\sum_{1 \leq j < i \leq n} t_i z_j w_j^{(i)} - \sum_{1 \leq i < j \leq n} t_i z_j w_i^{(j)} \right) \\ &= \partial \left(\sum_{i=1}^n \sum_{j=1}^{i-1} t_i z_j w_j^{(i)} - \sum_{i=1}^n \sum_{j=i+1}^n t_i z_j w_i^{(j)} \right) = \partial \left(\sum_{i=1}^n t_i \left[\sum_{j=1}^{i-1} z_j w_j^{(i)} - \sum_{j=i+1}^n z_j w_i^{(j)} \right] \right) \\ &= \partial \left(\sum_{i=1}^n t_i \left[\sum_{j=1}^{i-1} z_j w_j^{(i)} + \sum_{j=i+1}^n z_j w_j^{(i)} \right] \right) = \partial \left(\sum_{i=1}^n t_i \sum_{j=1}^n z_j w_j^{(i)} \right) = \partial \left(\sum_{i=1}^n t_i r_i \right) \\ &= (r_1, r_2, \dots, r_n)^t. \quad \square \end{aligned}$$

Notice that if $\partial g = (r_1, r_2, \dots, r_n)^t$ for some $g \in A^2$, then r_n is uniquely determined by $r_1, r_2, \dots, r_{n-1}$. In fact, if $r_i = \sum_{j=1}^n z_j w_j^{(i)}$, then, by Lemma 2, we obtain

$$(9) \quad r_n = \sum_{j=1}^{n-1} (-z_j w_n^{(j)}).$$

3. Jacobian matrices. The Jacobian matrix of an endomorphism $\phi = (f_1, \dots, f_n)$ is defined as

$$J(\phi) = \left[\frac{\partial f_j}{\partial x_i} \right]_{1 \leq i, j \leq n} = [\partial(f_1) \dots \partial(f_n)].$$

For any $f \in A$, let $\tilde{f}$ denote its image in $Y = A/A^2$. Every endomorphism $\phi = (f_1, \dots, f_n)$ of A induces an endomorphism of $Y = A/A^2$ that sends y_i to $\tilde{f}_i$

for all i . This endomorphism can be uniquely extended to an endomorphism of U that sends $z_i = R_{y_i}$ to $R_{\tilde{f}_i}$ for all i . Denote this endomorphism of U by $\tilde{\phi}$. For $f \in U$, the image of f under $\tilde{\phi}$ will be written as $f^{\tilde{\phi}}$.

If g is an arbitrary element of A , then it is easy to check that the following form of the “chain rule” holds:

$$\partial(\phi(g)) = \partial(g(f_1, \dots, f_n)) = \sum_{1 \leq i \leq n} \partial(f_i) \left(\frac{\partial g}{\partial x_i} \right)^{\tilde{\phi}}.$$

Recall that the composition of endomorphisms ϕ, ψ of A is defined as

$$\phi \circ \psi(a) = \phi(\psi(a)), \quad a \in A.$$

The chain rule immediately implies that

$$(10) \quad J(\phi \circ \psi) = J(\phi)(J(\psi))^{\tilde{\phi}}.$$

Consequently, if ϕ is an automorphism of A , then $J(\phi)$ is invertible over U . The converse of this statement is an analogue of the famous Jacobian Conjecture [10] for free metabelian anticommutative algebras [25, 27]. Following the arguments in [24, 26], one can show that the following lemma holds.

Lemma 3. *An endomorphism ϕ of A is an automorphism if and only if the Jacobian matrix $J(\phi)$ is invertible.*

We omit the proof of this lemma, as it is not needed in the present paper.

Denote by $\text{IAut}(A)$ the group of all automorphisms of A that induces the identity automorphism on $Y = A/A^2$. The elements of $\text{IAut}(A)$ are called *IA-automorphisms*. If $\phi \in \text{IAut}(A)$ then the Jacobian matrix of ϕ has the form

$$J(\phi) = I_n + R,$$

where I_n is the identity matrix of order n and every column of R satisfies the conditions (8). If $\phi \in \text{IAut}(A)$ then $\tilde{\phi} = \text{Id}$. Consequently, if $\phi, \psi \in \text{IAut}(A)$ then (10) implies that

$$J(\phi \circ \psi) = J(\phi)J(\psi).$$

Corollary 1. *The mapping*

$$J : \text{IAut}(A) \rightarrow SL_n(U)$$

is an embedding of groups.

It is obvious that J is one-to-one. Recall that $SL_n(U)$ denotes the set of all invertible matrices over U whose constant part has determinant one.

