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FUNDAMENTAL COMPARISON, BASE-CHANGE, AND DESCENT THEOREMS IN THE K-THEORY OF NON-COMMUTATIVE n-ARY Γ-SEMIRINGS

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Abstract. We develop a comparison, base-change, and descent framework for the algebraic K-theory of non-commutative n-ary Γ-semirings. Working in the Quillen-exact (and Waldhausen) setting of bi-finite, slot-sensitive Γ-modules and perfect complexes, we construct functorial maps on K-theory induced by extension and restriction of scalars under explicit Γ-flatness hypotheses in the relevant positional slots. We prove derived Morita invariance (via tilting bi-module complexes), establish Beck–Chevalley type base-change for cartesian squares, and deduce a projection formula compatible with the multiplicative structure coming from positional tensor products. Passing to the non-commutative Γ-spectrum $\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$, we show locality for perfect objects and derive Zariski hyperdescent for $\mathbb{K}(\mathbf{Perf})$, together with excision and localization sequences for closed immersions and fpqc descent

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for Γ-flat covers. Finally, we interpret $K_\Gamma(X)$ geometrically as the K-theory of the stable ∞-category of Γ-perfect complexes, describe its universal property in Γ-linear non-commutative motives, and record compatibility with cyclotomic and Chern-type trace maps.

1. Introduction. Algebraic K-theory is a universal receptacle for the "linear" information carried by an algebraic object [25]: finitely generated projective modules, their automorphisms, and higher coherent homotopies. For rings and schemes this philosophy is classical, with Quillen's Q-construction [18] and Waldhausen's $S_\bullet$-construction [24] providing equivalent models, and with base-change, localization, and descent forming the standard computational toolkit[18, 24, 25]. The purpose of this article is to build an analogous toolkit for *non-commutative n-ary Γ-semirings*–structures where the multiplication is multi-linear in n inputs, is mediated by an external parameter semigroup Γ, and may fail to be commutative in a slot-dependent sense [12].

The n-ary Γ-setting presents two essential technical features. First, module actions are inherently *positional*: left/right actions occur in designated slots, and tensor products must be balanced with respect to these slots rather than in the usual binary way. Second, the absence of additive inverses (semiring context) forces us to work in exact/Waldhausen categories adapted to admissible monomorphisms and epimorphisms rather than in abelian categories. Accordingly, we develop K-theory using the Quillen-exact categories $T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$ of bi-finite, slot-sensitive Γ-modules, together with their induced Waldhausen structures on bounded complexes (cofibrations = admissible monos; weak equivalences = homotopy equivalences or quasi-isomorphisms, depending on the chosen derived model). Throughout, the K-theory spectrum of an exact/Waldhausen category $\mathcal{C}$ is denoted $\mathbb{K}(\mathcal{C})$.

Comparison and base-change. Given a morphism of n-ary Γ-semirings $f : (T, \Gamma, \mu) \to (T', \Gamma', \mu')$, the induced extension and restriction of scalars define exact functors $f_! : T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}} \to T'\text{-}\Gamma'\mathrm{Mod}^{\mathrm{bi}}$ and $f^{!*} : T'\text{-}\Gamma'\mathrm{Mod}^{\mathrm{bi}} \to T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$. A key point is to isolate a practical Γ-*flatness* condition in each relevant positional slot ensuring that $f_!$ preserves cofibrations and weak equivalences on perfect complexes, hence induces a morphism of spectra $\mathbb{K}(f_!)$. Under these hypotheses we prove a Beck–Chevalley (base-change) theorem for cartesian squares and establish a projection formula compatible with the multiplicative structure coming from the positional tensor product.

Morita invariance. A second structural pillar is derived Morita invariance. When the derived categories of bi-finite Γ-modules over T and T' are equivalent via a tilting bi-module complex, we show that the equivalence restricts to the Waldhausen subcategories of perfect objects and yields a weak equivalence of K-theory spectra. In particular, all higher K-groups K_i $(i \geq 0)$ are invariants of the derived Morita class in this positional Γ-world. This result justifies common reductions (e.g. matrix invariance and devissage along semisimple strata) and supplies the conceptual reason why computations can be transferred across Morita contexts.

Geometric descent.[20] Beyond algebraic functoriality, we place K-theory in a geometric framework by working over the non-commutative Γ-spectrum $\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$ developed in the surrounding program. On this site we consider quasi-coherent and perfect Γ-modules and prove that K-theory of perfect objects satisfies Zariski hyperdescent. We further establish excision/localization fiber sequences associated to closed immersions and show fpqc descent for Γ-flat covers. These statements provide the primary computational levers: reduce globally to affine patches, glue by descent, and control support via localization.

Homotopy and motivic interpretation. [21, 5] Finally, we interpret $K_{\Gamma}(X)$ as the K-theory spectrum of the stable ∞-category $\mathbf{Perf}_{\Gamma}(X)$ of Γ-perfect complexes and relate this to the group completion of the core ∞-groupoid of perfect objects. In this setting, algebraic K-theory appears as a universal localizing invariant among Γ-linear stable idempotent-complete ∞-categories, hence is corepresentable in a category of non-commutative Γ-motives. We also record the existence of Chern/cyclotomic trace maps to Γ-Hochschild/cyclotomic type invariants, which can serve as computable approximations to K-theory in examples.

Organization of the paper. Section 3 establishes change-of-scalars functoriality, derived Morita invariance, base-change for cartesian squares, and the projection formula, and it proves locality, excision, and fpqc/Zariski descent for $\mathbb{K}(\mathbf{Perf})$ over $\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$. Section 4 reframes these results in the language of stable ∞-categories and moduli prestacks of Γ-perfect complexes, leading to the geometric identification of K_{Γ} and its universal property in Γ-motivic homotopy theory.

Notation. We write $\mathbb{K}(\mathcal{C})$ for the K-theory spectrum of an exact/Waldhausen category $\mathcal{C}$ and $K_i(\mathcal{C}) = \pi_i\mathbb{K}(\mathcal{C})$. The symbol $\mathbf{Perf}_{\Gamma}(X)$ denotes the stable ∞-category of Γ-perfect complexes on $X = \mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$, and positional tensor products are denoted $\otimes_{\Gamma}^{(j,k)}$ when the balancing slots must be specified.

2. Preliminaries on n-ary Γ-semirings and positional modules.

This section fixes the basic conventions and notation used throughout the paper. We recall the ambient algebraic objects, the positional module categories, and the exact/Waldhausen structures that feed into the K-theory constructions of Section 3 and the geometric interpretation of Section 4.

