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ON THE CONVERGENCE OF DOUBLE LEGENDRE SERIES WITH COEFFICIENTS OF GENERALIZED BOUNDED VARIATION

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Abstract. We investigate the convergence of double Legendre series whose coefficients belong to classes of bounded and generalized bounded variation. Under these assumptions, we establish pointwise, uniform, and L^r-convergence for $0 < r < 1$. In addition, we study Cesàro and de la Vallée Poussin summability and obtain Tauberian-type results linking summability to ordinary convergence. The results extend known one-dimensional convergence theorems to the setting of double orthogonal expansions.

1. Introduction. The study of convergence of orthogonal series has long occupied a central position in classical and modern analysis. Fourier series, in particular, play a fundamental role in harmonic analysis, approximation theory, and the solution of partial differential equations. It is well known that the Fourier series of an integrable function may diverge everywhere (see, e.g., [5, p. 164, last paragraph]), which naturally leads to the problem of identifying conditions on

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the coefficients that guarantee various modes of convergence such as pointwise, uniform, and L^r-convergence. The fundamental theorem in the theory of the uniform convergence of sine series is due to Chaundy and Jolliffe [4], which is as follows.

Theorem 1.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a non-negative non-increasing sequence. Then the sine series $(S) := \sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly on $[0, 2\pi]$ if and only if $na_n \to 0$ as $n \to \infty$.*

For the cosine series, we highlight the following obvious fact.

Theorem 1.2. *Let $\{a_n\}_{n=0}^{\infty}$ be a non-negative sequence. Then the cosine series $(C) := \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges uniformly on $[0, 2\pi]$ if and only if $\sum_{n=1}^{\infty} a_n$ converges.*

Beginning with the classical theorem of Chaundy and Jolliffe, a vast literature has developed around coefficient conditions weaker than monotonicity, most notably conditions involving bounded variation and its generalizations. These ideas have proven to be powerful tools in understanding convergence behavior while remaining flexible enough to encompass a wide class of series. Several authors gave conditions on the coefficients weaker than monotonicity to obtain similar results. For instance, many generalizations consider coefficients as a sequence of bounded variation. Let us recall the definition of a sequence of bounded variation.

Definition 1.3. *A real sequence $\{a_n\}_{n=0}^{\infty}$ is said to be of bounded variation (BV) if*

$$\sum_{n=0}^{\infty} |\Delta a_n| < \infty,$$

where $\Delta a_n = a_n - a_{n+1},\ n = 0, 1, 2, \ldots$.

Remark 1.4. If $\{a_n\}$ is a non-negative non-increasing sequence, then it is of bounded variation. This is because if $\{a_n\}$ is such a sequence, then $\{a_n\}$ converges, say, to a, and hence for any $N \in \mathbb{N}$,

$$\sum_{n=0}^{N} |\Delta a_n| = |\Delta a_0| + |\Delta a_1| + \cdots + |\Delta a_N|$$

$$\begin{aligned} &= a_0 - a_1 + a_1 - a_2 + \cdots + a_N - a_{N+1} \\ &= a_0 - a_{N+1} \\ &\to a_0 - a, \end{aligned}$$

as $N \to \infty$. Therefore, $\{a_n\}$ is of bounded variation.

From the above remark, the following theorem gives information about pointwise and uniform convergence of sine and cosine series with coefficients of bounded variation, which form a null-sequence, that is, $a_n = o(1)$ as $n \to \infty$, (see [5, 7.2.1]), generalizes Theorem 1.1.

Theorem 1.5. *For a real sequence $\{a_n\}_{n=0}^{\infty}$, suppose $a_n = o(1)$ as $n \to \infty$ and is of bounded variation. Then*

(*a*) (S) *is convergent everywhere;*

(*b*) (C) *and* (S) *are uniformly convergent for $\delta \le |x| \le \pi$ for any $0 < \delta \le \pi$.*

The following class BV^m $(m \in \mathbb{N})$ of sequences was defined by Garret et al. [6], which is wider than the class BV.

Definition 1.6. *A null-sequence $\{a_n\}_{n=0}^{\infty}$ is said to belong to the class BV^m for some integer $m \ge 1$, that is, it is said to be of bounded variation of order m, if*

$$\sum_{n=0}^{\infty} |\Delta^m a_n| < \infty, \tag{1}$$

where

$$\Delta^m a_n = \Delta^{m-1} a_n - \Delta^{m-1} a_{n+1}, \quad m \in \mathbb{N},$$

and $\Delta^0 a_n = a_n$, for $n = 0, 1, \ldots$.

Remark 1.7. For $m = 1$, $\Delta^1 a_n = a_n - a_{n+1}$, so $\sum |\Delta^1 a_n| < \infty$ is exactly the bounded variation condition of Definition 1.3. The only difference is that Definition 1.6 additionally requires $\{a_n\}$ to be a null sequence. Actually, BV alone does not force $a_n \to 0$; but in the context of Theorem 1.8 and later results, we need null sequences.

The classes BV^m increases with m, and inclusion relation is proper (see [6, p. 424]). Stanojevic (see [24, p. 371]) extends Theorem 1.5 to trigonometric series with (complex) coefficients of bounded variation of order m.

Theorem 1.8. *Let $\{a_n\}_{n\in\mathbb{Z}}$ be a null-sequence of complex numbers. If for some integer $m \geq 1$, (1) holds, then the pointwise limit of*

$$S_n(t) = \sum_{j=-n}^{n} a_j e^{ijt}$$

exists in $\mathbb{T} - \{0\}$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Later, Chang-Pao Chen [2] considered a condition for some non-negative integer m

$$\lim_{\lambda\to 1+0} \limsup_{n\to\infty} \sum_{|k|=n}^{\lambda_n} |\Delta^m a_k| = 0, \tag{2}$$

weaker than (1), where $\lambda_n = [\lambda n]$, integral part of λn, and studied the pointwise and uniform convergence of trigonometric series. Here $S_n(t_0)$ (defined as in Theorem 1.8) denotes the n-th partial sum and $\sigma_n(t_0)$ denotes the n-th Cesàro mean of the series, i.e., $\sigma_n(t_0) = \dfrac{1}{n+1}\sum_{j=0}^{n} S_j(t_0)$.

Theorem 1.9. *If $\{a_n\}_{n\in\mathbb{Z}}$ is a null-sequence of complex numbers satisfied (2) for some non negative integer m, then, for $0 < |t_0| \leq \pi$, $S_n(t_0)$ and $\sigma_n(t_0)$ converges together to the same limit, or they both diverge. Furthermore, let E be any subset of $\mathbb{T}$ bounded away from 0, i.e.,*

$$E \subset \{t : \delta \leq |t| \leq \pi\} \quad \text{for some } \ \delta > 0.$$

Then, $S_n(t)$ converges uniformly on E if and only if $\sigma_n(t)$ converges uniformly on E. Whenever $m = 0$, the conditions '$0 < |t_0| \leq \pi$' and 'E is bounded away from 0' can be eliminated.