4. The structure of Chein automorphisms.

Lemma 4. *Let $f \in A^2$. Then $\frac{\partial f}{\partial x_1} = 0$ if and only if f belongs to the ideal of A generated by all $x_i x_j$, where $i, j > 1$.*

Let $f \in A^2$. Then

$$f = \sum_{1 \leq j < i \leq n} (x_i x_j) u_{ij}, \quad u_{ij} \in U.$$

Consequently,

$$\frac{\partial f}{\partial x_1} = - \sum_{i=2}^n z_i u_{i1}.$$

If $\frac{\partial f}{\partial x_1} = 0$ in the free associative algebra, we immediately get $u_{21} = \dots = u_{n1} = 0$. Therefore

$$(11) \quad f = \sum_{2 \leq j < i \leq n} (x_i x_j) u_{ij},$$

satisfies the statement of the lemma. $\square$

Consider the endomorphisms of the form

$$(12) \quad \delta = (x_1 + f, x_2, \dots, x_n), \quad f \in A^2.$$

Lemma 5. *An endomorphism δ of A of the form (12) is an automorphism if and only if $\frac{\partial f}{\partial x_1} = 0$.*

Proof. We have

$$J(\delta) = \begin{bmatrix} 1 + \frac{\partial f}{\partial x_1} & 0 & \dots & 0 \\ \frac{\partial f}{\partial x_2} & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial f}{\partial x_n} & 0 & \dots & 1 \end{bmatrix}.$$

If $J(\delta)$ is invertible then $1 + \frac{\partial f}{\partial x_1}$ is right invertible in the free associative algebra U . Since $\frac{\partial f}{\partial x_1}$ belongs to the augmented ideal of U it follows that $\frac{\partial f}{\partial x_1} = 0$. By

Lemma 4, f has the form (11). Then δ is a product of endomorphisms

$$\delta_{ij} = (x_1 + (x_i x_j) u_{ij}, x_2, \dots, x_n), \quad 2 \leq j < i \leq n.$$

Since A is metabelian it follows that every δ_{ij} is an automorphism. $\square$

5. Automorphisms of algebras of rank 2. If $n = 1$ then the group of automorphisms $\text{Aut}(A)$ is isomorphic to $GL_1(K) \simeq K^* = K \setminus \{0\}$. If $n = 2$ then all elementary automorphisms are linear since A is anticommutative. Consequently, $\text{TAut}(A) \simeq GL_2(K)$. We show that A has no other automorphisms.

Lemma 6. *If $n = 2$ then $\text{Aut}(A) \simeq GL_2(K)$.*

Proof. It is sufficient to prove that $\text{IAut}(A) = \{\text{Id}\}$. Let $\phi = (x_1 + f_1, x_2 + f_2) \in \text{IAut}(A)$, where $f_1, f_2 \in A^2$. Then

$$J(\varphi) = I_2 + \begin{bmatrix} \partial(f_1) & \partial(f_2) \end{bmatrix} = I_2 + \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

By Lemma 2, we have

$$a = z_2 h, \quad c = -z_1 h, \quad h \in U.$$

The first column of $J(\varphi)$ is $[1 + z_2 h, -z_1 h]^t$. Since $J(\varphi)$ is invertible it follows that $[1 + z_2 h, -z_1 h]^t$ is left unimodular, that is, there exist $u_1, u_2 \in U$ such that

$$(13) \quad u_1(1 + z_2 h) + u_2(-z_1 h) = 1.$$

For any $u \in U$ denote by $\bar{u}$ its highest homogeneous part of u in U with respect to the degree function in the variables z_1, z_2 . If $h \neq 0$, then $\overline{1 + z_2 h} = z_2 \bar{h}$ and $\overline{-z_1 h} = -z_1 \bar{h}$. Then (13) implies that

$$\bar{u}_1 z_2 \bar{h} - \bar{u}_2 z_1 \bar{h} = 0,$$

that is, $z_2 \bar{h}$ and $z_1 \bar{h}$ are left dependent. It is well known that in free associative algebras this implies [6] that one of $z_2 \bar{h}$ or $z_1 \bar{h}$ belongs to the left ideal generated by the other. They must be linearly dependent since they have the same degrees. But they are linearly independent if $h \neq 0$.

Consequently, $h = 0$ and $a = c = 0$. Similarly, $b = d = 0$. Then $f_1 = f_2 = 0$ and $\phi = \text{Id}$. $\square$

6. Divergence in free anticommutative algebras. To prove the main result of the paper, we need the notion of the divergence of a derivation introduced in [21].

Let B be a free anticommutative algebra over K in the variables $\Xi = \{\xi_1, \dots, \xi_n\}$. Let Ξ^* be the set of all (nonempty) nonassociative monomials in the alphabet Ξ . Recall that every $u \in \Xi^*$ of degree ≥ 2 can be uniquely written as $u = u_1 u_2$ for some $u_1, u_2 \in \Xi^*$ [29]. Set $\xi_i < \xi_j$ if $i < j$. For any $u, v \in \Xi^*$ set $u < v$ if $\deg u < \deg v$. If $\deg u = \deg v \geq 2$ and $u = u_1 u_2$ and $v = v_1 v_2$, then set $u < v$ if either $u_1 < v_1$ or $u_1 = v_1$ and $u_2 < v_2$.