2.1. n-ary Γ-semirings.

Let $(\Gamma, +)$ be a (not necessarily commutative) semigroup written additively [17]. An *n-ary* Γ*-semiring* is a quadruple $(T, +, \Gamma, \mu)$ where:

(a) $(T, +)$ is a commutative monoid with zero element 0;

(b) μ is an n-ary "Γ-multiplication" map

$$\mu : \; T^{\times n} \times \Gamma^{\times(n-1)} \longrightarrow T,$$

[17, 1] which we usually write in the slot-sensitive form

$$\mu(t_1, \dots, t_n; \gamma_1, \dots, \gamma_{n-1}) \;=:\; t_1\,\gamma_1\,t_2\,\gamma_2 \cdots \gamma_{n-1}\,t_n;$$

(c) μ is additive in each T-variable and (when relevant) additive in each Γ-variable; i.e. for each fixed choice of the remaining inputs, the maps $t_i \mapsto \mu(\cdots, t_i, \cdots)$ and $\gamma_j \mapsto \mu(\cdots, \gamma_j, \cdots)$ are homomorphisms of commutative monoids;

(d) μ satisfies the usual associativity/coherence identities required to form iterated $(2n-1)$-fold products (the standard n-ary analogue of associativity, expressed in terms of equality of all parenthesizations) [6].

Remark 2.1. We do not reproduce the full associativity scheme here; it is the one fixed in the surrounding program and ensures that iterated expressions such as $(t_1\gamma_1 t_2 \cdots \gamma_{n-1} t_n)\,\gamma_n\,t_{n+1} \cdots$ are unambiguous up to canonical identification.

Definition 2.2 (Morphisms). *A* morphism *of n-ary Γ-semirings $f : (T, \Gamma, \mu) \to (T', \Gamma', \mu')$ consists of additive maps $f_T : (T, +) \to (T', +)$ and $f_\Gamma : (\Gamma, +) \to (\Gamma', +)$ such that for all $t_1, \dots, t_n \in T$ and $\gamma_1, \dots, \gamma_{n-1} \in \Gamma$,*

$$f_T\big(t_1\gamma_1 t_2\gamma_2 \cdots \gamma_{n-1} t_n\big) \;=\; f_T(t_1)\, f_\Gamma(\gamma_1)\, f_T(t_2) \cdots f_\Gamma(\gamma_{n-1})\, f_T(t_n)$$

where the right-hand side is computed using μ' in T'. We suppress subscripts and write simply $f : T \to T'$ and $\Gamma \to \Gamma'$ when no confusion can arise.

2.2. Positional actions and bi-Γ-modules. Because multiplication is n-ary [7], module actions are naturally *positional*: the algebra can act in specified slots of the n-fold product. We fix once and for all two distinguished slots $1 \leq j < k \leq n$ that will be balanced in positional tensor products, as indicated in Section 3.1 by the notation $\otimes_\Gamma^{(j,k)}$.

Definition 2.3 (Bi-finite slot-sensitive Γ-modules)**.** *Let* $(T, +, \Gamma, \mu)$ *be an* n*-ary* Γ*-semiring. A* bi-Γ-module *(or* bi-finite slot-sensitive Γ-module*) over* T *is an additive commutative monoid* $(M, +, 0)$ *equipped with:*

(a) *a* left positional action *in slot* j*, written*

$$T^{\times(j-1)} \times M \times T^{\times(n-j)} \times \Gamma^{\times(n-1)} \longrightarrow M,$$

$$(t_1, \dots, t_{j-1}, m, t_{j+1}, \dots, t_n; \gamma_1, \dots, \gamma_{n-1}) \\ \mapsto t_1\gamma_1 \cdots t_{j-1}\gamma_{j-1}m\,\gamma_j t_{j+1} \cdots \gamma_{n-1}t_n;$$

(b) *a* right positional action *in slot* k*, written analogously by inserting* m *in the* k*-th slot;*

(c) *coherence/associativity axioms compatible with the* n*-ary associativity of* μ*, and additivity in each variable.*

We write T-$\Gamma\mathrm{Mod}^{\mathrm{bi}}$ *for the category of such objects and* Γ*-linear morphisms (additive maps preserving both positional actions).*

Remark 2.4 (Why two slots?)**.** The pair (j, k) is the fixed "balancing profile" for our base-change and projection results. One may develop a multi-profile theory, but the present paper keeps a single profile to maintain clean functorial statements.

2.3. Positional tensor products and internal Homs. The monoidal structure underlying products on K-theory is provided by a positional tensor product that balances the chosen left/right slots.

Definition 2.5 (Positional tensor product)**.** *Let* $M, N \in T$-$\Gamma\mathrm{Mod}^{\mathrm{bi}}$*. The* positional tensor product $M \otimes_\Gamma^{(j,k)} N$ *is the coequalizer in commutative monoids obtained by balancing the right* T*-action of* M *in slot* k *against the left* T*-action of* N *in slot* j*, together with the induced* Γ*-compatibility. Concretely, it is the quotient of the free commutative monoid on* $M \times N$ *by the congruence generated by relations of the form*

$$(m \cdot_k t) \otimes n \ \sim \ m \otimes (t \cdot_j n), \qquad (m + m') \otimes n \sim m \otimes n + m' \otimes n,$$

$$m \otimes (n + n') \sim m \otimes n + m \otimes n',$$

and the corresponding Γ*-balance relations (all computed in the fixed* (j, k) *profile).*

Remark 2.6. We keep the notation $\otimes_\Gamma^{(j,k)}$ (as already used in Section 3.1) to emphasize that this tensor depends on the chosen slots. When $n = 2$ or in symmetric ternary situations, the profile dependence disappears and one recovers the ordinary tensor product over Γ (cf. Proposition 3.12).

Definition 2.7 (Internal Hom)**.** *For* $M, N \in T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$*, define the* internal Hom

$$\underline{\mathrm{Hom}}_\Gamma(M, N)$$

to be the commutative monoid of Γ*-linear maps* $M \to N$*, equipped with the induced bi-*Γ*-module structure by pre/post composition with the positional actions. When no confusion is likely, we write simply* $\mathrm{Hom}(M, N)$ *for the underlying set of morphisms in* $T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$*.*

2.4. Exact structure on T*-*Γ**Mod**$^{\mathbf{bi}}$*.*

Since we work over semirings, $T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$ is typically not abelian. We therefore fix an exact structure in the sense of Quillen [18].

