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AN UPPER BOUND FOR A CONDITION NUMBER THEOREM OF VARIATIONAL INEQUALITIES

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To the memory of Asen Dontchev

Abstract. Nonlinear variational inequalities in Banach spaces are considered. A notion of (absolute) condition number with respect to the right-hand side is introduced. A distance among variational inequalities is defined. We prove that the distance to suitably restricted ill-conditioned variational inequalities is bounded from above by a multiple of the reciprocal of the condition number. By using an analogous lower bound of the companion paper [14], we obtain a full condition number theorem for variational inequalities. The particular case of convex optimization problems is also considered. Known results dealing with optimization problems are thereby generalized.

1. Introduction. The condition number theorem of linear algebra states that, for a given non-singular matrix, the distance to the ill-conditioned (i.e. singular) matrices equals the reciprocal of its condition number (see [3, Theorem 1.7]). This is one of the simplest examples of a general property. To quote [2,

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p. 232]: "A very general theme in numerical analysis is a relationship between the condition number of a problem and the reciprocal of the distance to the set of ill-conditioned problems". For scalar optimization problems, such a relationship has been proved within several contexts in [8, 9, 10, 11, 12, 13]. For any optimization problem Q of a given class, after defining in a suitable way its (absolute) condition number cond Q and its distance d from the ill-conditioned problems, the above relationship is expressed by a pair of inequalities of the form

$$(*) \qquad \frac{c_1}{\text{cond } Q} \le d \le \frac{c_2}{\text{cond } Q}$$

for suitable positive constants c_1, c_2. Such a couple of inequalities will be called a condition number theorem for the given class.

Here we consider variational inequalities on Banach spaces, defined by nonlinear operators acting on a fixed closed set, with right-hand side from a given region. Aim of this paper is to find a proper setting in order to obtain a partial version of the condition number theorem, namely an upper bound as in $(*)$. Coupling it with the main result of the companion paper [14], we obtain a full condition number theorem for suitably restricted variational inequalities, in analogy with the above quoted known results about optimization problems.

To this aim, we need a definition of (absolute) condition number, and a notion of distance within variational inequalities. First we define here the (possibly infinite-valued) distance between two variational inequalities as the uniform distance between the corresponding operators, plus the Lipschitz norm of the difference between them. Next we consider the condition number of a variational inequality with respect to its right-hand side as the upper limit, when the perturbations vanish, of the ratio between the output and the input errors in the solutions due to small perturbations of the right-hand side. This definition follows a standard procedure in numerical analysis. Then we identify a suitable set of variational inequalities, within which an upper estimate as in $(*)$ holds true.

Such a new result on a condition number theorem is of interest in the analysis of computational complexity (see [2] and [3]) and of the stability and sensitivity for variational inequalities, moreover in the evaluation of the performance of numerical methods.

The paper is organized as follows. In Section 2 we collect notations and basic assumptions. Section 3 describes the class of variational inequalities we consider, the smoothness assumptions we require on the Banach space, and deals with an upper bound on the distance to ill-conditioning and on the condition number. An upper bound of the distance in terms of the reciprocal of the condition number is thereby obtained. In Section 4 we treat the particular case of convex optimization. In Section 5 we state a full condition number theorem by coupling the main result of this paper with the one obtained in the companion paper [14]. Section 6 collects the proofs. Moreover we present there a new proof of a known necessary condition for the Aubin property of a set-valued mapping in terms of the Lipschitz character of its distance function.

2. Notations, basic definitions and assumptions. We consider a real Banach space X with topological dual X^*, a closed set $K \subset X$, and a set $\Omega \subset X^*$, both containing 0 as a cluster point. We consider the set of all single-valued functions

$$A : K \to X^*. \tag{1}$$

For every A as in (1) and $p \in \Omega$ we consider the variational inequality, to find $u \in K$ such that

$$\langle A(u), x - u \rangle \geq \langle p, x - u \rangle \quad \text{for every } x \in K \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X^* and X. Given A, we denote by $Q = (A, K, \Omega)$ the set of all inequalities (2) as $p \in \Omega$, and by $B = B(K, \Omega)$ the set of all such variational inequalities Q. For each $p \in \Omega$ we denote by $S(p)$ the (possibly empty) set of all solutions of (2). The set-valued mapping

$$S : \Omega \to K$$

will be called the *solution mapping* of Q. We first need a notion of distance among variational inequalities. Given A_1, A_2 as in (1) with corresponding variational inequalities $Q_1, Q_2 \in B$, we consider

$$\hat{A} = A_1 - A_2$$

and define the distance between Q_1 and Q_2 as

$$\begin{aligned} \text{dist}(Q_1, Q_2) \\ = \sup\{\|\hat{A}(x)\| : x \in K\} + \sup\left\{ \frac{\|\hat{A}(x'') - \hat{A}(x')\|}{\|x'' - x'\|} : x', x'' \in K \right\}. \end{aligned} \tag{3}$$

In this way we obtain an extended real-valued distance on B.

