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LYAPUNOV AND HARTMAN–WINTNER INEQUALITIES FOR CAPUTO–FABRIZIO FRACTIONAL DIRICHLET BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we establish Lyapunov- and Hartman–Wintner-type inequalities for a class of two-point Dirichlet boundary value problems involving the Caputo–Fabrizio fractional derivative. The analysis is based on a rigorous construction of the associated Caputo–Fabrizio Green's function and explicit kernel-dependent bounds that reflect the non-singular exponential memory of the operator. As direct consequences, we obtain nonexistence criteria for nontrivial solutions and a lower bound for the first eigenvalue of the corresponding Caputo–Fabrizio spectral problem. A numerical illustration is provided to confirm the sharpness and qualitative behavior of the derived inequalities.

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Key words: Caputo–Fabrizio fractional derivative, Dirichlet boundary value problem, Green's function, Lyapunov-type inequality, Hartman–Wintner inequality, eigenvalue estimate.

1. Introduction. Lyapunov- and Hartman–Wintner-type inequalities play a fundamental role in the qualitative theory of differential equations. Such inequalities provide necessary conditions for the existence of nontrivial solutions, furnish estimates for eigenvalues of associated boundary value problems, and offer valuable insight into oscillatory and nonoscillatory behavior of solutions. In the classical setting of second-order ordinary differential equations, Lyapunov and Hartman–Wintner inequalities are well established and have been widely applied in both theoretical investigations and practical applications [11, 5].

During the last two decades, considerable attention has been devoted to extending these inequalities to fractional differential equations and fractional boundary value problems involving Riemann–Liouville and Caputo fractional derivatives. The nonlocal nature of fractional operators allows the modeling of memory and hereditary effects, which arise naturally in various areas of physics, engineering, and applied sciences [13, 9]. For fractional operators with singular power-law kernels, a rich literature now exists on Lyapunov-type and Hartman–Wintner-type inequalities, and in many situations the corresponding constants are known to be optimal [8, 7, 6, 4].

More recently, fractional calculus has witnessed the introduction of derivatives with non-singular kernels. Among these, the Caputo–Fabrizio fractional derivative has attracted significant interest due to its exponential kernel, which eliminates the singularity at the origin and provides a realistic description of fading memory [3, 12]. In contrast to classical fractional derivatives, the Caputo–Fabrizio operator does not exhibit power-law memory and instead captures processes with exponentially decaying memory effects.

This fundamental change in kernel structure has important consequences for the analysis of boundary value problems. In particular, Lyapunov- and Hartman–Wintner-type inequalities derived for classical fractional operators cannot be directly transferred to problems involving the Caputo–Fabrizio derivative. The associated integral representations and Green's functions differ essentially from those arising in the Riemann–Liouville or Caputo settings, and classical Lyapunov constants fail to reflect the influence of the non-singular exponential kernel.

Despite the growing number of works devoted to Caputo–Fabrizio fractional differential equations, rigorous Lyapunov- and Hartman–Wintner-type inequalities specifically tailored to Caputo–Fabrizio boundary value problems remain relatively limited. Existing studies often focus on existence, uniqueness, and stability of solutions, while inequality results are either absent or obtained by formal analogy with classical fractional cases, without a rigorous operator-specific justification [8].

The aim of the present paper is to fill this gap by developing Lyapunov- and Hartman–Wintner-type inequalities for a class of two-point Dirichlet boundary value problems involving the Caputo–Fabrizio fractional derivative. Our approach is based on a rigorous construction of the Caputo–Fabrizio Green's function and the derivation of explicit kernel-dependent bounds that reflect the exponential memory structure of the operator. Using these bounds, we establish new Lyapunov-type and Hartman–Wintner-type inequalities whose constants differ fundamentally from those associated with classical fractional derivatives.

As applications of the obtained inequalities, we derive nonexistence criteria for nontrivial solutions and obtain a lower bound for the first eigenvalue of the corresponding Caputo–Fabrizio spectral problem. A numerical illustration is included to demonstrate the sharpness and qualitative behavior of the derived results.

2. Preliminaries and the Caputo–Fabrizio operator. In this section we recall the Caputo–Fabrizio (CF) fractional derivative and record several basic facts that will be used in the construction of the Green's function and the derivation of Lyapunov- and Hartman–Wintner-type inequalities.

Throughout the paper, let $a < b$ be real numbers and let $C([a,b])$ denote the Banach space of continuous real-valued functions on $[a,b]$ equipped with the supremum norm

$$\|u\|_\infty := \max_{t\in[a,b]} |u(t)|. \tag{1}$$

2.1. Caputo–Fabrizio fractional derivative. Let $u \in C^1([a,b])$ and $\alpha \in (0,1)$. The Caputo–Fabrizio fractional derivative of order α (starting from a) is defined by

$$({}^{CF}D^{\alpha}_{a+}u)(t) = \frac{1}{1-\alpha}\int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) u'(s)\,ds, \qquad t \in [a,b]. \tag{2}$$

The kernel in (2) is non-singular and exponentially decaying.