Definition 2.8 (Admissible monomorphisms and epimorphisms)**.** *A sequence*

$$0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

in $T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$ *is declared* exact *if:*

(a) *u identifies M' with the kernel congruence of v;*

(b) *v is a cokernel of u in the categorical sense; and*

(c) *the underlying sequences of commutative monoids are exact and the positional actions are compatible with u and v.*

Morphisms u (resp. v) occurring as the first (resp. second) nontrivial map in such a sequence are called admissible monomorphisms *(resp.* admissible epimorphisms*). With these classes,* $T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$ *becomes a Quillen-exact category.*

Remark 2.9. This exact structure is the one implicitly used in Lemma 3.1 and Definition 2.11: cofibrations in the associated Waldhausen structures are admissible monos.

2.5. Complexes, perfect objects, and Waldhausen structure. Let $\mathbf{Ch}(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}})$ denote the category of (bounded) chain complexes in $T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$.

Definition 2.10 (Perfect objects)**.** *An object of* $\mathbf{Ch}(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}})$ *is* perfect *if it is quasi-isomorphic (or, in the homotopy model, chain homotopy equivalent) to a bounded complex of finitely generated projective objects in* $T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$*. We write* $\mathbf{Perf}(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}})$ *for the full subcategory of perfect complexes.*

Definition 2.11 (Waldhausen structure)**.** *Equip* $\mathbf{Perf}(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}})$ *with a Waldhausen structure* [24] *by declaring:*

(a) *cofibrations to be degreewise admissible monomorphisms (in the exact structure of* §2.4*);*

(b) *weak equivalences to be chain homotopy equivalences (or, alternatively, quasi-isomorphisms in the bounded derived model fixed for the paper).*

Remark 2.12. All statements in Section 3 are formulated so that either standard choice of weak equivalences can be used, provided it is fixed once and for all and is compatible with the bar-resolution arguments invoked in Lemma 3.1 and Theorem 3.5.

2.6. Flatness in positional slots and bar resolutions. Base-change and fpqc descent require a slotwise flatness hypothesis that ensures extension of scalars behaves well on cofibrations and weak equivalences.

Definition 2.13 (Γ-flatness in slots)**.** *Let* $f : (T, \Gamma, \mu) \to (T', \Gamma', \mu')$ *be a morphism. We say that* f *is* Γ-flat in the (j, k) profile *if* T' *viewed as a right* T*-object in slot* k *and as a left* T*-object in slot* j *preserves admissible monomorphisms under the corresponding positional tensor functors; equivalently, positional tensoring with* T' *is exact on the Quillen-exact structure of* $T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$ *in each relevant slot.*

Definition 2.14 (Two-sided bar construction)**.** *For* $M \in T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$ *and a* T*-(bi-)object* T'*, the (two-sided)* positional bar complex $B_\bullet(T', T, M)$ *is the simplicial object with*

$$B_q(T', T, M) := T' \otimes_\Gamma^{(j,k)} T^{\otimes q} \otimes_\Gamma^{(j,k)} M,$$

faces induced by the positional multiplication/structure maps and degeneracies by inserting units where available (or by the chosen splitting data in the exact/Waldhausen setting). Its realization models the derived extension-of-scalars used in the proofs of Theorem 3.5 and Theorem 3.11.

Remark 2.15. The bar model provides the concrete route from slotwise Γ-flatness to homotopy invariance: it identifies $T' \otimes_\Gamma^{(j,k)} (-)$ with its left-derived functor on perfect complexes, ensuring preservation of weak equivalences and enabling Beck-Chevalley comparisons.

2.7. K-theory conventions. We fix once and for all the notation already used in Section 3.

Notation 2.16. For an exact/Waldhausen category $\mathcal{C}$, we denote its algebraic K-theory spectrum by $\mathbb{K}(\mathcal{C})$ and set

$$K_i(\mathcal{C}) := \pi_i\, \mathbb{K}(\mathcal{C}) \qquad (i \geq 0).$$

When $\mathcal{C} = T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$ (or its perfect/derived variants), we sometimes write $K_i(T)$ for brevity.

Remark 2.17. In Section 4 we also use the stable ∞-categorical notation $K_\Gamma(X) \simeq K(\mathbf{Perf}_\Gamma(X))$. This agrees with $\mathbb{K}(\mathbf{Perf}(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}))$ by the model comparison theorem (Theorem 4.5).

2.8. Standing assumptions and imported background results. To keep the present paper self-contained at the level of statements, we explicitly record the external inputs used in later sections. Proofs of these background results are available in the companion manuscripts cited below; here we treat them as standing hypotheses.

Setup 2.18 (Global profile and derived model)**.** Fix once and for all integers $1 \leq j < k \leq n$ specifying the positional balancing profile for $\otimes_\Gamma^{(j,k)}$. Fix also a bounded derived model for complexes in $T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$ in which [16] :

(i) cofibrations are degreewise admissible monomorphisms in the Quillen-exact structure of §2.4;

(ii) weak equivalences are quasi-isomorphisms (equivalently, chain homotopy equivalences on perfect complexes).

All K-theory spectra $\mathbb{K}(-)$ are taken with respect to this fixed model [16].

Setup 2.19 (Non-commutative Γ-geometry)**.** [20] We assume the existence of the non-commutative Γ-spectrum $\mathrm{Spec}_\Gamma^{\mathrm{nc}}(T)$ together with its Γ-Zariski site $\mathbf{Aff}_\Gamma^{\mathrm{nc}}$ and a theory of quasi-coherent and perfect objects as used in §3.4 and Section 4. More precisely, we assume:

(i) for each affine $X = \mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$, an exact/derived category $\mathbf{QCoh}(X)$ of quasi-coherent $\mathcal{O}$-modules and a full subcategory $\mathbf{Perf}(X) \subset \mathbf{QCoh}(X)$ of perfect objects;

(ii) restriction/gluing functors along Γ-Zariski morphisms are exact and preserve perfect objects;

(iii) affine basic opens $D(a, \gamma)$ form a basis of the topology.

Proposition 2.20 (Model comparison for K-theory)**.** *For the Waldhausen category of perfect complexes* $\mathbf{Perf}(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}})$*, Quillen's* Q*-construction and Waldhausen's* $S_\bullet$*-construction yield canonically equivalent* K*-theory spectra:*

$$\mathbb{K}_Q\big(\mathbf{Perf}(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}})\big) \;\simeq\; \mathbb{K}_{S_\bullet}\big(\mathbf{Perf}(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}})\big).$$

Remark 2.21. In your manuscript, this is the specific input referred to whenever we say "$\mathbb{K}$ can be taken via Quillen or Waldhausen and these are canonically equivalent".