Next we need a notion of (absolute) condition number of any variational inequality $Q \in B$ with solution mapping S. We define the condition number of Q as

$$\operatorname{cond} Q = \limsup_{p', p'' \to 0} \frac{\operatorname{hausd}[S(p''), S(p')]}{\|p'' - p'\|}, \tag{4}$$

where of course $p', p'' \in \Omega$, and hausd denotes the Pompeiu-Hausdorff distance (see [6, p. 48] for the definition). The variational inequality Q will be called *well-conditioned* iff cond $Q < +\infty$, *ill-conditioned* iff cond $Q = +\infty$. The set of well-conditioned variational inequalities will be denoted by WC, and that of ill-conditioned by IC. Thus $Q \in WC$ iff Q has solutions for each sufficiently small $p \in \Omega$, and its solution mapping is Lipschitz near $p = 0$, according to the standard notion of Lipschitz continuity of set-valued mappings, see [6, Definition 1.40 (iii)]. The condition number of Q is then the Lipschitz modulus at $p = 0$ of its solution mapping. We have $Q \in IC$ if its solution mapping $S(p_n) = \emptyset$ for some sequence $p_n \in \Omega$ with $p_n \to 0$, or if $S(p) \neq \emptyset$ for each sufficiently small $p \in \Omega$ and S is not Lipschitz near $p = 0$.

Example 1. Suppose that X is Hilbert, $A : X \to X^*$ is Lipschitz on the convex set K, and strongly monotone with constant α, namely

$$\langle A(x) - A(y), x - y \rangle \geq \alpha \|x - y\|^2$$

for every $x, y \in K$. Then $S(p)$ is a singleton for each $p \in \Omega = X^*$ (see [1, Teorema 3.2]), and (as well-known)

$$\|S(p'') - S(p')\| \leq (1/\alpha)\|p'' - p'\|,$$

hence $Q \in WC$ with cond $Q \leq 1/\alpha$.

Example 2. Let $X = R$, $\Omega = K = [0, 1]$, $A(x) = x^2$. Then, as easily checked, $S(p) = \{\sqrt{p}\}$ for every p, hence $Q \in IC$.

3. Upper bound. In this section we obtain first an upper estimate of the distance from ill-conditioning in terms of the reciprocal of the condition number. In the next result we need the following condition:

$$0 \in \operatorname{int} \Omega, \ 0 \in \operatorname{int} K \tag{5}$$

where int denotes interior. Moreover we shall employ smoothness assumptions about the space X, as follows:

(6) X is reflexive with strictly convex norm, which is Gateaux differentiable off the origin with derivative $j(u) \in X^*$ at $u \neq 0$;

there exists a constant $\alpha \geq 1$ such that the single-valued function $D : K \to X^*$ given by

(7) $$D(0) = 0, D(u) = \|u\|^\alpha j(u) \text{ if } u \neq 0$$

is Lipschitz on every bounded set.

Conditions (6) and (7) are fulfilled if X is Hilbert, since then $j(u) = u/\|u\|$. In order to match with the lower bound from [14], and to prove a sharper upper bound of the form discussed above, we need to suitably restrict (in accordance with [14]) the class of variational inequalities we consider. Having fixed K and Ω, a given class D of variational inequalities $Q = (A, K, \Omega)$ with solution mapping S fulfills *property* Z if the following conditions are satisfied for each $Q \in D$. First, $S(p)$ is nonempty and closed for every sufficiently small $p \in \Omega$. Moreover, writing

$$\bar{K} = \{u \in K : ||u|| \leq \delta^*\}$$

we require that there exists a sufficiently small $\delta^* > 0$ such that

(8) $$A(\bar{K}) \subset \Omega;$$

furthermore

(9) $$S^{-1} \text{ is a single-valued function on } \bar{K}.$$

Finally, for every pair $Q_i = (A_i, K, \Omega) \in D$ with solution mapping S_i, $i = 1, 2$ we have $A_1(0) = A_2(0)$ and there exists a ball P of center 0 and positive radius such that

(10) $$S_1(P) \cup S_2(P) \subset \bar{K}.$$

Example 3. Let X be Hilbert, $K = X$, let Ω be a ball centered at 0 with positive radius, and let D be a set of variational inequalities corresponding to the unconstrained minimization of the convex quadratic forms

$$x \to \frac{1}{2}\langle Ax, x\rangle - \langle p, x\rangle, \; p \in \Omega,$$

where $A : X \to X^*$ is a linear, bounded and self-adjoint operator such that, for a fixed $\alpha > 0$,

$$\langle A(x) - A(y), x - y\rangle \;\geq\; \alpha\|x - y\|^2 \text{ for every } x, y \in X.$$

Then D fulfills property Z if the set of all linear operators A corresponding to the elements of D is bounded, see [14, Example 3.5] for details.