Parallel to the theory of trigonometric Fourier series, orthogonal polynomial expansions, the Walsh-Fourier series (see [23, 12, 13, 15, 16]), the Vilenkin-Fourier series (see [1, 26, 20]) have been extensively studied by numerous researchers. Legendre polynomials appear prominently in boundary value problems for elliptic partial differential equations, potential theory, spectral methods for numerical approximation, and mathematical physics.Consequently, Legendre series arise naturally in a wide range of theoretical and applied contexts. Legendre polynomials arise as solutions of the Legendre differential equation

$$(1 - x^2)\, y''(x) - 2x\, y'(x) + n(n+1)\, y(x) = 0,$$

where n is a non-negative integer, and may be defined explicitly by Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{3}$$

In particular, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$, etc. Classical developments and expansions in Legendre polynomials were studied many authors, for example, by A. L. Rabenstein [18], E. D. Rainville [19], and M. H. Stone [25]. The recurrence relation of the Legendre polynomials is given by

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

The Legendre polynomials are simply orthogonal over the interval $-1 < x < 1$ (see, e.g., [18, Section 6.7]), that is, for $m \neq n$ we have

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0.$$

Here we mention some representations and properties of Legendre polynomials (see [19, Chapter 10]). The Legendre polynomials $P_n(x)$, by the generating relation, is given by

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{j=0}^{\infty} P_j(x)t^j,$$

in which $\left(1-2xt+t^2\right)^{-\frac{1}{2}}$ denotes the particular branch which $\to 1$ as $t \to 0$. Hence $P_n(-x) = (-1)^n P_n(x)$, so that $P_n(x)$ is an odd function of x for n odd, an even function of x for n even. Laplace's first integral form of Legendre polynomials $P_n(x)$ (see [19, Chap. 10, Sec. 97, (5)]) is of the form

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x + (x^2-1)^{\frac{1}{2}}\cos\theta\right]^n d\theta. \tag{4}$$

Definition 1.10. *Let $P_n(x)$ be the Legendre polynomial of degree n normalized so that $P_n(1) = 1$. A series of the form*

$$\sum_{j=0}^{\infty} a_j P_j(x), \quad x \in (-1,1), \tag{5}$$

with coefficients a_j, is called a Legendre series. If the coefficients a_j of the above series, for a function $f \in L^1[-1,1]$, are given by

$$a_j := a_j(f) = \left(j+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_j(x)dx, \quad j = 0,1,2\ldots,$$

then the series (5) *is called the Fourier-Legendre series of* f *and coefficients* $a_j(f)$ $(j = 0, 1, 2, \dots)$ *are called the Fourier-Legendre coefficients of* f*.*

While the convergence theory of single-variable Legendre series has been studied extensively, the behavior of double Legendre series remains comparatively less explored. Double orthogonal expansions arise naturally when approximating functions of two variables, particularly in problems involving coupled systems, bivariate data approximation, and multidimensional spectral methods. Unlike the one-dimensional case, the analysis of double series introduces new challenges due to the interaction between summation indices and the need for suitable notions of regular convergence.

Motivated by earlier work on double trigonometric Fourier series (see [11, 9, 17, 8]), Walsh–Fourier series (see [14, 3]), and Vilenkin–Fourier series (see [22]), the present paper investigates the convergence of double Legendre series whose coefficients belong to classes of bounded and generalized bounded variation. These coefficient conditions provide a natural multivariate extension of classical bounded variation assumptions and allow one to derive convergence results without imposing restrictive monotonicity hypotheses.

The main objective of this paper is threefold. First, we establish pointwise, uniform, and L^r-convergence $(0 < r < 1)$ of double Legendre series under bounded variation assumptions on the coefficients. Second, we study summability methods such as Cesàro and de la Vallée Poussin means and obtain Tauberian-type results linking summability to ordinary convergence. Finally, our results extend known one-dimensional theorems to the two-variable setting, highlighting both the similarities and the essential analytical differences that arise in the study of double orthogonal series.

The results presented here contribute to the broader program of understanding convergence phenomena for multidimensional orthogonal expansions and complement existing work on double Fourier, Walsh, and Vilenkin series.

If g is an integrable function on I^2 $(I := [-1, 1])$, then the Fourier-Legendre series of g is the series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}(g)P_j(x)P_k(y),$$

where

$$a_{jk}(g) = \left(j + \frac{1}{2}\right)\left(k + \frac{1}{2}\right)\int_{-1}^{1}\int_{-1}^{1} g(u, v)P_j(u)P_k(v)dudv, \quad j, k = 0, 1, 2, \dots,$$

is the $(jk)^{th}$ Fourier-Legendre coefficient of the function g.

Definition 1.11. *For $m, n \geq 0$, the rectangular partial sums $S_{mn}(x, y)$, the first arithmetic (or Cesàro) means $\sigma_{mn}(x, y)$ and generalized de la Vallée Poussin means $\tau_{mn}^{\lambda}(x, y)$ of double Legendre series*

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}P_j(x)P_k(y), \quad x, y \in (-1, 1), \tag{6}$$

where $\{a_{jk}\}$ is a sequence of real (or complex) numbers, are defined as

$$S_{mn}(x, y) = \sum_{j=0}^{m}\sum_{k=0}^{n} a_{jk}P_j(x)P_k(y),$$

$$\sigma_{mn}(x, y) = \frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n} S_{jk}(x, y),$$

and

$$\tau_{mn}^{\lambda}(x, y) = \frac{1}{(\lambda_m - m)(\lambda_n - n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n} S_{jk}(x, y),$$

respectively, where $\lambda > 1$, $\lambda_m = [\lambda m]$, and $[r]$ denotes integral part of the real number r. If a_{jk} are the Fourier coefficients of some $g \in L^1(-1, 1)^2$ with respect to Legendre systems $\{P_j(x)P_k(y)\}_{i,j=0}^{\infty}$, we shall write $S_{mn}(g; x, y)$, $\sigma_{mn}(g; x, y)$ and $\tau_{mn}^{\lambda}(g; x, y)$ instead of $S_{mn}(x, y)$, $\sigma_{mn}(x, y)$ and $\tau_{mn}^{\lambda}(x, y)$ respectively.

Following Hardy [7], series (6) is said to be regularly convergent if it converges in Pringsheim's sense (see, e.g., [27, Vol. 2, Ch. 17]), and, in addition, each "row series" of (6) (i.e., when we delete $\sum_{k=0}^{\infty}$ in (6) and the summation is done only concerning j for each k) as well as each "column series" converges in the ordinary sense of convergence of single series. F. Móricz [10] rediscovered the notion of regular convergence by the following equivalent condition: the sum

$$S(R; x, y) = \sum_{j=m}^{M}\sum_{k=n}^{N} a_{jk}P_j(x)P_k(y) \tag{7}$$

tends to zero as $\max(m, n) \to \infty$, independently of the choices of $M(\geq m)$ and $N(\geq n)$, where $R = \{(j, k) : m \leq j \leq M \text{ and } n \leq k \leq N\}$.

We introduce the following three sums for $\lambda > 1$, $m, n \in \mathbb{N}$ and $x, y \in$

$(-1,1)$:

$$S_{10}^{\lambda}(m,n;x,y) := \sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}\frac{\lambda_m-j+1}{\lambda_m-m}a_{jk}P_j(x)P_k(y),$$

$$S_{01}^{\lambda}(m,n;x,y) := \sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}a_{jk}P_j(x)P_k(y),$$

$$S_{11}^{\lambda}(m,n;x,y) := \sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_m-j+1}{\lambda_m-m}\frac{\lambda_n-k+1}{\lambda_n-n}a_{jk}P_j(x)P_k(y).$$

Also we have

$$\begin{aligned}
S_{11}^{\lambda}(m,n;x,y) &= \sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_m-j+1}{\lambda_m-m}\frac{\lambda_n-k+1}{\lambda_n-n}a_{jk}P_j(x)P_k(y)\\
&= \frac{1}{\lambda_m-m}\left\{\sum_{j=m+1}^{\lambda_m}(\lambda_m-j+1)\right\}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}a_{jk}P_j(x)P_k(y)\\
&= \frac{1}{\lambda_m-m}\left\{\sum_{j=m+1}^{m+1}+\sum_{j=m+1}^{m+2}+\cdots+\sum_{j=m+1}^{\lambda_m}\right\}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}a_{jk}P_j(x)P_k(y)\\
&= \frac{1}{\lambda_m-m}\left\{\sum_{u=m+1}^{\lambda_m}\sum_{j=m+1}^{u}\right\}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}a_{jk}P_j(x)P_k(y)\\
&= \frac{1}{\lambda_m-m}\left\{\sum_{u=m+1}^{\lambda_m}\left(\sum_{j=0}^{u}-\sum_{j=0}^{m}\right)\right\}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}a_{jk}P_j(x)P_k(y)\\
&= \frac{1}{\lambda_m-m}\sum_{u=m+1}^{\lambda_m}\sum_{j=0}^{u}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}a_{jk}P_j(x)P_k(y)\\
&\qquad -\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}a_{jk}P_j(x)P_k(y)
\end{aligned}$$