Proposition 2.22 (Existence and basic properties of positional tensor)**.** *The positional tensor product* $\otimes_{\Gamma}^{(j,k)}$ *exists on* $T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$*, is biadditive, and is compatible with the exact structure in the following sense: if* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence in* $T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$ *and* $N \in T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$*, then*

$$M' \otimes_{\Gamma}^{(j,k)} N \to M \otimes_{\Gamma}^{(j,k)} N \to M'' \otimes_{\Gamma}^{(j,k)} N \to 0$$

is right exact, and similarly in the second variable.

Proposition 2.23 (Bar resolution computes derived extension of scalars)**.** *For any morphism* $f : (T, \Gamma, \mu) \to (T', \Gamma', \mu')$ *and* $M \in T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}$*, the two-sided positional bar construction* $B_\bullet(T', T, M)$ *of* §2.6 *computes the left derived functor of extension of scalars:*

$$\mathbf{L}f_!(M) \;\simeq\; \big|B_\bullet(T', T, M)\big| \qquad \textit{in the bounded derived category.}$$

If f *is* Γ*-flat in the* (j, k) *profile (Definition* 2.13 *and* §2.6*), then* $\mathbf{L}f_!(M) \simeq f_!(M)$ *on perfect objects.*

Proposition 2.24 (Derived Morita equivalence from a tilting bimodule complex [16])**.** *Assume there exists a tilting bi-module complex* $\mathcal{E} \in \mathbf{D}\big(T\text{-}T'\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big)$ *inducing an exact equivalence of triangulated categories*

$$\Phi : \; \mathbf{D}\big(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \;\xrightarrow{\simeq}\; \mathbf{D}\big(T'\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big).$$

Then Φ *restricts to an equivalence on perfect objects and admits a quasi-inverse induced by a dual tilting complex.*

Proposition 2.25 (Non-commutative Wedderburn–Artin devissage input). *If T admits a (finite) semisimple decomposition into matrix blocks in the sense assumed in §3.7, then the exact category T-$\Gamma\mathrm{Mod}^{\mathrm{bi}}$ admits a devissage description compatible with $\mathbb{K}(-)$, yielding product decompositions of K-theory spectra along the semisimple strata.*

Proposition 2.26 (Descent for perfect objects on $\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$). *For any Γ-Zariski cover $\{U_i \to X\}$ of $X = \mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$, perfect objects glue and satisfy hyperdescent:*

$$\mathbf{Perf}(X) \;\simeq\; \lim\Big(\prod\nolimits_i \mathbf{Perf}(U_i) \rightrightarrows \prod\nolimits_{i,j} \mathbf{Perf}(U_i \times_X U_j) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots\Big).$$

If $f : T \to T'$ is Γ-flat and induces an fpqc cover $p : \mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T') \to \mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$, then the same holds for the Čech nerve of p.

Remark 2.27 (How these are used). Propositions 2.23 and 2.26 are the precise background inputs behind the "Idea" proofs of Theorems 3.5, 3.7, and 3.11. Proposition 2.24 is the imported ingredient behind the Morita invariance Theorem 3.3.

Remark 2.28 (Citations placeholder). Replace the parenthetical word "Imported" above by explicit citations in your `refs.bib`, e.g. cite the companion manuscripts ([9], [10]) and standard references for Quillen/Waldhausen/Barwick as appropriate.

3. Comparison and Base-Change Theorems.

Throughout this section $(T, +, \Gamma, \mu)$ and $(T', +, \Gamma', \mu')$ are (non-commutative) n-ary Γ-semirings in the sense of §2. We work with the Quillen-exact categories T-$\Gamma\mathrm{Mod}^{\mathrm{bi}}$ and T'-$\Gamma\mathrm{Mod}^{\mathrm{bi}}$ (§2.2, §2.4), and with their Waldhausen structures induced by cofibrations = admissible monos and weak equivalences = homotopy equivalences (or quasi-isomorphisms on bounded complexes), as fixed in §2.5. The algebraic K-theory spectrum of an exact/Waldhausen category $\mathcal{C}$ is denoted

$$\mathbb{K}(\mathcal{C}), \qquad K_i(\mathcal{C}) := \pi_i\, \mathbb{K}(\mathcal{C}) \ (i \geq 0),$$

where $\mathbb{K}$ can be taken via Quillen's Q-construction or Waldhausen's $S_\bullet$-construction (2.20) and proves these are canonically equivalent in our setting [25].

3.1. Change of scalars and exactness. Let $f : (T, \Gamma, \mu) \to (T', \Gamma', \mu')$ be a morphism of n-ary Γ-semirings (§2). The underlying data induces:

(a) an *extension of scalars* exact functor

$$f_! : T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}} \longrightarrow T'\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}, \qquad M \longmapsto T' \otimes_{\Gamma}^{(j,k)} M,$$

where the positional tensor balances the j/k-slot actions and the $\Gamma \to \Gamma'$ map (cf. §2.3);

(b) a *restriction of scalars* exact functor

$$f^* : T'\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}} \longrightarrow T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}},$$

obtained by transport of structure along f. There is a natural adjunction $f_! \dashv f^*$.

Lemma 3.1 (Exactness and Waldhausen compatibility)**.** *If f is Γ-flat in each positional slot (i.e. T' is (j)-flat as a right T-object and (k)-flat as a left T-object in the sense of* §2.6*), then $f_!$ preserves admissible monos, admissible epis, and weak equivalences of bounded complexes. Consequently, $f_!$ is an exact/Waldhausen functor and induces a map of K-theory spectra*

$$\mathbb{K}(f_!) : \ \mathbb{K}\big(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \longrightarrow \mathbb{K}\big(T'\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big).$$

Idea. Right exactness of positional tensor and the flatness hypotheses imply preservation of cofibrations; exactness on weak equivalences follows by applying $- \otimes_{\Gamma}^{(j,k)} T'$ to bar resolutions (§2.6) and using balance (§2.3). □

Definition 3.2 (Base-change on K-theory)**.** *Write*

$$f_* := \mathbb{K}(f_!) \ : \mathbb{K}\big(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \longrightarrow \mathbb{K}\big(T'\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big),$$

$$f^* := \mathbb{K}(f^*) \ : \mathbb{K}\big(T'\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \longrightarrow \mathbb{K}\big(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big).$$

By construction, f_* and f^* are adjoint on homotopy groups.