Simple examples show that there are variational inequalities Q both in WC and in IC such that $\{Q\}$ fulfills property Z. Let $X = R$, $\Omega = K = [-1, 1]$. If $A_1(x) = x$, then $Q_1 = (A_1, K, \Omega) \in WC$ and $\{Q_1\}$ fulfills property Z. If $A_2(x) = x^3$, then $Q_2 = (A_2, K, \Omega) \in IC$ and $\{Q_2\}$ fulfills property Z.

Given $Q = (A, K, \Omega) \in B$ with a bounded set K we shall also consider

$$L = \sup\{\|(\alpha + 2)\|x\|D(x) - A(x)\| : x \in K\}; \tag{11}$$

moreover

$$K_0 = \sup\{\|x\| : x \in K\};$$

$$A_1 = \sup\left\{\frac{\|A(x'') - A(x')\|}{\|x'' - x'\|} : x', x'' \in K\right\};$$

$$D_1 = \sup\left\{\frac{\|D(x'') - D(x')\|}{\|x'' - x'\|} : x', x'' \in K\right\};$$

and

$$M = A_1 + (\alpha + 2)(K_0^\alpha + K_0 D_1), \tag{12}$$

where α and D are given by (7); of course $0 \leq A_1$, $L \leq +\infty$. In order to obtain an upper bound, we start with an upper estimate of the distance to ill-conditioning, as follows. Given K and Ω let $Z^* \subset B$ denote the set of all variational inequalities which satisfy property Z.

Theorem 4. *Let K be bounded and convex. Assume* (5), (6) *and* (7). *Then for every $Q = (A, K, \Omega) \in B$ we have*

$$\text{dist}(Q, IC) \leq \text{dist}(Q, IC \cap Z^*) \leq L + M, \tag{13}$$

where L is given by (11) *and M by* (12).

The last step is to obtain an upper bound of the condition number. Let $Q \in WC$ be with solution mapping S. We need the following assumption. There exists a positive constant C^* such that for every $p \in \Omega$ sufficiently small, every $u \in K$, every $w \in S(p)$ such that

$$\|u - w\| \leq \text{dist}[u, S(p)] + 1$$

and every $h \in X^*$ with $\|h\| \leq 1$, there exists a (single-valued) selection z of $t \to S(p + th)$, $t > 0$ and sufficiently small, such that

$$z(0) = w, \ \|z(t) - w\| \leq C^* t \ \text{ for every } t. \tag{14}$$

We assume, without loss of generality, that if $0 < L,\ M < +\infty, L$ given by (11) and M by (12), then $C^* > 1/(L + M)$.

Remark 5. A particular case of assumption (14) about the existence of C^* is related to Assumption 2.2 (ii) in [4], see also (i) of Remark 2.3 there, where it is shown that such condition has to do with differentiability properties of the solution mapping.

Example 6. Suppose that X is Hilbert and $A : X \to X^*$ is strongly monotone on K with constant α, namely

$$\langle A(x) - A(y), x - y\rangle \geq \alpha\|x - y\|^2 \ \text{ for every } x, y \in K.$$

If K is convex and A is Lipschitz on K, then the corresponding variational inequality (2) has exactly one solution $S(p)$ for each $p \in X^*$ (see [1, Teorema 3.2]). Given $p \in \Omega = X^*, h \in X^*$ with $\|h\| \leq 1$ and $w = S(p)$, we consider

$$z(t) = S(p + th), t > 0$$

and we get, for every $x \in K$,

$$\langle A[z(t)], x - z(t)\rangle \geq \langle p + th, x - z(t)\rangle, \ \langle A(w), x - w\rangle \geq \langle p, x - w\rangle$$

hence

$$\langle A[z(t)] - A(w), w - z(t)\rangle \geq t\langle h, w - z(t)\rangle$$

whence

$$\alpha\|z(t) - w\|^2 \leq t\|z(t) - w\|$$

so that (14) is fulfilled with $C^* = 1/\alpha$.