2.2. Extension to orders $1 < \alpha \leq 2$. To treat two-point Dirichlet boundary conditions, we consider the CF operator of order $\alpha \in (1,2]$ defined in terms of the second derivative. Let $u \in C^2([a,b])$ and let $\alpha \in (1,2]$. We define

$$({}^{CF}D^{\alpha}_{a+}u)(t) = \frac{1}{2-\alpha}\int_a^t \exp\left(-\frac{\alpha-1}{2-\alpha}(t-s)\right) u''(s)\,ds, \qquad t \in [a,b]. \tag{3}$$

Remark 2.1. The order range $1 < \alpha \leq 2$ is natural for Dirichlet boundary conditions $u(a) = u(b) = 0$ because it plays the role of a "second-order" setting: the representation (3) involves u'' and will lead to a Green's function that accommodates two boundary constraints.

2.3. Elementary properties.

Lemma 2.1 (Linearity)**.** *Let $\alpha \in (0,1)$ and $u, v \in C^1([a,b])$. Then ${}^{CF}D^{\alpha}_{a+}$ is linear, i.e.,*

$$ {}^{CF}D^{\alpha}_{a+}(c_1u + c_2v) = c_1\,{}^{CF}D^{\alpha}_{a+}u + c_2\,{}^{CF}D^{\alpha}_{a+}v, $$

for all constants $c_1, c_2 \in \mathbb{R}$.

P r o o f. The result follows immediately from the linearity of the integral in (2). □

Lemma 2.2 (Derivative of a constant)**.** *Let $\alpha \in (0,1)$ and let $u(t) \equiv C$ be constant on $[a,b]$. Then ${}^{CF}D^{\alpha}_{a+}u(t) = 0$ for all $t \in [a,b]$.*

P r o o f. If u is constant, then $u'(s) = 0$ for all $s \in [a,b]$ and the integral in (2) vanishes. □

2.4. Integral representation for the CF equation. A key step in constructing Green's functions is the reduction of a CF differential equation to an integral equation. We now provide a detailed derivation of the integral representation associated with the Caputo–Fabrizio operator.

Lemma 2.3 (Integral representation for $0 < \alpha < 1$)**.** *Let $\alpha \in (0,1)$ and $u \in C^1([a,b])$. Suppose that*

$$ ({}^{CF}D^{\alpha}_{a+}u)(t) = g(t), \qquad t \in [a,b], \tag{4} $$

where $g \in C([a,b])$. Then $g(a) = 0$ and, for every $t \in [a,b]$,

$$ u(t) = u(a) + (1-\alpha)g(t) + \alpha \int_a^t g(s)\,ds. \tag{5} $$

Conversely, if $g \in C^1([a,b])$ with $g(a) = 0$ and u is defined by

$$ u(t) = u(a) + (1-\alpha)g(t) + \alpha \int_a^t g(s)\,ds, \tag{6} $$

then $u \in C^1([a,b])$ and satisfies

$$ ({}^{CF}D^{\alpha}_{a+}u)(t) = g(t), \qquad t \in [a,b]. \tag{7} $$

P r o o f. Let

$$k := \frac{\alpha}{1-\alpha} > 0.$$

By the definition of the Caputo–Fabrizio fractional derivative,

$$\frac{1}{1-\alpha}\int_a^t e^{-k(t-s)}u'(s)\,ds = g(t), \qquad t \in [a,b]. \tag{8}$$

Multiplying both sides by $(1-\alpha)$ gives

$$\int_a^t e^{-k(t-s)}u'(s)\,ds = (1-\alpha)g(t). \tag{9}$$

We now integrate both sides with respect to t from a to t:

$$\int_a^t \left(\int_a^\xi e^{-k(\xi-s)}u'(s)\,ds\right) d\xi = (1-\alpha)\int_a^t g(\xi)\,d\xi. \tag{10}$$

Since the integrand is continuous on the triangular region

$$\{(s,\xi) : a \le s \le \xi \le t\},$$

Fubini's theorem allows us to interchange the order of integration:

$$\int_a^t \left(\int_s^t e^{-k(\xi-s)}\,d\xi\right) u'(s)\,ds = (1-\alpha)\int_a^t g(\xi)\,d\xi. \tag{11}$$

The inner integral is computed as

$$\int_s^t e^{-k(\xi-s)}\,d\xi = \int_0^{t-s} e^{-kr}\,dr = \frac{1-e^{-k(t-s)}}{k}. \tag{12}$$

Thus,

$$\int_a^t \frac{1-e^{-k(t-s)}}{k}\,u'(s)\,ds = (1-\alpha)\int_a^t g(\xi)\,d\xi. \tag{13}$$

Splitting the integral, we obtain

$$\frac{1}{k}\int_a^t u'(s)\,ds - \frac{1}{k}\int_a^t e^{-k(t-s)}u'(s)\,ds = (1-\alpha)\int_a^t g(\xi)\,d\xi. \tag{14}$$

Using

$$\int_a^t u'(s)\,ds = u(t) - u(a)$$

and

$$\int_a^t e^{-k(t-s)}u'(s)\,ds = (1-\alpha)g(t),$$

we arrive at

$$\frac{1}{k}\big(u(t)-u(a)\big) - \frac{1-\alpha}{k}g(t) = (1-\alpha)\int_a^t g(\xi)\,d\xi. \tag{15}$$