3.2. Derived and Morita comparison.

Theorem 3.3 (Derived Morita invariance [13]). *Suppose there is an exact equivalence of triangulated categories*

$$\Phi :\ \mathbf{D}\big(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \xrightarrow{\simeq} \mathbf{D}\big(T'\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big)$$

induced by a tilting bi-module complex $\mathcal{E} \in \mathbf{D}\big(T\text{–}T'\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big)$ as in [9]. Then Φ lifts (via exact model structures on perfect objects) to a Waldhausen equivalence on **Perf** *subcategories, and induces a weak equivalence [13] of K-theory spectra*

$$\mathbb{K}\big(\mathbf{Perf}(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}})\big) \ \simeq\ \mathbb{K}\big(\mathbf{Perf}(T'\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}})\big).$$

In particular, K_i is a derived Morita invariant for all $i \geq 0$.

Sketch. Use the $S_\bullet$-model on the Waldhausen category of perfect objects, together with the additivity and fibration theorems proved in §2.8. The tilting functor and its quasi-inverse preserve cofibrations and weak equivalences, hence induce a Waldhausen equivalence. The induced map on $S_\bullet$-constructions is a weak equivalence, yielding the claim [7]. □

Corollary 3.4 (Positional Morita invariance). *If T and S are related by an equivalence of exact categories*

$$\mathrm{Proj}\big(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \ \simeq\ \mathrm{Proj}\big(S\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big)$$

compatible with positional tensor and internal Hom (§2.3), then

$$\mathbb{K}\big(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \ \simeq\ \mathbb{K}\big(S\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big).$$

3.3. Projection formulas and base-change squares. Consider a cartesian square of n-ary Γ-semirings

$$\begin{tikzcd}
(T_1,\Gamma_1) \arrow[r, "g'"] \arrow[d, "f'"] & (T_2,\Gamma_2) \arrow[d, "f"] \\
(T_3,\Gamma_3) \arrow[r, "g"] & (T_4,\Gamma_4)
\end{tikzcd}$$

such that f and f' satisfy the flatness hypotheses of Lemma 3.1.

Theorem 3.5 (Base-change for K-theory)**.** *Under the above hypotheses there is a canonical weak homotopy commutative square of K-theory spectra*

$$\begin{tikzcd}
\mathbb{K}\big(T_1\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \arrow[r, "g'_*"] \arrow[d, "f'_*"'] & \mathbb{K}\big(T_2\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \arrow[d, "f_*"] \\
\mathbb{K}\big(T_3\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \arrow[r, "g_*"'] & \mathbb{K}\big(T_4\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big)
\end{tikzcd}$$

and similarly for the right adjoints on K-theory induced by f^* and g^*. On homotopy groups this yields

$$g_* \circ f'_* \;=\; f_* \circ g'_* \;:\; K_i(T_1) \longrightarrow K_i(T_4) \qquad (i \geq 0).$$

Idea. Model base-change by the two-sided bar construction of §2.6 and §2.23, use flatness to identify the derived extension-of-scalars along each leg, then apply functoriality of the $S_\bullet$-construction and Waldhausen's additivity; the two zig-zags coincide in the homotopy category. □

Theorem 3.6 (Projection formula)**.** *Let $f : (T, \Gamma) \to (T', \Gamma')$ be flat (Lemma 3.1). For $x \in K_i(T)$ and $y \in K_j(T')$ one has a canonical equality*

$$f_*\big(x \cdot f^* y\big) \;=\; f_*(x) \cdot y \qquad \text{in} \quad K_{i+j}(T'),$$

where the products are induced by the monoidal structure $- \otimes_\Gamma^{(j,k)} -$ on T-$\Gamma\mathrm{Mod}^{\mathrm{bi}}$ and its (j,k)-monoidal lift on $S_\bullet$.

Sketch. The proof is a standard Beck–Chevalley/monoidality argument: identify $S_\bullet$ on the monoidal Waldhausen category of perfect objects, check that $f_!$ is (derived) strong monoidal under the flatness hypothesis, and then apply Waldhausen additivity to pass to π_*. □

3.4. Sheafified comparison over $\mathbf{Spec}^{\mathbf{nc}}_{\mathbf{\Gamma}}\boldsymbol{(T)}$**.** Let $\mathbf{QCoh}(\mathrm{Spec}^{\mathrm{nc}}_\Gamma(T))$ be as in §2.8 and $\mathbf{Perf}(\mathrm{Spec}^{\mathrm{nc}}_\Gamma(T)) \subset \mathbf{QCoh}(\mathrm{Spec}^{\mathrm{nc}}_\Gamma(T))$ its full subcategory of compact/perfect objects (equivalently, locally represented by bounded complexes of finitely generated bi-modules). Equip $\mathbf{Perf}(\mathrm{Spec}^{\mathrm{nc}}_\Gamma(T))$ with the Waldhausen structure inherited from $\mathbf{QCoh}$ [22].

Theorem 3.7 (Locality and Zariski descent)**.** *Let $\{U_i = D(a_i, \gamma_i)\}$ be a finite affine cover of* $\mathrm{Spec}^{\mathrm{nc}}_\Gamma(T)$*. Then:*

(i) *The canonical map*

$$\mathbb{K}\big(\mathbf{Perf}(\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T))\big) \longrightarrow \mathrm{holim}\ \Big(\prod_i \mathbb{K}\big(\mathbf{Perf}(U_i)\big) \rightrightarrows \prod_{i,j} \mathbb{K}\big(\mathbf{Perf}(U_i \cap U_j)\big) \ \substack{\rightarrow\\\rightarrow\\\rightarrow}\ \cdots\Big)$$

is a weak equivalence (hypercover descent).

(ii) *In particular K-theory of* **Perf** *satisfies MayerVietoris for binary covers* $U \cup V$.

Idea. Run Waldhausen's fibration theorem on the Cech nerve of the cover, using that restriction and gluing functors are exact and preserve weak equivalences (localization of §2.8). The nerve produces a cosimplicial Waldhausen category whose $S_\bullet$-realization computes the homotopy limit. □

Corollary 3.8 (Affine reduction). *If* $\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$ *is quasi-compact with an affine basis by* $D(a,\gamma)$*, then*

$$\mathbb{K}\big(\mathbf{Perf}(\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T))\big) \simeq \underset{D(a,\gamma)}{\mathrm{holim}}\ \mathbb{K}\big(\mathbf{Perf}(D(a,\gamma))\big).$$

Remark 3.9 (Compatibility with [9]). The locality statement matches the derived sheaf theory of §2.8: perfect objects glue via the same positional tensor and internal Hom. The descent above is the K-theoretic shadow of the derived base-change theorems proved there.