(8)

$$= \frac{1}{\lambda_m-m}\sum_{u=m+1}^{\lambda_m}\left\{S_{01}^{\lambda}(u,n;x,y)-S_{01}^{\lambda}(m,n;x,y)\right\}.$$

Similarly, we get

$$S_{11}^{\lambda}(m,n;x,y)=\frac{1}{\lambda_n-n}\sum_{v=n+1}^{\lambda_n}\left\{S_{10}^{\lambda}(m,v;x,y)-S_{10}^{\lambda}(m,n;x,y)\right\}. \tag{9}$$

That is, S_{11}^{λ} is the average of

$$S_{01}^{\lambda}(u,n;x,y)-S_{01}^{\lambda}(m,n;x,y)\ \ (m<u\le\lambda_m)$$

or

$$S_{10}^{\lambda}(m,v;x,y)-S_{10}^{\lambda}(m,n;x,y)\ \ (n<v\le\lambda_n).$$

Also, we have

$$\left|S_{11}^{\lambda}(m,n;x,y)\right|\le\begin{cases}2\ \sup\limits_{m\le u\le\lambda_m}|S_{01}^{\lambda}(u,n;x,y)|,\\ 2\ \sup\limits_{n\le v\le\lambda_n}|S_{10}^{\lambda}(m,v;x,y)|.\end{cases} \tag{10}$$

Definition 1.12. *A double sequence $\{a_{jk}\}_{j,k=0}^{\infty}$ is said to be a null sequence if*

$$a_{jk}\to 0\quad as\quad \max(j,k)\to\infty. \tag{11}$$

Definition 1.13. *Let $\{a_{jk}\}_{j,k=0}^{\infty}$ be a double sequence of complex numbers. Then for non-negative integers p and q, the finite order differences $\Delta_{pq}a_{jk}$ are defined by*

$$\begin{aligned}&\Delta_{00}a_{jk}=a_{jk};\\ &\Delta_{pq}a_{jk}=\Delta_{p-1,q}a_{jk}-\Delta_{p-1,q}a_{j+1,k}\ \ (p\ge 1);\\ &\Delta_{pq}a_{jk}=\Delta_{p,q-1}a_{jk}-\Delta_{p,q-1}a_{j,k+1}\ \ (q\ge 1),\end{aligned}$$

and the sequence $\{a_{jk}\}_{j,k=0}^{\infty}$ is said to be of bounded variation if

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{11}a_{jk}|<\infty. \tag{12}$$

2. Main results. Our first theorem concerns the point-wise, uniform, and L^r (where $0<r<1$) convergence of double Legendre series with coefficients of bounded variation. For one variable similar result proved in [21, Theorem 1].

Theorem 2.1. *Let $\{a_{jk}\}_{j,k=0}^{\infty}$ be a null sequence of bounded variation. Then the series* (6) *converges regularly to some function g for all $x,y\in(-1,1)$.*

Also, for $0 < r < 1$, $\|S_{mn} - g\|_r \to 0$ *as* $\min(m,n) \to \infty$, *where* $\|f\|_r = \left(\int_{-1}^{1}\int_{-1}^{1} |f(x,y)|^r \, dxdy\right)^{\frac{1}{r}}$.

Our second main result is as follows.

Theorem 2.2. *(i) Let* $E \subset (-1,1)^2$. *Assume that the following conditions are satisfied:*

$$\lim_{\lambda\to 1+} \limsup_{m,n\to\infty} \left(\sup_{(x,y)\in E} \left| S_{10}^{\lambda}(m,n;x,y) \right| \right) = 0, \tag{13}$$

$$\lim_{\lambda\to 1+} \limsup_{m,n\to\infty} \left(\sup_{(x,y)\in E} \left| S_{01}^{\lambda}(m,n;x,y) \right| \right) = 0. \tag{14}$$

If $\tau_{mn}^{\lambda}(x,y)$ *converges uniformly on* E *to* $f(x,y)$ *for every* $\lambda > 1$, *then so does* $S_{mn}(x,y)$.

(ii) Assume that the following conditions are satisfied for $0 < r < 1$*:*

$$\lim_{\lambda\to 1+} \limsup_{m,n\to\infty} \left(\left\| S_{10}^{\lambda}(m,n;x,y) \right\|_r \right) = 0, \tag{15}$$

$$\lim_{\lambda\to 1+} \limsup_{m,n\to\infty} \left(\left\| S_{01}^{\lambda}(m,n;x,y) \right\|_r \right) = 0. \tag{16}$$

If $\left\| \tau_{mn}^{\lambda} - f \right\|_r \to 0$ *as* $\min(m,n) \to \infty$, *then* $\|S_{mn} - f\|_r \to 0$ *as* $\min(m,n) \to \infty$.

Remark 2.3. (1) The conditions (13) and (14) involving $\lim_{\lambda\to 1+}$ are strictly weaker than assuming the limit superior is zero for a single fixed $\lambda > 1$. The $\lim_{\lambda\to 1+}$ condition is a direct double-sequence analogue of the optimal Tauberian condition introduced by Chen [2] for single sequences.

(2) Furthermore, in view of Lemma 3.2, the assumption regarding the uniform convergence of τ_{mn}^{λ} in Theorem 2.2 can be equivalently replaced by the uniform convergence of the Cesàro means $\sigma_{mn}(x,y)$. This provides a more flexible criterion, as the uniform convergence of σ_{mn} is often easier to verify directly in practical applications.

(3) Here the limit superior of a double sequence $\{a_{jk}\}_{j,k=0}^{\infty}$ of extended real numbers is defined as

$$\limsup_{m,n\to\infty} a_{mn} = \inf_{m,n\geq 0} \left(\sup_{j\geq m,\, k\geq n} a_{jk} \right) = \lim_{m\to\infty} \left(\sup_{j\geq m,\, k\geq m} a_{jk} \right).$$

The following Theorem generalizes Theorem 2.2.

Theorem 2.4. *Let $E \subset (-1,1)^2$. Assume that a null sequence $\{a_{jk}\}_{j,k=0}^{\infty}$ satisfies following conditions:*

$$\sum_{j=0}^{\infty} |\Delta_{10} a_{jk}| \text{ is finite for each } k \text{ and tends to } 0 \text{ as } k \to \infty, \tag{17}$$

$$\sum_{k=0}^{\infty} |\Delta_{01} a_{jk}| \text{ is finite for each } j \text{ and tends to } 0 \text{ as } j \to \infty, \tag{18}$$

$$\lim_{\lambda \to 1+} \limsup_{n \to \infty} \sum_{j=0}^{\infty} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\Delta_{11} a_{jk}| = 0, \tag{19}$$

and

$$\lim_{\lambda \to 1+} \limsup_{m \to \infty} \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{\infty} \frac{\lambda_m - j + 1}{\lambda_m - m} |\Delta_{11} a_{jk}| = 0. \tag{20}$$

Then

(i) Let $E \subset [-1, 1-\delta]^2$ for some $\delta > 0$. If $\tau_{mn}^{\lambda}(x,y)$ converges uniformly on E to $f(x,y)$ as $\min(m,n) \to \infty$, then so does $S_{mn}(x,y)$.