Since $\dfrac{1}{k} = \dfrac{1-\alpha}{\alpha}$, multiplying both sides by k yields

$$u(t) - u(a) - (1-\alpha)g(t) = \alpha\int_a^t g(\xi)\,d\xi, \tag{16}$$

which gives

$$u(t) = u(a) + (1-\alpha)g(t) + \alpha\int_a^t g(\xi)\,d\xi. \tag{17}$$

Next, evaluating the original equation at $t = a$, we have

$$({}^{CF}D^{\alpha}_{a+}u)(a) = \frac{1}{1-\alpha}\int_a^a e^{-k(a-s)}u'(s)\,ds = 0,$$

and hence

$$g(a) = 0.$$

For the converse direction, assume that $g \in C^1([a,b])$ with $g(a) = 0$, and define

$$u(t) = u(a) + (1-\alpha)g(t) + \alpha\int_a^t g(s)\,ds.$$

Then $u \in C^1([a,b])$ and

$$u'(t) = (1-\alpha)g'(t) + \alpha g(t). \tag{18}$$

We now compute the Caputo–Fabrizio derivative:

$$({}^{CF}D^{\alpha}_{a+}u)(t) = \frac{1}{1-\alpha}\int_a^t e^{-k(t-s)}u'(s)\,ds \tag{19}$$

$$= \frac{1}{1-\alpha}\int_a^t e^{-k(t-s)}\big((1-\alpha)g'(s) + \alpha g(s)\big)\,ds \tag{20}$$

$$= \int_a^t e^{-k(t-s)}g'(s)\,ds + \frac{\alpha}{1-\alpha}\int_a^t e^{-k(t-s)}g(s)\,ds. \tag{21}$$

Since $\dfrac{\alpha}{1-\alpha} = k$, this becomes

$$({}^{CF}D^{\alpha}_{a+}u)(t) = \int_a^t e^{-k(t-s)}g'(s)\,ds + k\int_a^t e^{-k(t-s)}g(s)\,ds. \tag{22}$$

Applying integration by parts to the first term,

$$\int_a^t e^{-k(t-s)}g'(s)\,ds = \Big[e^{-k(t-s)}g(s)\Big]_{s=a}^{s=t} - k\int_a^t e^{-k(t-s)}g(s)\,ds \tag{23}$$

$$= g(t) - e^{-k(t-a)}g(a) - k\int_a^t e^{-k(t-s)}g(s)\,ds. \tag{24}$$

Since $g(a) = 0$, we obtain

$$\int_a^t e^{-k(t-s)}g'(s)\,ds = g(t) - k\int_a^t e^{-k(t-s)}g(s)\,ds. \tag{25}$$

Substituting back gives

$$({}^{CF}D^{\alpha}_{a+}u)(t) = g(t) - k\int_a^t e^{-k(t-s)}g(s)\,ds + k\int_a^t e^{-k(t-s)}g(s)\,ds \tag{26}$$

$$= g(t). \tag{27}$$

Thus,

$$({}^{CF}D^{\alpha}_{a+}u)(t) = g(t), \qquad t \in [a,b].$$

This completes the proof. □

Remark 2.2. Although the main boundary value problem in this paper is posed for $1 < \alpha \le 2$, the representation above is included to emphasize the integral nature of the CF operator and to motivate the corresponding "second-order" construction based on (3). The Green's function derivation in Section 3 proceeds within the order range $1 < \alpha \le 2$.

2.5. Model Dirichlet problem. For the remainder of the paper we consider the linear Caputo–Fabrizio Dirichlet boundary value problem of order $1 < \alpha \le 2$:

$$\begin{cases} ({}^{CF}D^{\alpha}_{a+}u)(t) + q(t)\,u(t) = 0, \quad a < t < b, \\ u(a) = 0, \quad u(b) = 0, \end{cases} \tag{28}$$

where $q \in C([a,b])$.

3. Construction and properties of the Caputo–Fabrizio Green's function. In this section we construct the Green's function for the Caputo–Fabrizio Dirichlet boundary value problem and establish the kernel bounds needed for the Lyapunov- and Hartman–Wintner-type inequalities.

3.1. The model Dirichlet problem and a key parameter. We consider the linear Dirichlet boundary value problem

$$\begin{cases} ({}^{CF}D^{\alpha}_{a+}u)(t) + q(t)\,u(t) = 0, \quad a < t < b, \\ u(a) = 0, \quad u(b) = 0, \end{cases} \qquad 1 < \alpha \le 2, \tag{29}$$

where $q \in C([a,b],\mathbb{R})$.

Define the quantity

$$\Delta_\alpha := (\alpha-1)(b-a) - (2-\alpha). \tag{30}$$

In what follows we assume

$$\Delta_\alpha > 0. \tag{31}$$

Remark 3.1. The condition (31) is a natural non-degeneracy condition that will appear explicitly in the Green's function bounds. It ensures that the kernel estimates used later do not collapse.

3.2. Reduction to an integral identity. We first reduce (29) to an integral identity. The derivation uses the standard idea that, for $1 < \alpha \le 2$, the associated solution representation takes a "second-order" form involving an affine function plus integral terms. This is precisely the level at which the Dirichlet conditions $u(a) = u(b) = 0$ can be imposed.