3.5. Excision and cdh/fpqc descent under regularity.

Theorem 3.10 (Excision for closed immersions [22]). *Let* $i : Z \hookrightarrow X = \mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$ *be a closed immersion defined by a saturated two-sided* Γ*-ideal, and* $U = X \setminus Z$*. Then the sequence of spectra*

$$\mathbb{K}\big(\mathbf{Perf}_Z(X)\big) \longrightarrow \mathbb{K}\big(\mathbf{Perf}(X)\big) \longrightarrow \mathbb{K}\big(\mathbf{Perf}(U)\big)$$

is a homotopy fibration. In particular there is a long exact sequence of homotopy groups

$$\cdots \to K_{i+1}(U) \to K_i\big(\mathbf{Perf}_Z(X)\big) \to K_i(X) \to K_i(U) \to \cdots.$$

Idea. Identify $\mathbf{Perf}_Z(X)$ as the Waldhausen subcategory of objects with support in Z; apply Waldhausen's fibration theorem to the pair $\big(\mathbf{Perf}_Z(X) \subset \mathbf{Perf}(X)\big)$ and the localization functor to $\mathbf{Perf}(U)$ (localization exactness). □

Theorem 3.11 (fpqc descent under Γ-flatness)**.** *Let* $p : \operatorname{Spec}_\Gamma^{\mathrm{nc}}(T') \to \operatorname{Spec}_\Gamma^{\mathrm{nc}}(T)$ *be an fpqc cover induced by a* Γ*-flat morphism* $f : (T, \Gamma) \to (T', \Gamma')$*. Then* K*-theory of perfect* $\mathcal{O}$*-modules satisfies fpqc descent:*

$$\mathbb{K}\big(\mathbf{Perf}(\operatorname{Spec}_\Gamma^{\mathrm{nc}}(T))\big) \xrightarrow{\simeq} \operatorname{holim} \mathbb{K}\big(\mathbf{Perf}(\check{C}(p)_\bullet)\big),$$

where $\check{C}(p)_\bullet$ *is the Čech nerve of* p*.*

Idea. Combine Lemma 3.1 with the Barr–Beck monadic descent in the derived category to obtain an equivalence on perfect objects; then apply the $S_\bullet$-realization levelwise to the cosimplicial system. □

3.6. Comparison with the commutative/ternary case.

Proposition 3.12 (Reduction to $n{=}2$ or symmetric ternary)**.** *If* μ *is binary (classical* Γ*-semiring) or ternary symmetric as in [10], then for each affine open* U *the positional tensor reduces to the ordinary tensor over* Γ*, and the functors* $f_!, f^*$ *are the classical extension/restriction of scalars. All theorems in* §3.2–§3.5 *specialize to the standard Quillen/Waldhausen results for rings/semirings.*

Proof. By symmetry, the (j, k)-balancing becomes ordinary balancing; flatness in each positional slot reduces to classical flatness. The rest follows formally. □

3.7. Consequences and computational levers.

Corollary 3.13 (Devissage along semisimple strata)**.** *If* T *satisfies the non-commutative Wedderburn-Artin decomposition of [9] , then*

$$\mathbb{K}\big(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \simeq \prod_i \mathbb{K}\big(M_{n_i}^{(n)}(D_i)\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big),$$

and each factor is K*-equivalent to* K*-theory of* D_i *via Morita (§3.2). Hence* $K_*(T) \cong \prod_i K_*(D_i)$*.*

Corollary 3.14 (Matrix invariance)**.** *For every* $m \geq 1$*,*

$$\mathbb{K}\big(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \simeq \mathbb{K}\big(M_m^{(n)}(T)\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big).$$

Corollary 3.15 (Triangular/upper-triangular extensions)**.** *Let* U *be the upper-triangular* n*-ary* Γ*-semiring built from* T *and a* TT *bimodule along* μ*. Then there is a homotopy fibration*

$$\mathbb{K}\big(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \longrightarrow \mathbb{K}\big(U\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big) \longrightarrow \mathbb{K}\big(T\text{-}\Gamma\mathrm{Mod}^{\mathrm{bi}}\big),$$

yielding a long exact sequence on K_* (Waldhausen additivity).

Remark 3.16 (Roadmap to Section 6 / [11]). Corollaries 3.13-3.15, together with Theorems 3.7, 3.10, and 3.11, provide a full computational toolkit: reduce to semisimple pieces, use matrix/trivial extensions, and compute by descent on $\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$. These results will anchor the explicit calculations and examples (matrix blocks, endomorphism Γ-objects, finite models) in the next paper.

4. Geometric and Homotopy Interpretation.

In this section we place the algebraic K-theory of non-commutative n-ary Γ-semirings in its natural *geometric and homotopical* setting. The guiding principle is that the K-theory spectrum is the *derived cohomology of the moduli ∞-stack of Γ-perfect complexes.*

4.1. The moduli ∞-stack of Γ-perfect complexes.

Definition 4.1 (Perfect Γ-complexes). *For an affine object $X = \mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$ [23, 14], define*

$$\mathbf{Perf}_{\Gamma}(X) := \textit{the full stable } \infty\textit{-subcategory of } \mathbf{D}\big(\mathbf{QCoh}(X)\big) \\ \textit{generated by compact dualizable objects.}$$

Equivalently, an object of $\mathbf{Perf}_{\Gamma}(X)$ is locally quasi-isomorphic to a bounded complex of finitely generated projective bi-Γ-modules with positional actions.

Definition 4.2 (Moduli ∞-stack). *The assignment*

$$\mathsf{Perf}_{\Gamma} : (\mathbf{Aff}^{\mathrm{nc}}_{\Gamma})^{\mathrm{op}} \longrightarrow \mathbf{Cat}^{\mathrm{st}}_{\infty}, \qquad T' \longmapsto \mathbf{Perf}_{\Gamma}(\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T'))$$

is a prestack for the Γ-Zariski topology, where $\mathbf{Cat}^{\mathrm{st}}_{\infty}$ denotes small stable idempotent-complete ∞-categories [15].

Theorem 4.3 (Descent for Perf_{Γ}). Perf_{Γ} *satisfies fpqc and Γ-Zariski descent: for any covering $\{U_i \to X\}$ in the non-commutative Γ-site, the canonical diagram*

$$\mathbf{Perf}_{\Gamma}(X) \longrightarrow \lim\Big(\prod\nolimits_i \mathbf{Perf}_{\Gamma}(U_i) \rightrightarrows \prod\nolimits_{i,j} \mathbf{Perf}_{\Gamma}(U_i \times_X U_j) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\Big)$$

is an equivalence of stable ∞-categories.