(ii) If $\left\| \tau_{mn}^{\lambda} - f \right\|_r \to 0$ for $0 < r < 1$, then $\|S_{mn} - f\|_r \to 0$ as $\min(m,n) \to \infty$.

For one variable similar result proved in [21, Theorem 2].

3. Lemmas. We required the following lemmas for the proof of our results.

Lemma 3.1. *Let $x \in (-1,1)$ and $D_n(x) = \sum_{j=0}^{n} P_j(x)$, $n \in \mathbb{N}$. Then*

$$|D_n(x)| \leq \frac{2}{1-x}.$$

P r o o f. Let $x \in (-1,1)$, n be any positive integer, and $1 \leq j \leq n$. Then, from Laplace's first integral form (4), of the Legendre polynomial, we have

$$P_j(x) = \frac{1}{\pi} \int_0^{\pi} \left[x + \sqrt{x^2 - 1} \cos\theta \right]^j d\theta = \frac{1}{\pi} \int_0^{\pi} \left[x + i\sqrt{1 - x^2} \cos\theta \right]^j d\theta.$$

Summing both sides over j from 0 to n, we get

$$\sum_{j=0}^{n} P_j(x) = \frac{1}{\pi}\int_0^{\pi} \sum_{j=0}^{n} \left[x + i\sqrt{1-x^2}\cos\theta\right]^j d\theta. \tag{21}$$

For $-1 < x < 1$ and $0 < \theta < \pi$, we have

$$\begin{aligned} |x + i\sqrt{1-x^2}\cos\theta| &= \sqrt{x^2 + (1-x^2)\cos^2\theta} \\ &= \sqrt{\cos^2\theta + x^2\sin^2\theta} \\ &= \sqrt{1-(1-x^2)\sin^2\theta} < 1. \end{aligned} \tag{22}$$

Therefore, we have

$$\sum_{j=0}^{n}[x + i\sqrt{1-x^2}\cos\theta]^j = \frac{1-[x+i\sqrt{1-x^2}\cos\theta]^{n+1}}{1-[x+i\sqrt{1-x^2}\cos\theta]}.$$

Using this in (21), we get

$$\sum_{j=0}^{n} P_j(x) = \frac{1}{\pi}\int_0^{\pi} \frac{1-\left[x+i\sqrt{1-x^2}\cos\theta\right]^{n+1}}{1-\left[x+i\sqrt{1-x^2}\cos\theta\right]} d\theta.$$

Therefore, we have

$$\begin{aligned} \left|\sum_{j=0}^{n} P_j(x)\right| &\le \frac{1}{\pi}\int_0^{\pi} \left|\frac{1-\left[x+i\sqrt{1-x^2}\cos\theta\right]^{n+1}}{1-\left[x+i\sqrt{1-x^2}\cos\theta\right]}\right| d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} \frac{\left|1-\left[x+i\sqrt{1-x^2}\cos\theta\right]^{n+1}\right|}{\left|1-\left[x+i\sqrt{1-x^2}\cos\theta\right]\right|} d\theta \\ &\le \frac{1}{\pi}\int_0^{\pi} \frac{1+\left|x+i\sqrt{1-x^2}\cos\theta\right|^{n+1}}{\left|(1-x)-i\sqrt{1-x^2}\cos\theta\right|} d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} \frac{1+\left[x^2+(1-x^2)\cos^2\theta\right]^{\frac{n+1}{2}}}{[(1-x)^2+(1-x^2)\cos^2\theta]^{\frac{1}{2}}} d\theta. \end{aligned}$$

Since $(1-x^2)\cos^2\theta \ge 0$ and from (22), we have

$$|D_n(x)| = \left|\sum_{j=0}^{n} P_j(x)\right| \le \frac{1}{\pi}\int_0^{\pi} \frac{1+1^{n+1}}{[(1-x)^2]^{\frac{1}{2}}} d\theta = \frac{2}{1-x}.$$

This completes the proof. □

Lemma 3.2. *If* $\{\sigma_{mn}(x,y)\}$ *converges pointwise, uniformy or in* L^r*-quasinorm,* $(0<r\le 1)$*, to* $\phi(x,y)$*, then so does* $\{\tau^{\lambda}_{mn}(x,y)\}$*.*

P r o o f. By definition of $\sigma_{mn}(x,y)$ and $\tau^{\lambda}_{mn}(x,y)$, we have

$$\begin{aligned}
&(\lambda_m+1)(\lambda_n+1)\sigma_{\lambda_m,\lambda_n}(x,y)+(m+1)(n+1)\sigma_{mn}(x,y)\\
&-(m+1)(\lambda_n+1)\sigma_{m,\lambda_n}(x,y)-(\lambda_m+1)(n+1)\sigma_{\lambda_m,n}(x,y)\\
&=\left\{\sum_{j=0}^{\lambda_m}\sum_{k=0}^{\lambda_n}+\sum_{j=0}^{m}\sum_{k=0}^{n}-\sum_{j=0}^{m}\sum_{k=0}^{\lambda_n}-\sum_{j=0}^{\lambda_m}\sum_{k=0}^{n}\right\}S_{jk}(x,y)\\
&=\left\{\sum_{j=0}^{\lambda_m}\sum_{k=0}^{\lambda_n}-\sum_{j=0}^{m}\sum_{k=0}^{n}-\left(\sum_{j=0}^{m}\sum_{k=0}^{\lambda_n}-\sum_{j=0}^{m}\sum_{k=0}^{n}\right)-\left(\sum_{j=0}^{\lambda_m}\sum_{k=0}^{n}-\sum_{j=0}^{m}\sum_{k=0}^{n}\right)\right\}S_{jk}(x,y)\\
&=\left\{\sum_{j=0}^{\lambda_m}\sum_{k=0}^{\lambda_n}-\sum_{j=0}^{m}\sum_{k=0}^{n}-\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}-\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}\right\}S_{jk}(x,y)\\
&=\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}S_{jk}(x,y)=(\lambda_m-m)(\lambda_n-n)\tau^{\lambda}_{mn}(x,y).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\tau^{\lambda}_{mn}(x,y)=&\frac{\lambda_m+1}{\lambda_m-m}\cdot\frac{\lambda_n+1}{\lambda_n-n}\sigma_{\lambda_m,\lambda_n}(x,y)+\frac{m+1}{\lambda_m-m}\cdot\frac{n+1}{\lambda_n-n}\sigma_{mn}(x,y)\\
&-\frac{m+1}{\lambda_m-m}\cdot\frac{\lambda_n+1}{\lambda_n-n}\sigma_{m,\lambda_n}(x,y)-\frac{\lambda_m+1}{\lambda_m-m}\cdot\frac{n+1}{\lambda_n-n}\sigma_{\lambda_m,n}(x,y).
\end{aligned}\tag{23}$$

For fixed $\lambda>1$, we have $\dfrac{[\lambda m]}{m}\to\lambda$ $\left(\dfrac{[\lambda n]}{n}\to\lambda\right)$ as $m\to\infty$ $(n\to\infty)$. Also, $\lambda_m\to\infty$ $(\lambda_n\to\infty)$ if $m\to\infty$ $(n\to\infty)$. Given that $\{\sigma_{mn}(x,y)\}$ converges to $\phi(x,y)$, we get

$$\begin{aligned}
&\lim_{m,n\to\infty}\tau^{\lambda}_{mn}(x,y)\\
&=\left\{\frac{\lambda\lambda}{(\lambda-1)(\lambda-1)}+\frac{1}{(\lambda-1)(\lambda-1)}-\frac{\lambda}{(\lambda-1)(\lambda-1)}-\frac{\lambda}{(\lambda-1)(\lambda-1)}\right\}\phi(x,y)\\
&=\phi(x,y).
\end{aligned}$$

Convergence in above depends on convergence of $\{\sigma_{mn}(x,y)\}$.