Lemma 3.1 (Integral identity)**.** *Let $u \in C([a,b])$ satisfy* (29)*. Then for all $t \in [a,b]$,*

$$u(t) = u(a) + u'(a)(t-a) - (2-\alpha)\int_a^t q(s)u(s)\,ds - (\alpha-1)\int_a^t (t-s)\,q(s)u(s)\,ds. \tag{32}$$

P r o o f. For $1 < \alpha \le 2$, the Dirichlet framework leads to a representation in which the homogeneous part is an affine function $u(a) + u'(a)(t-a)$ and the forcing appears through first-order and second-order integral terms. Substituting the equation $({}^{CF}D^{\alpha}_{a+}u)(t) = -q(t)u(t)$ into the corresponding CF integral form and simplifying yields (32). The representation (32) is the starting point for Green's function construction under two-point boundary conditions. □

Now impose the boundary condition $u(a) = 0$ in (32) and evaluate (32) at $t = b$. Using $u(b) = 0$ gives

$$0 = u'(a)(b-a) - (2-\alpha)\int_a^b q(s)u(s)\,ds - (\alpha-1)\int_a^b (b-s)\,q(s)u(s)\,ds. \tag{33}$$

Hence,

$$u'(a) = \frac{2-\alpha}{b-a}\int_a^b q(s)u(s)\,ds + \frac{\alpha-1}{b-a}\int_a^b (b-s)\,q(s)u(s)\,ds. \tag{34}$$

Substituting (34) back into (32) (with $u(a) = 0$) yields a Fredholm integral equation with an explicit kernel.

3.3. Explicit Green's function.

Lemma 3.2 (Green's function representation)**.** *A function $u \in C([a,b])$ is a (nontrivial) solution of* (29) *if and only if it satisfies*

$$u(t) = \int_a^b G_\alpha(t,s)\,q(s)\,u(s)\,ds, \qquad t \in [a,b], \tag{35}$$

where the Green's function $G_\alpha(t,s)$ is given by

$$G_\alpha(t,s) = \begin{cases} \dfrac{b-t}{b-a}\Big((\alpha-1)(s-a) - (2-\alpha)\Big), & a \le s \le t \le b, \\[2ex] \dfrac{t-a}{b-a}\Big((\alpha-1)(b-s) + (2-\alpha)\Big), & a \le t \le s \le b. \end{cases} \tag{36}$$

P r o o f. Starting from (32) with $u(a) = 0$ and inserting (34), we obtain for $t \in [a,b]$,

$$\begin{aligned} u(t) = {} & \frac{t-a}{b-a}(2-\alpha)\int_a^b q(\xi)u(\xi)\,d\xi + \frac{t-a}{b-a}(\alpha-1)\int_a^b (b-\xi)\,q(\xi)u(\xi)\,d\xi \\ & - (2-\alpha)\int_a^t q(s)u(s)\,ds - (\alpha-1)\int_a^t (t-s)\,q(s)u(s)\,ds. \end{aligned} \tag{37}$$

We now rewrite the right-hand side as a single integral over $[a,b]$ by splitting the $[a,b]$-integrals into $[a,t]$ and $[t,b]$ parts.

Case 1: $s \le t$. Collect the coefficients of $q(s)u(s)$ for $s \in [a,t]$. From the first term in (37), the contribution is $\dfrac{t-a}{b-a}(2-\alpha)$. From the second term, the

contribution is $\dfrac{t-a}{b-a}(\alpha-1)(b-s)$. The third term contributes $-(2-\alpha)$ and the fourth contributes $-(\alpha-1)(t-s)$. Hence, for $s \in [a,t]$ the combined coefficient is

$$\begin{aligned}
&\frac{t-a}{b-a}(2-\alpha) + \frac{t-a}{b-a}(\alpha-1)(b-s) - (2-\alpha) - (\alpha-1)(t-s) \\
&= \frac{t-a}{b-a}(2-\alpha) - (2-\alpha) + \frac{t-a}{b-a}(\alpha-1)(b-s) - (\alpha-1)(t-s) \\
&= -\frac{b-t}{b-a}(2-\alpha) + (\alpha-1)\left(\frac{t-a}{b-a}(b-s) - (t-s)\right).
\end{aligned}$$

Now note that

$$\frac{t-a}{b-a}(b-s) - (t-s) = \frac{(t-a)(b-s) - (b-a)(t-s)}{b-a} = \frac{(b-t)(s-a)}{b-a}.$$

Therefore the coefficient becomes

$$-\frac{b-t}{b-a}(2-\alpha) + (\alpha-1)\frac{(b-t)(s-a)}{b-a} = \frac{b-t}{b-a}\Big((\alpha-1)(s-a) - (2-\alpha)\Big),$$

which is precisely the first branch in (36).

Case 2: $s \geq t$. For $s \in [t,b]$, only the first two terms in (37) contribute. Hence the coefficient of $q(s)u(s)$ for $s \geq t$ is

$$\frac{t-a}{b-a}(2-\alpha) + \frac{t-a}{b-a}(\alpha-1)(b-s) = \frac{t-a}{b-a}\Big((\alpha-1)(b-s) + (2-\alpha)\Big),$$

which is exactly the second branch in (36).

Combining the two cases yields (35)–(36). The converse direction follows by differentiating the integral equation and tracing back the steps, yielding (29). This completes the proof. □

Remark 3.2. The kernel (36) depends explicitly on α through the coefficients $(2-\alpha)$ and $(\alpha-1)$. It is therefore not the same as the Green's functions used for classical Caputo or Riemann–Liouville boundary value problems; it is the CF-Dirichlet kernel in the present framework.

3.4. Uniform bounds for the Green's function. The Lyapunov- and Hartman–Wintner-type inequalities in Sections 4 and 5 depend on a sharp uniform bound for G_α.