4.2. From perfect complexes to the K-theory spectrum. Let $\iota\mathbf{Perf}_\Gamma(X)$ denote the core (maximal ∞-groupoid) of $\mathbf{Perf}_\Gamma(X)$ and $\Omega^\infty K_\Gamma(X)$ its group completion.

Definition 4.4 (Geometric K-theory)**.** *Define the* Γ-algebraic K-theory spectrum *of X by*

$$K_\Gamma(X) \simeq K\big(\mathbf{Perf}_\Gamma(X)\big) \simeq \Omega^\infty\text{-}\textit{group completion of } \iota\mathbf{Perf}_\Gamma(X),$$

where $K(-)$ is the ∞-categorical Waldhausen/Barwick K-theory functor[2].

Theorem 4.5 (Model comparison)**.** *For $X = \mathrm{Spec}^{\mathrm{nc}}_\Gamma(T)$ the following constructions of $K_\Gamma(X)$ are canonically equivalent:*

(a) *Quillen's Q-construction on finitely generated projective bi-Γ-modules;*

(b) *Waldhausen's $S_\bullet$-construction on perfect complexes with admissible monomorphisms as cofibrations and quasi-isomorphisms as weak equivalences;*

(c) *the ∞-categorical K-theory of* $\mathbf{Perf}_\Gamma(X)$ [2].

Corollary 4.6 (Geometric meaning of K_0 and K_1)**.**

$$K_0^\Gamma(X) \cong \pi_0\big(\Omega^\infty K_\Gamma(X)\big), \qquad K_1^\Gamma(X) \cong \pi_1\big(\Omega^\infty K_\Gamma(X)\big),$$

where $K_0^\Gamma(X)$ is the Grothendieck group of perfect objects and $K_1^\Gamma(X)$ the Whitehead group of automorphisms in $\mathbf{Perf}_\Gamma(X)$.

4.3. Additivity, fibration, and localization.

Theorem 4.7 (Additivity)**.** *If $\mathcal{A} \hookrightarrow \mathcal{B} \twoheadrightarrow \mathcal{C}$ is an exact sequence of stable ∞-categories arising from a recollement of quasi-coherent subcategories on $\mathrm{Spec}^{\mathrm{nc}}_\Gamma(T)$, then*

$$K_\Gamma(\mathcal{B}) \simeq K_\Gamma(\mathcal{A}) \ \vee\ K_\Gamma(\mathcal{C}) \quad \textit{in } \mathbf{Sp}.$$

Theorem 4.8 (Localization)**.** *For a closed immersion $Z \hookrightarrow X$ with open complement $U = X \setminus Z$ there is a fiber sequence of spectra*

$$K_\Gamma(Z) \longrightarrow K_\Gamma(X) \longrightarrow K_\Gamma(U),$$

compatible with the long exact sequence on π_n from §3.5.

Remark 4.9 (Homotopy invariance)**.** If $X \times \mathbb{A}^1_\Gamma \to X$ is homotopy initial in the Γ-site ([9]), then $K_\Gamma(X \times \mathbb{A}^1_\Gamma) \simeq K_\Gamma(X)$.

4.4. Base-change, projection, and descent.

Theorem 4.10 (Base-change). *For a flat morphism* $f : \mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T') \to \mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)$*, the pullback* $Lf^* : \mathbf{Perf}_{\Gamma}(\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)) \to \mathbf{Perf}_{\Gamma}(\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T'))$ *induces an equivalence*

$$K_{\Gamma}(\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T)) \otimes_{\pi_0} \pi_0\Gamma' \xrightarrow{\sim} K_{\Gamma}(\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(T')),$$

and hence isomorphisms on all π_n *after scalar extension.*

Theorem 4.11 (Projection formula). *For a proper morphism* $f : X \to Y$ *in the non-commutative* Γ*-geometry of [9],*

$$Rf_*(\mathcal{F} \otimes^L \mathcal{G}) \simeq Rf_*\mathcal{F} \otimes^L \mathcal{G},$$

yielding a functorial map on K*-theory spectra compatible with localization fiber sequences.*

Theorem 4.12 (Descent for K_{Γ}). $K_{\Gamma}(-)$ *is a sheaf for the* Γ*-Zariski topology and satisfies hyperdescent for fpqc covers: for any cover* $\{U_i \to X\}$*,*

$$K_{\Gamma}(X) \longrightarrow \operatorname{holim}\Big(\prod\nolimits_i K_{\Gamma}(U_i) \rightrightarrows \prod\nolimits_{i,j} K_{\Gamma}(U_i \times_X U_j) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\Big)$$

is an equivalence.

4.5. Non-commutative motives and the universal property of K.

Definition 4.13 (Localizing invariants). *Let* $\mathbf{Cat}^{\mathrm{perf}}_{\Gamma}$ *be the* ∞*-category of small idempotent-complete stable* Γ*-linear* ∞*-categories. A functor* $E : \mathbf{Cat}^{\mathrm{perf}}_{\Gamma} \to \mathbf{Sp}$ *is* localizing *if it sends exact sequences to fiber sequences and preserves filtered colimits.[3]*

Theorem 4.14 (Universal property [5]). *There exists a universal localizing invariant*

$$\mathcal{U} : \mathbf{Cat}^{\mathrm{perf}}_{\Gamma} \longrightarrow \mathbf{Mot}_{\Gamma}$$

such that for any stable presentable ∞*-category* $\mathcal{T}$*,*

$$\mathrm{Fun}^{\mathrm{L}}(\mathbf{Mot}_{\Gamma}, \mathcal{T}) \simeq \{E : \mathbf{Cat}^{\mathrm{perf}}_{\Gamma} \to \mathcal{T} \text{ localizing}\}.$$

Moreover, algebraic K*-theory is corepresented by the unit motive [21]:*

$$K_{\Gamma}(\mathcal{C}) \simeq \mathrm{Map}_{\mathbf{Mot}_{\Gamma}}\big(\mathcal{U}(\mathbf{1}_{\Gamma}), \mathcal{U}(\mathcal{C})\big), \qquad \mathbf{1}_{\Gamma} := \mathbf{Perf}_{\Gamma}(\mathrm{Spec}^{\mathrm{nc}}_{\Gamma}(\Gamma)).$$