Also, for $x, y \in (-1, 1)$, we have

$$\begin{aligned}
&\left|\tau_{mn}^{\lambda}(x,y) - \phi(x,y)\right| \\
&\leq \frac{1}{(\lambda_m - m)(\lambda_n - n)}\Big\{(\lambda_m+1)(\lambda_n+1)\left|\sigma_{\lambda_m,\lambda_n}(x,y) - \phi(x,y)\right| \\
&+ (m+1)(n+1)\left|\sigma_{mn}(x,y) - \phi(x,y)\right| - m(\lambda_n+1)\left|\sigma_{m,\lambda_n}(x,y) - \phi(x,y)\right| \\
&+ (\lambda_m+1)(n+1)\left|\sigma_{\lambda_m,n}(x,y) - \phi(x,y)\right|\Big\}.
\end{aligned}$$

For any fixed $\lambda > 1$ and $0 < r \leq 1$, $\{\sigma_{mn}(x,y)\}$ converges to $\phi(x,y)$ in L^r-norm, we have

$$\lim_{m,n\to\infty}\int_0^1\int_0^1 |\tau_{mn}^{\lambda}(x,y) - \phi(x,y)|^r dxdy = 0.$$

This completes the proof of the lemma. □

Lemma 3.3 ([14, Lemma 2]). *Let $\{c_{jk}\}_{i,j=0}^{\infty}$ and $\{d_{jk}\}_{i,j=0}^{\infty}$ be two double sequences of numbers and $D_{mn} = \sum_{j=0}^{m}\sum_{k=0}^{n} d_{jk}$ $(m, n = 0, 1, \dots)$ be the rectangular partial sums of $\{d_{jk}\}$. Then, for all $0 \leq m \leq M$ and $0 \leq n \leq N$,*

$$\begin{aligned}
\sum_{j=m}^{M}\sum_{k=n}^{N} c_{jk}d_{jk} &= \sum_{j=m}^{M}\sum_{k=n}^{N}\Delta_{11}c_{jk}D_{jk} + \sum_{j=m}^{M}\Delta_{10}c_{j,N+1}D_{jN} - \sum_{j=m}^{M}\Delta_{10}c_{jn}D_{j,n-1} \\
&+ \sum_{k=n}^{N}\Delta_{01}c_{M+1,k}D_{Mk} - \sum_{k=n}^{N}\Delta_{01}c_{mk}D_{m-1,k} + c_{M+1,N+1}D_{MN} \\
&- c_{M+1,n}D_{M,n-1} - c_{m,N+1}D_{m-1,N} + c_{mn}D_{m-1,n-1}.
\end{aligned}$$

Lemma 3.4 ([14, Lemma 1]). *If $\{a_{jk}\}_{i,j=0}^{\infty}$ satisfies the condition* (11) *and for $p, q \geq 1$,*

$$C_{pq} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{pq}a_{jk}| < \infty \tag{24}$$

then

$$\sum_{j=0}^{\infty}|\Delta_{p,q-1}a_{jk}| \leq C_{pq} \quad (k = 0, 1, 2, \dots),$$

$$\sum_{j=0}^{\infty}|\Delta_{p,q-1}a_{jk}| \to 0 \text{ as } k \to \infty,$$

$$\sup_k \sum_{j=m}^{\infty} |\Delta_{p,q-1} a_{jk}| \to 0 \text{ as } m \to \infty.$$

Analogous statements hold for $\Delta_{p-1,q} a_{jk}$ *under the same conditions* (11) *and* (24) *if the roles of* j *and* k *are interchanged.*

Lemma 3.5 ([14, Lemma 3])**.** *Let* $\{c_{jk}\}_{i,j=0}^{\infty}$ *be a null sequence and* $\{d_{jk}\}_{i,j=0}^{\infty}$ *be a sequence of numbers and* $D_{mn} = \sum_{j=0}^{m} \sum_{k=0}^{n} d_{jk}$ $(m, n = 1, 2, 3, \dots)$ *be the rectangular partial sums of* $\{d_{jk}\}$*. Then for all* $0 \le m \le M$ *and* $0 \le n \le N$*,*

$$\sum_{R_{mn}} c_{jk} d_{jk} = \sum_{R_{mn}} \Delta_{11} c_{jk} D_{jk} - \sum_{j=0}^{m} \Delta_{10} c_{j,n+1} D_{jn} - \sum_{k=0}^{n} \Delta_{01} c_{m+1,k} D_{mk} - c_{m+1,n+1} D_{mn},$$

where $R_{mn} = \{(j,k) \in \mathbb{Z}_+ \cup \{0\} \times \mathbb{Z}_+ \cup \{0\} : \text{ either } j \ge m+1 \text{ or } k \ge n+1\}$ *and* $\sum_{R_{mn}}$ *stands for* $\sum_{(j,k) \in R_{mn}} \sum$*. Where* $\mathbb{Z}_+$ *denotes the set of the positive integers. We note that the definition* R_{mn} *already allows* $j = 0$ *(when* $k \ge n+1$*) and* $k = 0$ *(when* $j \ge m+1$*), since the "or" condition requires only one index to exceed the threshold.*

The following lemma, for trigonometric series, can be found in [8, Lemma 3.1]; here we give a different proof of the lemma.

Lemma 3.6. *For* $m, n \ge 0$*, we have the following representation*

$$\tau_{mn}^{\lambda}(x,y) - S_{mn}(x,y) = S_{10}^{\lambda}(m,n;x,y) + S_{01}^{\lambda}(m,n;x,y) + S_{11}^{\lambda}(m,n;x,y).$$

P r o o f. By definition of $\tau_{mn}^{\lambda}(x,y)$ and $S_{mn}(x,y)$, we have

$$\begin{aligned}
&\tau_{mn}^{\lambda}(x,y) - S_{mn}(x,y) \\
&= \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} S_{jk}(x,y) - S_{mn}(x,y) \\
&= \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \sum_{l_1=0}^{j} \sum_{l_2=0}^{k} a_{l_1 l_2} P_{l_1}(x) P_{l_2}(y) \\
&- \sum_{l_1=0}^{m} \sum_{l_2=0}^{n} a_{l_1 l_2} P_{l_1}(x) P_{l_2}(y)
\end{aligned}$$

$$= \frac{1}{(\lambda_m - m)(\lambda_n - n)} \Bigg[\sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \Bigg\{ \sum_{l_1=m+1}^{j} \sum_{l_2=n+1}^{k} + \sum_{l_1=0}^{m} \sum_{l_2=n+1}^{k} + \sum_{l_1=m+1}^{j} \sum_{l_2=0}^{n} + \sum_{l_1=0}^{m} \sum_{l_2=0}^{n} \Bigg\} \Bigg] a_{l_1 l_2} P_{l_1}(x) P_{l_2}(y) - \sum_{l_1=0}^{m} \sum_{l_2=0}^{n} a_{l_1 l_2} P_{l_1}(x) P_{l_2}(y)$$

$$= \frac{1}{(\lambda_m - m)(\lambda_n - n)} \Bigg[\sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \sum_{l_1=m+1}^{j} \sum_{l_2=n+1}^{k} + (\lambda_m - m) \sum_{k=n+1}^{\lambda_n} \sum_{l_1=0}^{m} \sum_{l_2=n+1}^{k} + (\lambda_n - n) \sum_{j=m+1}^{\lambda_m} \sum_{l_1=m+1}^{j} \sum_{l_2=0}^{n} + (\lambda_m - m)(\lambda_n - n) \sum_{l_1=0}^{m} \sum_{l_2=0}^{n} \Bigg] a_{l_1 l_2} P_{l_1}(x) P_{l_2}(y) - \sum_{l_1=0}^{m} \sum_{l_2=0}^{n} a_{l_1 l_2} P_{l_1}(x) P_{l_2}(y).$$