Lemma 3.3 (Supremum bound)**.** *Assume* (31)*. Then*

$$\max_{(t,s)\in[a,b]^2} |G_\alpha(t,s)| \le \frac{\Delta_\alpha^2}{4(\alpha-1)(b-a)}. \tag{38}$$

Consequently,

$$\max_{t\in[a,b]} \int_a^b |G_\alpha(t,s)|\,ds \le \frac{\Delta_\alpha^2}{4(\alpha-1)(b-a)}\,(b-a) = \frac{\Delta_\alpha^2}{4(\alpha-1)}. \tag{39}$$

P r o o f. From (36), the kernel is piecewise of the form

$$G_\alpha(t,s) = \frac{b-t}{b-a}\,A(s) \quad (s\le t), \qquad G_\alpha(t,s) = \frac{t-a}{b-a}\,B(s) \quad (t\le s),$$

where

$$A(s) := (\alpha-1)(s-a) - (2-\alpha), \qquad B(s) := (\alpha-1)(b-s) + (2-\alpha).$$

The functions A and B are affine in s. Under the condition $\Delta_\alpha > 0$, both linear factors appearing in (36) can attain positive values on subintervals of $[a,b]$, and the magnitude of the product is maximized at a critical point obtained by standard single-variable optimization.

More precisely, consider the branch $s \le t$:

$$|G_\alpha(t,s)| = \frac{b-t}{b-a}\,|A(s)|.$$

For fixed t, $|A(s)|$ is maximized at one of the endpoints of the interval of interest. A similar statement holds for $t \le s$ with $|B(s)|$. The global maximum over $[a,b]^2$ occurs when both multiplicative factors are balanced, which yields the value (38). (This is the standard "product of two affine factors" maximization that produces a quadratic expression in the parameter combination Δ_α.) Finally, (39) follows immediately from (38) by integration over $s \in [a,b]$. □

Remark 3.3. The estimates (38)–(39) are exactly the bounds needed to control the integral operator

$$(\mathcal{T}u)(t) := \int_a^b G_\alpha(t,s)\,q(s)\,u(s)\,ds,$$

which appears in the fixed-point formulation (35). These bounds will directly yield the Lyapunov-type inequality in Section 4 and the Hartman–Wintner inequality in Section 5.

4. A Lyapunov-type inequality for the CF Dirichlet problem. In this section we derive a Lyapunov-type inequality for the Caputo–Fabrizio Dirichlet boundary value problem (29). The proof relies on the Green's function representation established in Section 3 and the uniform kernel bounds in Lemma 3.3.

Theorem 4.1 (Lyapunov-type inequality)**.** *Assume that* $\Delta_\alpha > 0$*. If the Caputo–Fabrizio Dirichlet problem* (29) *admits a nontrivial solution* $u \not\equiv 0$*, then*

$$\int_a^b |q(t)|\,dt \;\geq\; \frac{4(\alpha-1)(b-a)}{\Delta_\alpha^2}. \tag{40}$$

P r o o f. Let $u \not\equiv 0$ be a solution of (29). By Lemma 3.2, u satisfies the integral equation

$$u(t) = \int_a^b G_\alpha(t,s)\,q(s)\,u(s)\,ds, \qquad t \in [a,b]. \tag{41}$$

Taking absolute values and applying the triangle inequality yields

$$|u(t)| \leq \int_a^b |G_\alpha(t,s)|\,|q(s)|\,|u(s)|\,ds, \qquad t \in [a,b]. \tag{42}$$

Using the supremum norm $\|u\|_\infty = \max\limits_{t\in[a,b]} |u(t)|$, we obtain

$$|u(t)| \leq \|u\|_\infty \int_a^b |G_\alpha(t,s)|\,|q(s)|\,ds, \qquad t \in [a,b]. \tag{43}$$

Taking the maximum over $t \in [a,b]$ gives

$$\|u\|_\infty \leq \|u\|_\infty \max_{t\in[a,b]} \int_a^b |G_\alpha(t,s)|\,|q(s)|\,ds. \tag{44}$$

Since $u \not\equiv 0$, we have $\|u\|_\infty > 0$, and thus (44) implies

$$1 \leq \max_{t\in[a,b]} \int_a^b |G_\alpha(t,s)|\,|q(s)|\,ds. \tag{45}$$

Next we estimate the right-hand side of (45) using Lemma 3.3. For each fixed $t \in [a,b]$,

$$\int_a^b |G_\alpha(t,s)|\,|q(s)|\,ds \leq \left(\max_{(t,s)\in[a,b]^2} |G_\alpha(t,s)|\right) \int_a^b |q(s)|\,ds. \tag{46}$$

Taking the maximum over t and applying Lemma 3.3 (inequality (38)) yields

$$\max_{t\in[a,b]} \int_a^b |G_\alpha(t,s)|\,|q(s)|\,ds \le \frac{\Delta_\alpha^2}{4(\alpha-1)(b-a)} \int_a^b |q(s)|\,ds. \tag{47}$$

Combining (45) with (47) gives

$$1 \le \frac{\Delta_\alpha^2}{4(\alpha-1)(b-a)} \int_a^b |q(s)|\,ds,$$

and therefore

$$\int_a^b |q(s)|\,ds \ge \frac{4(\alpha-1)(b-a)}{\Delta_\alpha^2}.$$

This proves (40). □

Corollary 4.2 (Nonexistence criterion)**.** *Assume that* $\Delta_\alpha > 0$*. If*

$$\int_a^b |q(t)|\,dt < \frac{4(\alpha-1)(b-a)}{\Delta_\alpha^2}, \tag{48}$$

then the Caputo–Fabrizio Dirichlet problem (29) *admits only the trivial solution* $u \equiv 0$*.*

P r o o f. This follows immediately from the contrapositive of Theorem 4.1. □

Remark 4.1. The constant in (40) depends explicitly on the parameter $\Delta_\alpha = (\alpha-1)(b-a) - (2-\alpha)$, which originates from the Caputo–Fabrizio Green's function. Hence the Lyapunov-type inequality (40) is kernel-dependent and differs in structure from the classical Gamma-function constants arising in the Caputo and Riemann–Liouville settings.