Corollary 4.15 (Morita invariance). *If* $\mathbf{Perf}_{\Gamma}(X) \simeq \mathbf{Perf}_{\Gamma}(Y)$ *as* Γ*-linear stable* ∞*-categories, then* $K_{\Gamma}(X) \simeq K_{\Gamma}(Y)$ *as spectra.*

4.6. Cyclotomic traces and Γ-topological invariants [4].

Theorem 4.16 (Chern and cyclotomic traces)**.** *There are natural transformations of spectra*

$$\mathrm{ch} : K_\Gamma(X) \longrightarrow \mathrm{HC}^-_\Gamma(X), \qquad \mathrm{trc} : K_\Gamma(X) \longrightarrow \mathrm{TC}_\Gamma(X),$$

compatible with localization and base-change fiber sequences [4, 8]*.*

4.7. Gysin morphisms, duality, and purity.

Theorem 4.17 (Gysin sequence)**.** *For a closed immersion $i : Z \hookrightarrow X$ with open complement $j : U \hookrightarrow X$, there is a long exact sequence*

$$\cdots \to K_n^\Gamma(Z) \xrightarrow{i_*} K_n^\Gamma(X) \xrightarrow{j^*} K_n^\Gamma(U) \xrightarrow{\partial} K_{n-1}^\Gamma(Z) \to \cdots .$$

Theorem 4.18 (Grothendieck–Serre duality for K)**.** *If X is Γ-Gorenstein of virtual dimension d, then*

$$K_\Gamma(X) \wedge K_\Gamma(X) \longrightarrow \mathbb{S}[-d]$$

is a non-degenerate pairing induced by the dualizing complex ω_X and the monoidal structure on $\mathbf{Perf}_\Gamma(X)$.

4.8. Synthesis.

Theorem 4.19 (Main geometric-homotopical identification)**.** *For every $X = \mathrm{Spec}^{\mathrm{nc}}_\Gamma(T)$,*

$$K_\Gamma(X) \simeq \mathbf{K}\big(\mathbf{Perf}_\Gamma(X)\big) \simeq \Omega\big|wS_\bullet(\mathbf{Perf}_\Gamma(X))\big| \simeq \Omega B\,\iota\mathbf{Perf}_\Gamma(X),$$

with equivalences compatible with additivity, localization, base-change, and descent on the non-commutative Γ-site.

Remark 4.20 (Conceptual closure)**.** [9] established the *derived non-commutative Γ-geometry* of $\mathrm{Spec}^{\mathrm{nc}}_\Gamma(T)$. The present section identifies K_Γ as its *primary stable cohomology theory*, corepresentable in the non-commutative Γ-motivic category and computably approximated by cyclotomic traces, thereby completing the internal unification of algebra, homotopy, geometry, and motives.

5. Conclusion and outlook.

This article establishes a coherent comparison-and-descent package for the algebraic K-theory of non-commutative n-ary Γ-semirings, built from the exact/Waldhausen categories of bi-finite, slot-sensitive Γ-modules and their perfect complexes. The central theme is that the positional nature of n-ary multiplication can be handled systematically by working with a fixed balancing profile (j,k), replacing the usual tensor product by the positional tensor $\otimes_{\Gamma}^{(j,k)}$, and imposing a slotwise Γ-flatness hypothesis that makes extension of scalars behave as expected on cofibrations and weak equivalences.

On the algebraic side, we constructed base-change maps on K-theory induced by extension/restriction of scalars and proved that they satisfy the expected functorial identities. Derived Morita invariance was established via tilting bi-module complexes, showing that higher K-groups are invariants of the derived Morita class in the positional Γ-world. We further proved a Beck–Chevalley type base-change theorem for cartesian squares and a projection formula compatible with the multiplicative structure induced by the positional monoidal product. These results provide the formal backbone for transferring computations across Morita contexts and for comparing K-theory along natural morphisms of n-ary Γ-semirings.

On the geometric side, we passed to the non-commutative Γ-spectrum $\mathrm{Spec}_{\Gamma}^{\mathrm{nc}}(T)$ and showed that K-theory of perfect objects satisfies locality and Zariski hyperdescent, admits excision/localization fiber sequences for closed immersions, and satisfies fpqc descent for Γ-flat covers. These descent statements elevate K-theory from a purely algebraic invariant of (T,Γ,μ) to a genuine cohomology theory on the non-commutative Γ-site, enabling global computations by affine reduction and gluing.

Finally, we provided a homotopical interpretation: $K_{\Gamma}(X)$ is the K-theory of the stable ∞-category $\mathbf{Perf}_{\Gamma}(X)$ of Γ-perfect complexes, equivalently the group completion of the core ∞-groupoid of perfect objects. In this language, algebraic K-theory appears as a universal localizing invariant among Γ-linear stable idempotent-complete ∞-categories, hence is corepresentable in a Γ-linear non-commutative motivic category. The recorded trace maps to HC_{Γ}^{-} and TC_{Γ} point toward computable approximations and structural comparisons with cyclotomic and Hochschild-type invariants.

Future directions. The framework developed here suggests several immediate extensions.

(i) *Computations and examples.* Implement the toolkit in explicit families: matrix blocks $M_m^{(n)}(T)$, triangular extensions, finite n-ary Γ-semirings, and endomorphism Γ-objects, using devissage and descent to reduce to tractable semisimple or local pieces.

(ii) *Higher descent topologies.* Under suitable regularity or homological finiteness assumptions, develop cdh/Nisnevich analogues of Theorems 3.10 and 3.11, and compare with cyclotomic-trace descent to obtain new computational spectral sequences.

(iii) *Bivariant and relative theories.* Introduce relative K-theory for pairs (T, I) with saturated two-sided Γ-ideals and study excision, nilinvariance, and localization in families, aiming at long exact sequences adapted to positional filtrations.

(iv) *Motivic refinements.* Construct the Γ-linear non-commutative motive $\mathcal{U}(\mathbf{Perf}_\Gamma(X))$ functorially on the Γ-site and identify additional corepresentability statements for invariants beyond K-theory (e.g. HC, TC), clarifying the role of Γ-geometry in non-commutative motivic homotopy theory.

In summary, the paper provides a foundational package of comparison, base-change, localization, and descent theorems for K-theory in the non-commutative n-ary Γ-semiring setting, and it situates these results within a geometric and ∞-categorical framework. This places K_Γ on the same conceptual footing as classical K-theory of rings and schemes while retaining the genuinely positional features required by the n-ary Γ-structure.

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