Therefore, we get

$$\begin{aligned} & \tau_{mn}^{\lambda\mu}(x,y) - S_{mn}(x,y) \\ & = \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \sum_{l_1=m+1}^{j} \sum_{l_2=n+1}^{k} a_{l_1 l_2} P_{l_1}(x) P_{l_2}(y) \\ & \quad + \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \sum_{l_1=0}^{m} \sum_{l_2=n+1}^{k} a_{l_1 l_2} P_{l_1}(x) P_{l_2}(y) \\ & \quad + \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{l_1=m+1}^{j} \sum_{l_2=0}^{n} a_{l_1 l_2} P_{l_1}(x) P_{l_2}(y) \\ & = J_1 + J_2 + J_3, \text{ say.} \end{aligned} \tag{25}$$

Now, by interchanging the order of summation, we get

$$\begin{aligned} J_1 &= \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{l_1=m+1}^{j} P_{l_1}(x) \Bigg\{ \sum_{k=n+1}^{\lambda_n} \sum_{l_2=n+1}^{k} a_{l_1 l_2} P_{l_2}(y) \Bigg\} \\ &= \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{l_1=m+1}^{j} P_{l_1}(x) \Bigg\{ \sum_{k=n+1}^{\lambda_n} (\lambda_n - k + 1) a_{l_1 k} P_k(y) \Bigg\} \\ &= \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} (\lambda_m - j + 1)(\lambda_n - k + 1) a_{jk} P_j(x) P_k(y) \end{aligned}$$

$$= \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_m - j + 1}{\lambda_m - m} \frac{\lambda_n - k + 1}{\lambda_n - n} a_{jk} P_j(x) P_k(y) = S_{11}^{\lambda}(m, n; x, y).$$

Similarly, we have

$$J_2 = \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} a_{jk} P_j(x) P_k(y) = S_{01}^{\lambda}(m, n; x, y),$$

and

$$J_3 = \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m - m} a_{jk} P_j(x) P_k(y) = S_{10}^{\lambda}(m, n; x, y).$$

Hence, from (25), get

$$\tau_{mn}^{\lambda}(x, y) - S_{mn}(x, y) = S_{11}^{\lambda}(m, n; x, y) + S_{01}^{\lambda}(m, n; x, y) + S_{10}^{\lambda}(m, n; x, y). \quad \square$$

The following lemma for trigonometric series can be found in [8, Lemma 3.4].

Lemma 3.7. *For $m, n \geq 0$ and $\lambda > 1$, we have the following representation:*

$$\begin{aligned} S_{10}^{\lambda}(m, n; x, y) &= \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m - m} \Delta_{11} a_{jk} D_j(x) D_k(y) \\ &+ \sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m} \Delta_{10} a_{j,n+1} D_j(x) D_n(y) \\ &+ \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \Delta_{01} a_{j+1,k} D_j(x) D_k(y) \\ &+ \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} a_{j+1,n+1} D_j(x) D_n(y) - \sum_{k=0}^{n} \Delta_{01} a_{m+1,k} D_m(x) D_k(y) \\ &- a_{m+1,n+1} D_m(x) D_n(y). \end{aligned}$$

P r o o f. By the definition of $S_{10}^{\lambda}(m, n; x, y)$ and using summation by parts, we have

$$\begin{aligned} &(\lambda_m - m) S_{10}^{\lambda}(m, n; x, y) \\ &= \sum_{k=0}^{n} \left\{ \sum_{j=m+1}^{\lambda_m} (\lambda_m - j + 1) a_{jk} P_j(x) \right\} P_k(y) \end{aligned}$$

$$= \sum_{k=0}^{n} \Bigg\{ \sum_{j=m+1}^{\lambda_m} [(\lambda_m - j + 1)a_{jk} - \{\lambda_m - (j+1) + 1\}a_{j+1,k}]\, D_j(x) - (\lambda_m - (m+1) + 1)a_{m+1,k} D_m(x) \Bigg\} P_k(y)$$

$$= \sum_{k=0}^{n} \Bigg\{ \sum_{j=m+1}^{\lambda_m} (\lambda_m - j + 1)\Delta_{10} a_{jk} D_j(x) + \sum_{j=m+1}^{\lambda_m} a_{j+1,k} D_j(x) - (\lambda_m - m) a_{m+1,k} D_m(x) \Bigg\} P_k(y)$$

$$= \sum_{j=m+1}^{\lambda_m} (\lambda_m - j + 1) D_j(x) \sum_{k=0}^{n} \Delta_{10} a_{jk} P_k(y) + \sum_{j=m+1}^{\lambda_m} D_j(x) \sum_{k=0}^{n} a_{j+1,k} P_k(y) - (\lambda_m - m) D_m(x) \sum_{k=0}^{n} a_{m+1,k} P_k(y)$$

$$= \sum_{j=m+1}^{\lambda_m} (\lambda_m - j + 1) D_j(x) \left\{ \sum_{k=0}^{n} \Delta_{11} a_{jk} D_k(y) + \Delta_{10} a_{j,n+1} D_n(y) \right\} + \sum_{j=m+1}^{\lambda_m} D_j(x) \left\{ \sum_{k=0}^{n} \Delta_{01} a_{j+1,k} D_k(y) + a_{j+1,n+1} D_n(y) \right\} - (\lambda_m - m) D_m(x) \left\{ \sum_{k=0}^{n} \Delta_{01} a_{m+1,k} D_k(y) + a_{m+1,n+1} D_n(y) \right\}$$

$$= \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} (\lambda_m - j + 1)\Delta_{11} a_{jk} D_j(x) D_k(y) + \sum_{j=m+1}^{\lambda_m} (\lambda_m - j + 1)\Delta_{10} a_{j,n+1} D_j(x) D_n(y) + \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \Delta_{01} a_{j+1,k} D_j(x) D_k(y) + \sum_{j=m+1}^{\lambda_m} a_{j+1,n+1} D_j(x) D_n(y) - (\lambda_m - m) \sum_{k=0}^{n} \Delta_{01} a_{m+1,k} D_m(x) D_k(y) - (\lambda_m - m) a_{m+1,n+1} D_m(x) D_n(y).$$

Hence, by dividing both the sides by $(\lambda_m - 1)$, we get the required representation

of $S_{10}^{\lambda}(m,n;x,y)$. This completes the proof of the lemma.