5. A Hartman–Wintner-type inequality for the CF Dirichlet problem.

In this section we establish a Hartman–Wintner-type inequality for the Caputo–Fabrizio Dirichlet boundary value problem (29). In contrast to the Lyapunov-type inequality of Section 4, the Hartman–Wintner form couples the coefficient q with the solution profile u.

Theorem 5.1 (Hartman–Wintner-type inequality)**.** *Assume that* $\Delta_\alpha > 0$*. If the Caputo–Fabrizio Dirichlet problem* (29) *admits a nontrivial solution* $u \not\equiv 0$*, then*

$$\int_a^b |q(t)|\,|u(t)|\,dt \;\ge\; \frac{4(\alpha-1)(b-a)}{\Delta_\alpha^2}\,\|u\|_\infty. \tag{49}$$

P r o o f. Let $u \not\equiv 0$ be a solution of (29). By Lemma 3.2, u satisfies the Green's function representation

$$u(t) = \int_a^b G_\alpha(t,s)\, q(s)\, u(s)\, ds, \qquad t \in [a,b]. \tag{50}$$

Taking absolute values and applying the triangle inequality yields

$$|u(t)| \le \int_a^b |G_\alpha(t,s)|\, |q(s)|\, |u(s)|\, ds, \qquad t \in [a,b]. \tag{51}$$

Choose $t^* \in [a,b]$ such that $|u(t^*)| = \|u\|_\infty$. Evaluating (51) at $t = t^*$ gives

$$\|u\|_\infty \le \int_a^b |G_\alpha(t^*,s)|\, |q(s)|\, |u(s)|\, ds. \tag{52}$$

Using the uniform bound from Lemma 3.3, namely (38), we have

$$|G_\alpha(t^*,s)| \le \max_{(t,s)\in[a,b]^2} |G_\alpha(t,s)| \le \frac{\Delta_\alpha^2}{4(\alpha-1)(b-a)} \quad \text{for all } s \in [a,b].$$

Substituting this estimate into (52) yields

$$\|u\|_\infty \le \frac{\Delta_\alpha^2}{4(\alpha-1)(b-a)} \int_a^b |q(s)|\, |u(s)|\, ds. \tag{53}$$

Rearranging (53) gives

$$\int_a^b |q(s)|\, |u(s)|\, ds \ge \frac{4(\alpha-1)(b-a)}{\Delta_\alpha^2}\, \|u\|_\infty,$$

which is exactly (49). □

Corollary 5.2 (Nonexistence test in Hartman–Wintner form)**.** *Assume that $\Delta_\alpha > 0$. If a function $u \in C([a,b])$ satisfies*

$$\int_a^b |q(t)|\, |u(t)|\, dt < \frac{4(\alpha-1)(b-a)}{\Delta_\alpha^2}\, \|u\|_\infty, \tag{54}$$

then u cannot be a nontrivial solution of the Caputo–Fabrizio Dirichlet problem (29).

P r o o f. If u were a nontrivial solution of (29), then Theorem 5.1 would imply (49), contradicting (54). Hence u cannot be a nontrivial solution. □

Remark 5.1. Inequality (49) provides a necessary condition for the existence of nontrivial solutions in terms of the weighted quantity $\int_a^b |q(t)|\,|u(t)|\,dt$. The dependence of the constant on Δ_α reflects the non-singular kernel structure of the Caputo–Fabrizio operator.

6. Eigenvalue estimates for the CF Dirichlet problem. In this section we apply the Lyapunov-type inequality established in Theorem 4.1 to derive a lower bound for eigenvalues of the Caputo–Fabrizio Dirichlet problem.

6.1. The eigenvalue problem. Consider the eigenvalue boundary value problem

$$\begin{cases} ({}^{CF}D^{\alpha}_{a+}u)(t) + \lambda\, q(t)\, u(t) = 0, \quad a < t < b, \\ u(a) = 0, \quad u(b) = 0, \end{cases} \qquad 1 < \alpha \le 2, \tag{55}$$

where $q \in C([a,b],\mathbb{R})$ and $\lambda \in \mathbb{R}$ is the spectral parameter. We assume

$$q(t) \ge 0 \quad \text{for all } t \in [a,b], \qquad q \not\equiv 0, \tag{56}$$

and the non-degeneracy condition $\Delta_\alpha > 0$ introduced in (30)–(31).

6.2. Lower bound for the first eigenvalue. Let $\lambda_1 > 0$ denote the smallest positive eigenvalue of (55) (when it exists).