Example 7. Let

$$X = R^2,\ K = \left\{(x_1, x_2) \in R^2 : |x_1| \le 1, |x_2| \le 1\right\},\ \Omega = \{(p_1, 0) : |p_1| \le 1\},$$

and let $A : R^2 \to R^2$ be the linear operator corresponding to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, given $p_1 \in [-1, 1]$, (2) is equivalent to find $u_1 \in [-1, 1]$ such that $(u_1 - p_1)(x_1 - u_1) \ge 0$ for every $x_1 \in [-1, 1]$. Thus

$$S(p_1, 0) = \{(p_1, x_2) : |x_2| \le 1\}$$

for every $p \in \Omega$. Therefore, given $p \in \Omega, u \in K, w = (w_1, w_2)$ with $w_1 = p_1$ and $h = (h_1, h_2)$, we see that $h_2 = 0$ if $p + th \in \Omega$, and

$$z(t) = (p_1 + th_1, w_2)$$

fulfills (14) with $C^* = 1$.

The required estimate about the condition number is given in the next result.

Theorem 8. *Let $Q \in WC$. Assume* (6), (14) *and let $0 \in$ int Ω. Then*

$$\operatorname{cond} Q \le C^* \tag{15}$$

Remark 9. Strict convexity of the norm is not needed in Theorem 8, as we shall see in its proof.

Summarizing, the upper bound we obtain is the following. Let $Q = (A, K, \Omega)$ be with M given by (12) and L by (11).

Corollary 10. *Suppose that K is bounded and convex, let $Q \in WC$ and assume* (5), (6), (7) *and* (14). *Then, if cond $Q > 0$, we have*

$$\operatorname{dist}(Q, IC \cap Z^*) \le \frac{(L + M)C^*}{\operatorname{cond} Q}.$$

Corollary 10 follows immediately from Theorems 4 and 8.

4. Convex optimization. In this section we assume that K is bounded and convex in the reflexive Banach space X. Denote by T the set of all real-valued functions f defined on some open set containing K, which are lower semicontinuous and convex on K, and Gateaux differentiable. For each $f \in T$ we consider the optimization problems Q, to minimize $f(x) - \langle p, x \rangle$ on K for each $p \in \Omega$, with solution mapping S. So we consider tilt perturbations of f constrained by K. As well-known, $u \in S(p)$ iff u solves (2) with $A = \nabla f$ the Gateaux gradient of f. Therefore the set of these optimization problems can be considered as corresponding to a subclass B_m of B. We keep the previous definitions (3) and (4) of distance and condition number within B_m. Let us write

$$WC_m = B_m \cap WC, IC_m = B_m \cap IC, Z_m^* = B_m \cap Z^*.$$

Then we get the following

Theorem 11. *Let* $Q = (A, K, \Omega) \in WC_m \cap Z^*$*, where* $A = \nabla f$ *for some* $f \in T$*. Assume* (6)*,* (7) *and* (14)*. Let* $0 \in \operatorname{int} K$ *and let* Ω *be a ball centered at* 0*. Then, if cond* $Q > 0$*, we have*

$$\frac{1}{\operatorname{cond} Q} \le \operatorname{dist}(Q, IC_m \cap Z_m^*) \le \frac{(L+M)C^*}{\operatorname{cond} Q},$$

M *given by* (12) *and* L *by* (11)*.*

5. A condition number theorem for variational inequalities. We recall here the main result of the companion paper [14] about a lower bound of the distance to ill-conditioning in terms of the reciprocal of the condition number. By coupling such a result with those of the present paper we end up with a full condition number theorem for variational inequalities of the form discussed in the introduction.

According to Corollary 3.9 of [**?**], we have the following lower bound.

Theorem 12. *Assume that* Ω *is a ball of center* 0 *with positive radius. Let* $Q \in WC \cap Z^*$ *be such that cond* $Q > 0$*. Then*

$$\operatorname{dist}(Q, IC \cap Z^*) \ge \frac{1}{\operatorname{cond} Q}.$$

Corollary 10 and Theorem 12 we obtain immediately the following condition number theorem.