Similarly we can have representation for $S_{01}^{\lambda}(m,n;x,y)$. □

4. Proof of theorems.

Proof of Theorem 2.1. Let $0 \le m \le M$ and $0 \le n \le N$. Using the notation (7) and Lemma 3.3, we have

$$\begin{aligned}
&S(R;x,y)\\
&\quad=\sum_{j=m}^{M}\sum_{k=n}^{N}\Delta_{11}a_{jk}D_j(x)D_k(y)+\sum_{j=m}^{M}\Delta_{10}a_{j,N+1}D_j(x)D_N(y)\\
&\qquad-\sum_{j=m}^{M}\Delta_{10}a_{jn}D_j(x)D_{n-1}(y)+\sum_{k=n}^{N}\Delta_{01}a_{M+1,k}D_M(x)D_k(y)\\
&\qquad-\sum_{k=n}^{N}\Delta_{01}a_{mk}D_{m-1}(x)D_k(y)+a_{M+1,N+1}D_M(x)D_N(y)\\
&\qquad-a_{M+1,n}D_M(x)D_{n-1}(y)-a_{m,N+1}D_{m-1}(x)D_N(y)+a_{mn}D_{m-1}(x)D_{n-1}(y).
\end{aligned}$$

By using Lemma 3.1, for $x,y\in(-1,1)$, we have

$$\begin{aligned}
&4^{-1}(1-x)(1-y)\,|S(R;x,y)|\\
&\quad\le\sum_{j=m}^{M}\sum_{k=n}^{N}|\Delta_{11}a_{jk}|+\sum_{j=m}^{M}|\Delta_{10}a_{j,+1N}|+\sum_{j=m}^{M}|\Delta_{10}a_{jn}|+\sum_{k=n}^{N}|\Delta_{01}a_{M+1,k}|\\
&\qquad+\sum_{k=n}^{N}|\Delta_{01}a_{mk}|+|a_{M+1,N+1}|+|a_{M+1,n}|+|a_{m,N+1}|+|a_{mn}|.
\end{aligned}$$

With the help of (12), Lemma 3.4 (for $p=q=1$), and (11), the right-hand side of the above inequality tends to 0 as $\max(m,n)\to\infty$. Therefore, for $x,y\in(-1,1)$, $S(R;x,y)\to 0$ as $\max(m,n)\to\infty$. Hence, the series (6) converges regularly to some function g.

The series (6) converges to the function $g(x,y)$, that is,

$$g(x,y)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{jk}P_j(x)P_k(y),$$

for all $x,y\in(-1,1)$. Therefore

$$g(x,y)-S_{mn}(x,y)=\sum_{R_{mn}}a_{jk}P_j(x)P_k(y),$$

where $\sum_{R_{mn}}$ is defined as in Lemma 3.5. The use of Lemma 3.5 gives

$$\sum_{R_{mn}} a_{jk}P_j(x)P_k(y) = \sum_{R_{mn}} \Delta_{11}a_{jk}D_j(x)D_k(y) - \sum_{j=0}^{m} \Delta_{10}a_{j,n+1}D_j(x)D_n(y) - \sum_{k=0}^{n} \Delta_{01}a_{m+1,k}D_m(x)D_k(y) - a_{m+1,n+1}D_m(x)D_n(y).$$

In view of Lemma 3.1, for $x, y \in (-1, 1)$, we get

$$4^{-1}(1-x)(1-y)\,|g(x,y) - S_{mn}(x,y)| \leq \sum_{R_{mn}} |\Delta_{11}a_{jk}| + \sum_{j=0}^{m} |\Delta_{10}a_{j,n+1}| + \sum_{k=0}^{n} |\Delta_{01}a_{m+1,k}| + |a_{m+1,n+1}|\,.$$

The right-hand side of above inequality is of $o(1)$ as $\min(m,n) \to \infty$, (that is, tends to zero as $\min(m,n) \to \infty$), by virtue of (12), Lemma 3.4 (for $p = q = 1$), and (11). Now, for $x, y \in (-1, 1)$ and $0 < r < 1$, we obtain

$$|g(x,y) - S_{mn}(x,y)|^r \leq \{o(1)\}^r \left\{\frac{4}{(1-x)(1-y)}\right\}^r$$

and hence

$$\int_{-1}^{1}\int_{-1}^{1} |g(x,y) - S_{mn}(x,y)|^r\,dxdy \leq \{o(1)\}^r \int_{-1}^{1}\int_{-1}^{1} \frac{4^r}{(1-x)^r(1-y)^r}dxdy = \{o(1)\}^r \frac{4}{(1-r)^2}.$$

Hence, we have $\left(\int_{-1}^{1}\int_{-1}^{1} |g(x,y) - S_{mn}(x,y)|^r\,dxdy\right)^{\frac{1}{r}} \to 0$ as $\min(m,n) \to \infty$. That is, for $0 < r < 1$, $\|S_{mn} - g\|_r \to 0$ as $\min(m,n) \to \infty$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2 Proof of (i). With the help of (10), we find that (13) (or (14)) implies

$$\lim_{\lambda \to 1+} \limsup_{m,n\to\infty} \left(\sup_{(x,y)\in E} \left| S_{11}^{\lambda}(m,n;x,y) \right| \right) = 0.$$

Assume that $\tau_{mn}^{\lambda}(x,y)$ converges uniformly on E to $f(x,y)$. So we have

$$\sup_{(x,y)\in E} \left| \tau_{mn}^{\lambda}(x,y) - f(x,y) \right| \to 0 \tag{26}$$

as $m, n \to \infty$. By Lemma 3.6, we get

$$\begin{aligned}
&\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|\tau_{mn}^{\lambda}(x,y)-S_{mn}(x,y)\right|\right)\\
&\quad\le \limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|S_{01}^{\lambda}(m,n;x,y)\right|\right)+\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|S_{10}^{\lambda}(m,n;x,y)\right|\right)\\
&\qquad+\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|S_{11}^{\lambda}(m,n;x,y)\right|\right).
\end{aligned}$$

After taking $\lim_{\lambda\to 1+}$, we have

$$\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|\tau_{mn}^{\lambda}(x,y)-S_{mn}(x,y)\right|\right)=0. \tag{27}$$

We have the inequality

$$\begin{aligned}
&\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}|S_{mn}(x,y)-f(x,y)|\right)\\
&\qquad\le \limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|\tau_{mn}^{\lambda}(x,y)-S_{mn}(x,y)\right|\right)\\
&\qquad\quad+\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|\tau_{mn}^{\lambda}(x,y)-f(x,y)\right|\right)
\end{aligned}$$

By using (26), (27), and as $\lambda \to 1+$, we have

$$\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}|S_{mn}(x,y)-f(x,y)|\right)=0.$$

That is, $S_m n(x,y)$ converges uniformly on E to $f(x,y)$.

Proof of (ii). By (9) (or (8)), for $0 < r < 1$, we have

$$\begin{aligned}
&\left\|S_{11}^{\lambda}(m,n;x,y)\right\|_r\\
&\le \frac{1}{\lambda_n-n}\sum_{v=n+1}^{\lambda_n}\left\{\left\|S_{10}^{\lambda}(m,v;x,y)\right\|_r+\left\|S_{10}^{\lambda}(m,n;x,y)\right\|_r\right\}\\
&\le 2\left(\sup_{n\le v\le\lambda_n}\left(\left\|S_{10}^{\lambda}(m,v;x,y)\right\|_r\right)\right).
\end{aligned}$$

Thus, (15) implies

$$\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\left\|S_{11}^{\lambda}(m,n;x,y)\right\|_r\right)=0.$$

We replace $\sup_{(x,y)\in E}$ by $\|\cdot\|_r$ in the proof of (i) above. Then the preceding arguments verify (ii). This completes the proof of Theorem 2.2. □

[Proof of Theorem 2.4] Proof of (i). Using the Lemma 3.7, we can write the expression for $S_{10}^{\lambda}(m,n;x,y)$ as

$$\begin{aligned}S_{10}^{\lambda}(m,n;x,y)&=\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}\frac{\lambda_m-j+1}{\lambda_m-m}\Delta_{11}a_{jk}D_j(x)D_k(y)\\&+\sum_{j=m+1}^{\lambda_m}\frac{\lambda_m-j+1}{\lambda_m-m}\Delta_{10}a_{j,n+1}D_j(x)D_n(y)\\&+\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}\Delta_{01}a_{j+1,k}D_j(x)D_k(y)\\&+\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}a_{j+1,n+1}D_j(x)D_n(y)\\&-\sum_{k=0}^{n}\Delta_{01}a_{m+1,k}D_m(x)D_k(y)-a_{m+1,n+1}D_m(x)D_n(y)\\&=I_1^{\lambda}+I_2^{\lambda}+I_3^{\lambda}+I_4^{\lambda}+I_5+I_6,\ \text{ say.}\end{aligned}\tag{28}$$