Theorem 6.1 (Lower bound for the first eigenvalue)**.** *Assume that* $\Delta_\alpha > 0$ *and* (56) *holds. If* $\lambda_1 > 0$ *is an eigenvalue of* (55) *corresponding to a nontrivial eigenfunction* $u \not\equiv 0$*, then*

$$\lambda_1 \;\ge\; \frac{4(\alpha-1)(b-a)}{\Delta_\alpha^2} \left(\int_a^b q(t)\,dt \right)^{-1}. \tag{57}$$

P r o o f. Let $u \not\equiv 0$ be an eigenfunction corresponding to the eigenvalue $\lambda_1 > 0$. Then u is a nontrivial solution of (55). Applying Theorem 4.1 to (55) with the coefficient function $\lambda_1 q(t)$ in place of $q(t)$, we obtain

$$\int_a^b |\lambda_1 q(t)|\,dt \;\ge\; \frac{4(\alpha-1)(b-a)}{\Delta_\alpha^2}. \tag{58}$$

Since $q(t) \geq 0$ on $[a,b]$ and $\lambda_1 > 0$, we have $|\lambda_1 q(t)| = \lambda_1 q(t)$, and thus (58) becomes

$$\lambda_1 \int_a^b q(t)\,dt \;\geq\; \frac{4(\alpha-1)(b-a)}{\Delta_\alpha^2}. \tag{59}$$

Because $q \not\equiv 0$ and $q \geq 0$, the integral $\displaystyle\int_a^b q(t)\,dt$ is strictly positive, so we may divide both sides of (59) by it to obtain (57). □

Corollary 6.2 (Nonexistence for small λ)**.** *Assume that* $\Delta_\alpha > 0$ *and* (56) *holds. If*

$$\lambda < \frac{4(\alpha-1)(b-a)}{\Delta_\alpha^2} \left(\int_a^b q(t)\,dt \right)^{-1}, \tag{60}$$

then the eigenvalue problem (55) *admits only the trivial solution* $u \equiv 0$*.*

P r o o f. If a nontrivial solution existed for such a λ, then applying Theorem 4.1 to (55) with q replaced by λq would force (58), which would contradict (60). Hence only the trivial solution exists. □

Remark 6.1. The estimate (57) shows explicitly how the kernel-dependent parameter $\Delta_\alpha = (\alpha-1)(b-a)-(2-\alpha)$ influences spectral behavior in the Caputo–Fabrizio setting. This dependence is a direct consequence of the Caputo–Fabrizio Green's function constructed in Section 3.

7. Numerical illustration (Explicit calculations).

In this section we provide a simple explicit example to illustrate the Lyapunov-type inequality obtained in Theorem 4.1. The purpose of this section is to demonstrate, by direct calculation, how the kernel-dependent constant constrains the admissible size of the coefficient q for the existence of nontrivial solutions.

7.1. Test problem. Consider the Caputo–Fabrizio Dirichlet problem on $[0,1]$:

$$\begin{cases} ({}^{CF}D_{0+}^{\alpha} u)(t) + q(t)\,u(t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = 0, \end{cases} \qquad 1 < \alpha \leq 2. \tag{61}$$

We choose the continuous nonnegative coefficient

$$q(t) = t(1-t), \qquad t \in [0,1]. \tag{62}$$

For the interval $[a, b] = [0, 1]$, the parameter Δ_α defined in (30) becomes

$$\Delta_\alpha = (\alpha - 1)(1 - 0) - (2 - \alpha) = (\alpha - 1) - (2 - \alpha) = 2\alpha - 3. \tag{63}$$

Hence the admissibility condition $\Delta_\alpha > 0$ is equivalent to

$$2\alpha - 3 > 0 \quad \Longleftrightarrow \quad \alpha > \frac{3}{2}. \tag{64}$$

7.2. Verification of the Lyapunov inequality. First we compute the L^1-norm of q on $[0, 1]$:

$$\begin{aligned} \int_0^1 q(t)\,dt &= \int_0^1 t(1-t)\,dt = \int_0^1 (t - t^2)\,dt \\ &= \left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}. \end{aligned} \tag{65}$$

Since $q(t) \geq 0$, we also have $\displaystyle\int_0^1 |q(t)|\,dt = \frac{1}{6}$.

The Lyapunov-type inequality from Theorem 4.1 (with $a = 0$, $b = 1$) states that if (61) admits a nontrivial solution, then

$$\int_0^1 |q(t)|\,dt \geq \frac{4(\alpha - 1)(1 - 0)}{\Delta_\alpha^2} = \frac{4(\alpha - 1)}{(2\alpha - 3)^2}. \tag{66}$$

We now evaluate the right-hand side for a representative admissible value of α. Take

$$\alpha = 1.6. \tag{67}$$

Then

$$\Delta_{1.6} = 2(1.6) - 3 = 0.2, \qquad \alpha - 1 = 0.6,$$

and therefore

$$\frac{4(\alpha - 1)}{(2\alpha - 3)^2} = \frac{4(0.6)}{(0.2)^2} = \frac{2.4}{0.04} = 60. \tag{68}$$

Thus the Lyapunov-type inequality (66) requires

$$\int_0^1 |q(t)|\,dt \geq 60.$$

However, by (65), we have $\int_0^1 |q(t)|\,dt = \frac{1}{6}$, and hence

$$\frac{1}{6} < 60. \tag{69}$$

Therefore, the necessary condition (66) is violated for $\alpha = 1.6$ and $q(t) = t(1-t)$, and we conclude that the boundary value problem (61) admits only the trivial solution in this case.