A nonassociative monomial $u \in \Xi^*$ is called *regular* if it does not have submonomials of the form vw , where $v, w \in \Xi^*$ and $v \leq w$. The set of all regular words is a linear basis of B [22].

The universal (multiplicative) enveloping algebra $U(B) = U_{\mathfrak{M}}(B)$ is the free associative algebra with identity generated by all universal operators of right multiplication R_u , where u runs over the set of all regular monomials in the alphabet Ξ [25].

Let

$$\Omega_B = d\xi_1 U(B) \oplus \dots \oplus d\xi_n U(B)$$

be a free right $U(B)$ -module with free generators $d\xi_1, \dots, d\xi_n$. The linear map

$$D : B \rightarrow \Omega_B$$

defined by $D(\xi_i) = d\xi_i$ for all i and such that

$$D(ab) = aD(b) + D(a)b = -D(b)R_a + D(a)R_b$$

for all $a, b \in B$, is called the *universal derivation* of B and Ω_B is called the *universal differential module* of B [25, 27].

For any $a \in B$ there exist unique elements $u_1, u_2, \dots, u_n \in U(B)$ such that

$$D(a) = d\xi_1 u_1 + \dots + d\xi_n u_n.$$

The elements $u_i = \frac{\partial a}{\partial \xi_i}$ are called the *Fox derivatives* of $a \in B$ [25, 27]. Set also

$$\partial(a) = \left(\frac{\partial a}{\partial \xi_1}, \dots, \frac{\partial a}{\partial \xi_n} \right)^t.$$

Consider the natural grading

$$B = B_1 \oplus \dots \oplus B_k \oplus \dots$$

of B with respect to the standard function $\deg$, that is, B_k is the linear span of (non-associative) monomials of degree k .

Let $L = \text{Der}(B)$ be the Lie algebra of all derivations of B . For any n -tuple $F = (f_1, \dots, f_n) \in B^n$ there exists a unique derivation D of B such that $D(\xi_i) = f_i$ for all $1 \leq i \leq n$. Denote this derivation by

$$(14) \quad D = f_1 \partial_1 + \dots + f_n \partial_n = D_F.$$

For any derivation of this form define its Jacobian matrix by

$$J(D) = \left[\frac{\partial f_j}{\partial \xi_i} \right]_{1 \leq i, j \leq n} = [\partial(f_1) \dots \partial(f_n)].$$

We say that D from (14) is homogeneous of degree i if $f_1, \dots, f_n \in B_{i+1}$. Let L_i be the space of all homogeneous derivations of degree i . Then

$$L = L_0 \oplus L_1 \oplus \dots \oplus L_k \oplus \dots, \quad [L_i, L_j] \subseteq L_{i+j},$$

is a grading of L . Notice that L_0 is isomorphic to the matrix algebra $\mathfrak{gl}(n)$ with the matrix units $e_{ij} = \xi_i \partial_j$, where $1 \leq i, j \leq n$.

The *divergence* $\text{div}(D)$ of D is defined as the image of the trace of the Jacobian matrix $J(D)$ in the quotient space $U(B)/[U(B), U(B)]$ [21]. This agrees with the definition of the divergence given in [21, Definition 1], since the Jacobson radical $R = R(U)$ of the free associative algebra U is zero. It is shown in [21] that the space

$$S = S_0 \oplus S_1 \oplus \dots \oplus S_k \oplus \dots,$$

of all derivations of B with zero divergence is a graded subalgebra of L and called the *special Lie algebra of derivations* of B .

Let $\text{IE}(k) = \text{IE}(B, k)$ be the monoid of all endomorphisms of B that induces the identity automorphism on the factor-algebra B/B^{k+1} . Then we get the descending series of monoids

$$\text{IE}(1) \supseteq \text{IE}(2) \supseteq \dots \supseteq \text{IE}(k) \supseteq \dots$$

If $\phi \in \text{IE}(i) \setminus \text{IE}(i+1)$ for some $i \geq 1$, then

$$\phi = (\xi_1 + f_1 + F_1, \dots, \xi_n + f_n + F_n),$$

where $f_1, \dots, f_n$ are homogeneous elements of degree $i + 1$, with at least one of them nonzero, and $F_j \in B^{i+2}$ for all $1 \leq j \leq n$. Set

$$T(\phi) = f_1 \partial_1 + \dots + f_n \partial_n.$$

The derivation $T(\phi)$ is referred to as the tangent derivation of the endomorphism ϕ (with respect to the power series topology) [21].