Now by using (11), Lemma 3.1, (17), (18), (20), and noting that for $E\subset[-1,1-\delta]^2$, we have the following estimates:

$$\left|I_1^{\lambda}\right|\le\frac{4}{(1-x)(1-y)}\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}\frac{\lambda_m-j+1}{\lambda_m-m}\left|\Delta_{11}a_{jk}\right|.$$

First take supremum and then use $(1-x)(1-y)\ge\delta^2>0$, we get

$$\begin{aligned}\sup_{(x,y)\in E}\left|I_1^{\lambda}\right|&\le\frac{4}{\delta^2}\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}\frac{\lambda_m-j+1}{\lambda_m-m}\left|\Delta_{11}a_{jk}\right|\\&\le\frac{4}{\delta^2}\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{\infty}\frac{\lambda_m-j+1}{\lambda_m-m}\left|\Delta_{11}a_{jk}\right|.\end{aligned}$$

Now taking $\limsup_{m,n\to\infty}$ and then $\lim_{\lambda\to 1+}$, we get

$$\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|I_1^{\lambda}\right|\right)\leq\frac{4}{\delta^2}\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{\infty}\frac{\lambda_m-j+1}{\lambda_m-m}\left|\Delta_{11}a_{jk}\right|.$$

Now using (20), We have $\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|I_1^{\lambda}\right|\right)=0.$

Similarly, we get

$$\begin{aligned}\left|I_2^{\lambda}\right|&\leq\frac{4}{(1-x)(1-y)}\sum_{j=m+1}^{\lambda_m}\frac{\lambda_m-j+1}{\lambda_m-m}\left|\Delta_{10}a_{j,n+1}\right|\\&\leq\frac{4}{(1-x)(1-y)}\sum_{j=m+1}^{\lambda_m}\left|\Delta_{10}a_{j,n+1}\right|\\&\to 0\ \text{ as }\ \min(m,n)\to\infty.\end{aligned}$$

Thus $\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|I_2^{\lambda}\right|\right)=0.$

Similarly, we get

$$\begin{aligned}\left|I_3^{\lambda}\right|&\leq\frac{4}{(1-x)(1-y)}\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}\left|\Delta_{01}a_{j+1,k}\right|\\&\leq\frac{4}{(1-x)(1-y)}\sup_{m<j\leq\lambda_m}\sum_{k=0}^{n}\left|\Delta_{01}a_{j+1,k}\right|\\&\to 0\ \text{ as }\ \min(m,n)\to\infty.\end{aligned}$$

So $\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|I_3^{\lambda}\right|\right)=0.$

Similarly, we have

$$\begin{aligned}\left|I_4^{\lambda}\right|&\leq\frac{4}{(1-x)(1-y)}\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}\left|a_{j+1,n+1}\right|\\&\leq\frac{4}{(1-x)(1-y)}\sup_{m<j\leq\lambda_m}\left|a_{j+1,n+1}\right|\\&\to 0\ \text{ as }\ \min(m,n)\to\infty.\end{aligned}$$

We have $\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|I_4^{\lambda}\right|\right)=0.$

$$\left|I_5^{\lambda}\right| \le \frac{4}{(1-x)(1-y)}\sum_{k=0}^{n}|\Delta_{01}a_{m+1,k}|$$
$$\to 0 \text{ as } \min(m,n)\to\infty.$$

Thus $\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|I_5^{\lambda}\right|\right)=0.$

Similarly, we get

$$\left|I_6^{\lambda}\right| \le \frac{4}{(1-x)(1-y)}\,|a_{m+1,n+1}|$$
$$\to 0 \text{ as } \min(m,n)\to\infty.$$

We have $\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|I_6^{\lambda}\right|\right)=0.$

Thus, combining all these estimates, we have

$$\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|S_{10}^{\lambda}(m,n;x,y)\right|\right)=0. \tag{29}$$

Similarly, (11), Lemma 3.1, (17), (18), and (19) gives

$$\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left|S_{01}^{\lambda}(m,n;x,y)\right|\right)=0. \tag{30}$$

Thus, by (29) and (30), the first part of the theorem follows from Theorem 2.2.

Proof of (ii). Assume that $\left\|\tau_{mn}^{\lambda}-f\right\|_r\to 0$ as $\min(m,n)\to\infty$ for some $0<r<1$. We have

$$\|S_{mn}-f\|_r^r \le \left\|S_{mn}-\tau_{mn}^{\lambda}\right\|_r^r+\left\|\tau_{mn}^{\lambda}-f\right\|_r^r,$$

and so it suffices to show that $\left\|S_{mn}-\tau_{mn}^{\lambda}\right\|_r^r\to 0$ as $\min(m,n)\to\infty$. By Lemma 3.6, we have

$$\left\|S_{mn}-\tau_{mn}^{\lambda}\right\|_r^r \le \left\|S_{10}^{\lambda}(m,n;x,y)\right\|_r^r+\left\|S_{01}^{\lambda}(m,n;x,y)\right\|_r^r+\left\|S_{11}^{\lambda}(m,n;x,y)\right\|_r^r.$$

Now in order to estimate $\left\|S_{10}^{\lambda}(m,n;x,y)\right\|_r^r$, we first find $\left\|I_1^{\lambda}\right\|_r^r$, $\left\|I_2^{\lambda}\right\|_r^r$, $\left\|I_3^{\lambda}\right\|_r^r$,

$\left\|I_4^\lambda\right\|_r^r$, $\|I_5\|_r^r$, $\|I_6\|_r^r$, see (28). For $0 < r < 1$, we have

$$\left\|I_1^\lambda\right\|_r^r \leq \frac{4}{(1-r)^2}\left(\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}\frac{\lambda_m - j + 1}{\lambda_m - m}\left|\Delta_{11}a_{jk}\right|\right).$$

$$\left\|I_2^\lambda\right\|_r^r \leq \frac{4}{(1-r)^2}\left(\sum_{j=m+1}^{\lambda_m}\left|\Delta_{10}a_{j,n+1}\right|\right),$$

$$\left\|I_3^\lambda\right\|_r^r \leq \frac{4}{(1-r)^2}\left(\sup_{m<j\leq\lambda_m}\sum_{k=0}^{n}\left|\Delta_{01}a_{j+1,k}\right|\right),$$

$$\left\|I_4^\lambda\right\|_r^r \leq \frac{4}{(1-r)^2}\left(\sup_{m<j\leq\lambda_m}\left|a_{j+1,n+1}\right|\right),$$

$$\left\|I_5^\lambda\right\|_r^r \leq \frac{4}{(1-r)^2}\left(\sum_{k=0}^{n}\left|\Delta_{01}a_{m+1,k}\right|\right),$$

and

$$\left\|I_6^\lambda\right\|_r^r \leq \frac{4}{(1-r)^2}\left(\left|a_{m+1,n+1}\right|\right).$$

By (11), Lemma 3.1, (17), (18), and (20), we conclude that

$$\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left\|S_{10}^\lambda(m,n;x,y)\right\|_r^r\right) = 0.$$

Similarly by conditions (11), Lemma 3.1, and (17)-(19), we get

$$\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left\|S_{01}^\lambda(m,n;x,y)\right\|_r^r\right) = 0.$$

Also by (10), we have

$$\lim_{\lambda\to 1+}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in E}\left\|S_{11}^\lambda(m,n;x,y)\right\|_r^r\right) = 0.$$

Thus $\left\|S_{mn} - \tau_{mn}^\lambda\right\|_r \to 0$ as $\min(m,n) \to \infty$, and the desired result follows. This completes the proof of Theorem 2.4, □

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