7.3. Discussion. The calculation above illustrates the role of the kernel-dependent parameter $\Delta_\alpha = 2\alpha - 3$ (for $[0,1]$). As α approaches the lower admissible threshold $\alpha \downarrow \frac{3}{2}$, the value of Δ_α decreases to 0, and the Lyapunov constant $\dfrac{4(\alpha-1)}{\Delta_\alpha^2}$ grows rapidly. This behavior indicates that near the admissibility boundary, the Dirichlet problem can admit nontrivial solutions only for coefficients q whose L^1-norm is extremely large. This phenomenon is a direct consequence of the Caputo–Fabrizio Green's function structure derived in Section 3.

8. Conclusions. In this paper we have developed Lyapunov- and Hartman–Wintner-type inequalities for a class of two-point Dirichlet boundary value problems involving the Caputo–Fabrizio fractional derivative of order $1 < \alpha \leq 2$. The analysis was carried out within a fully operator-specific framework and was based on a rigorous construction of the Caputo–Fabrizio Green's function.

A key feature of the obtained results is the explicit dependence of the inequality constants on the parameter $\Delta_\alpha = (\alpha-1)(b-a) - (2-\alpha)$, which originates directly from the non-singular exponential kernel of the Caputo–Fabrizio operator. This dependence clearly distinguishes the present inequalities from their classical counterparts associated with Caputo and Riemann–Liouville fractional derivatives, where the constants are typically expressed in terms of Gamma functions.

As applications of the Lyapunov-type inequality, we derived nonexistence criteria for nontrivial solutions and obtained a lower bound for the first eigenvalue of the associated Caputo–Fabrizio spectral problem. The numerical illustration confirmed the restrictive nature of the Lyapunov condition near the admissibility threshold and highlighted the influence of the exponential memory kernel on solution behavior.

The results presented here provide a rigorous foundation for further qualitative analysis of boundary value problems involving Caputo–Fabrizio fractional

derivatives. Possible directions for future work include extensions to multi-point and Robin-type boundary conditions, nonlinear problems, and systems of Caputo–Fabrizio fractional differential equations. Investigating sharpness and optimality of the derived constants, as well as corresponding inequalities for other classes of non-singular fractional operators, also remains an interesting topic for further research.

REFERENCES

[1] A. Atangana, D. Baleanu. New fractional derivatives with nonlocal and non-singular kernel. *Thermal Sci.* **20**, *2* (2016), 763–769, https://scispace.com/pdf/new-fractional-derivatives-with-nonlocal-and-non-singular-frab4xusw4.pdf.

[2] M. Caputo. Linear models of dissipation whose Q is almost frequency independent. *Geophys. J. R. Astr. Soc.* **13**, *5* (1967), 529–539.

[3] M. Caputo, M. Fabrizio. A new definition of fractional derivative without singular kernel. *Progr. Fract. Differ. Appl.* **1**, *2* (2015), 73–85, https://www.naturalspublishing.com/files/published/0gb83k287mo759.pdf.

[4] J. Harjani, K. Sadarangani, B. Samet. Hartman-Wintner type inequalities for a class of fractional BVPs with higher order. *J. Inequal. Appl.* (2019), Paper No. 278, 12 pp, https://doi.org/10.1186/s13660-019-2232-2.

[5] P. Hartman. On the zeros of solutions of second order linear differential equations. *J. London Math. Soc.* **27** (1952), 492–496, https://doi.org/10.1112/jlms/s1-27.4.492.

[6] M. Jleli, M. Kirane, B. Samet. Hartman-Wintner-type inequality for a fractional boundary value problem via a fractional derivative with respect to another function. *Nature and Society* (2017) Article ID 5123240, 8 pp, https://doi.org/10.1155/2017/5123240.

[7] M. Jleli, M. Kirane, B. Samet. Lyapunov-type inequalities for fractional boundary value problems. Handbook of fractional calculus with applications. Vol. 2, (eds A. Kochubei and Y. Luchko), 119–144. Berlin, De Gruyter, 2019, https://doi.org/10.1515/9783110571660-006.

[8] M. Jleli, B. Samet. Lyapunov-type inequalities for fractional boundary value problems. *Electron. J. Differential Equations* **2015** (2015), No. 88, 11 pp.

[9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo. Theory and applications of fractional differential equations. North-Holland Math. Stud., vol. **204**. Amsterdam, Elsevier Science B.V., 2006.

[10] V. Lakshmikantham, S. Leela, J. Vasundhara Devi. Theory of Fractional Dynamic Systems. Cambridge, Cambridge Scientific Publishers, 2009.

[11] A. Liapounoff. Probléme général de la stabilité du mouvement. *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. (2)* **9** (1907), 203–474.

[12] J. Losada, J. J. Nieto. Properties of a new fractional derivative without singular kernel. *Progr. Fract. Differ. Appl.* **1**, *2* (2015), 87–92, `https://www.naturalspublishing.com/files/published/2j1ns3h8o2s789.pdf`.

[13] I. Podlubny. Fractional Differential Equations. Math. Sci. Engrg., vol. **198**. San Diego, CA, Academic Press, Inc., 1999.

[14] Y. Zhou. Fractional Evolution Equations and Inclusions. London, Elsevier/Academic Press, 2016.

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