Consider the endomorphism

$$(15) \quad \sigma = (\xi_1 + (\xi_3 \xi_2) \xi_1, \xi_2, \dots, \xi_n)$$

Lemma 7. $\operatorname{div}(T(\sigma)) \neq 0$.

Proof. We have $T(\sigma) = ((\xi_3 \xi_2) \xi_1) \partial_1$. Then the first entry on the main diagonal of $J(T(\sigma))$ is

$$\frac{\partial((\xi_3 \xi_2) \xi_1)}{\partial \xi_1},$$

and all the other diagonal entries are zero. Consequently, the trace of the matrix $J(T(\sigma))$ is

$$\frac{\partial((\xi_3 \xi_2) \xi_1)}{\partial \xi_1} = -R_{\xi_3 \xi_2} \notin [U(B), U(B)].$$

Therefore $\operatorname{div}(T(\sigma)) \neq 0$. $\square$

7. An absolutely wild automorphism. In this section we show that the Chein automorphism

$$(16) \quad \tau = (x_1 + (x_3 x_2) x_1, x_2, \dots, x_n)$$

of the free metabelian anticommutative algebra A is absolutely wild for all $n \geq 3$.

First, we recall the precise definitions of approximately tame and absolutely wild automorphisms from [21].

Let $\operatorname{IA}(k) = \operatorname{IA}(A, k)$ be the set of all automorphisms of A that induces the identity automorphism on the factor-algebra A/A^{k+1} . Consider the descending central series

$$\operatorname{IAut}(A) = \operatorname{IA}(1) \supseteq \operatorname{IA}(2) \supseteq \dots \supseteq \operatorname{IA}(k) \supseteq \dots$$

An automorphism $\phi \in \operatorname{Aut}(A)$ is called *approximately tame* if there exists a sequence of tame automorphisms $\{\psi_k\}_{k \geq 0}$ such that $\phi \psi_k^{-1} \in \operatorname{IA}(k)$. The topology

defined by this definition on $\text{Aut}(A)$ is called the *formal power series topology* [1]. An automorphism ϕ will be called *absolutely wild* if it is not approximately tame.

In 1981 Shafarevich [20] and in 1983 Anick [1] independently proved that every automorphism of the polynomial algebra $K[x_1, x_2, \dots, x_n]$ over a field K of characteristic zero is approximately tame. In 1993 Bryant and Drensky [4] (see also [11]) proved that every automorphism of M of rank $n \geq 4$ is approximately tame in characteristic zero.

The following result is a part of Theorem 5 from [21].

(F) Let $\mathfrak{M}$ be a Nielsen-Schreier variety of algebras and B is a free algebra of $\mathfrak{M}$ of rank n . Let $\mathfrak{N}$ be any subvariety of $\mathfrak{M}$ and let $I \subseteq B$ be the ideal of all identities of $\mathfrak{N}$ in $\mathfrak{M}$ in n variables. Suppose that an endomorphism $\epsilon \in \text{IE}_i(A) \setminus \text{IE}_{i+1}(A)$ for some $i \geq 1$ induces an automorphism ϕ of $B = A/I$ and the ideal I does not contain elements of degree $\leq i + 1$. If $\text{div}(T(\epsilon)) \neq 0$ then ϕ is absolutely wild.

Theorem 1. *The automorphism τ from (16) is an absolutely wild automorphism of the free metabelian anticommutative algebra A of rank $n \geq 3$.*

Proof. We apply (F). Let $\mathfrak{M}$ be the variety of all anticommutative algebras. It is well known that $\mathfrak{M}$ is Nielsen-Schreier [22]. Let B be the free anticommutative algebra in the variables $\Xi = \{\xi_1, \dots, \xi_n\}$. Let $\mathfrak{N}$ be the subvariety of $\mathfrak{M}$ defined by the identity (6). Then I is the ideal of B generated by all elements of the form $(uv)(wq)$, where $u, v, w, q \in \Xi^*$.

Notice that the endomorphism σ of B induces the automorphism τ of $A = B/I$. We have $\sigma \in \text{IE}_2(A) \setminus \text{IE}_3(A)$. The ideal I does not contain elements of degree ≤ 3 . Moreover, $\text{div}(T(\sigma)) \neq 0$ by Lemma 7. By (F), τ is an absolutely wild automorphism of A .

Note that, in general, all results in [21] are stated for fields of characteristic zero. By carefully repeating the proof of Theorem 5 from [21] for these specific varieties $\mathfrak{M}$ and $\mathfrak{N}$, we see that the assumption of characteristic zero is not required. Thus, τ is an absolutely wild automorphism of A of rank $n \geq 3$ over an arbitrary field. $\square$

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