Theorem 13. *Let $Q \in WC \cap Z^*$, let K be bounded and convex, and let Ω be a ball with center* 0 *and positive radius. Assume* (5), (6), (7) *and* (14). *Then if cond $Q > 0$ we have*

$$\frac{1}{\operatorname{cond} Q} \le \operatorname{dist}(Q, IC \cap Z^*) \le \frac{(L+M)C^*}{\operatorname{cond} Q}.$$

6. Proofs and auxiliary results. The proof of Theorem 4 requires the following example.

Example 14. Assume (5), (6) and (7). Let K be convex. Given $p \in \Omega$ we minimize on K

$$\bar{f}(x) = \|x\|^{\alpha+2} - \langle p, x \rangle$$

with α given by (7). This function is strictly convex and, by standard reasoning, has a unique minimizer $u \in K$. Moreover the Gateaux gradient of $\bar{f}$ is $\nabla \bar{f}(u) = (\alpha+2)\, u\|D(u) - p$, where D is given by (7), hence for every p the corresponding minimizer u is the solution of the following variational inequality $\bar{Q}$, namely

$$(\alpha+2)\langle \|u\| D(u), x-u \rangle \ge \langle p, x-u \rangle \text{ for each } x \in K. \tag{16}$$

We want to show that $\bar{Q} \in IC$. Denote by S the solution mapping of $\bar{Q}$. Arguing by contradiction, suppose that $\bar{Q} \in WC$. Then there exists a constant $c_1 > 0$ such that if $u = S(p)$ we have

$$\|u\| \le c_1 \|p\| \tag{17}$$

for each p sufficiently small (remembering that $S(0) = 0$). Then by (16), (7) and (17)

$$\langle p, x-u \rangle \le (\alpha+2)\, \|u\|\, \|D(u)\|\, \|x-u\| \ \le c_2\, \|x-u\|\, \|p\|^{\alpha+1}$$

for some constant c_2, hence if $x \ne u$, $p \ne 0$,

$$\left\langle \frac{p}{\|p\|}, \frac{x-u}{\|x-u\|} \right\rangle \le c_2\, \|p\|^{\alpha}. \tag{18}$$

As well-known, given $x \in K$ with $x \ne 0$ there exists $p \in X^*$ with $\|p\| = 1$ such that $\langle p, x \rangle = \|x\|$. Then consider $p_n = p/n$, so that $p_n \in \Omega$ by (5) for every sufficiently large integer n. Let $u_n = S(p_n)$, then by (18)

$$\left\langle p, \frac{x-u_n}{\|x-u_n\|} \right\rangle \le \frac{c_2}{n^{\alpha}}.$$

Letting $n \to +\infty$ we get $u_n \to 0$ by (17), hence the contradiction

$$\left\langle p, \frac{x}{\|x\|} \right\rangle = 1 \leq 0$$

whence $\bar{Q} \in IC$. Now we prove that $\bar{Q} = (\bar{A}, K, \Omega)$ fulfills the conditions required for property Z. By (5) there exists $r > 0$ such that $p \in X^*, \|p\| \leq r$ imply $p \in \Omega$. Then, if $\delta^* > 0$ and $\|u\| \leq \delta^*$, we have

$$\|\bar{A}(u)\| \leq (\alpha + 2)(\delta^*)^{\alpha+1} \leq r$$

if δ^* is sufficiently small, hence (8) is fulfilled. If $u \in \bar{K}$ and $p, q \in \bar{S}^{-1}(u)$, then $u = S(p) = S(q)$ and if δ^* is sufficiently small, then by (5) u is an interior minimizer corresponding to both p and q, hence $\bar{A}(u) = p = q$, thus (9) holds. Finally, if $u \in S(p)$ then $\bar{f}(u) \leq \bar{f}(0)$, whence $\|u\|^{\alpha+1} \leq \|p\|$, so (10) holds provided the radius of P is $\leq (\delta^*)^{\alpha+1}$. Summarizing,

$$\bar{Q} \in IC \cap Z^*. \tag{19}$$

Remark 15. In Example 14, the assumption $0 \in \operatorname{int} K$ of (5) cannot be omitted to get $\bar{Q} \in Z^*$. Consider X =R, $K = [0, 1]$, $\Omega = [-1, 1]$, then if $p < 0$ we have $0 \in S(p)$, hence (9) fails. Of course, different choices of the variational inequality $\bar{Q} \in IC \cap Z^*$ may lead to different conditions on problem's data, and to different estimates of the distance to ill-conditioning.

P r o o f o f T h e o r e m 4. We consider $\bar{Q}$ of Example 14, so that by (19) we need to prove that

$$\operatorname{dist}(Q, \bar{Q}) \leq L + M. \tag{20}$$

Consider $\bar{A}(x) = (\alpha + 2)\|x\|D(x), x \in X$, then by (3) and (16)

$$\operatorname{dist}(Q, \bar{Q}) = \sup\{\|A(x) - \bar{A}(x)\| : x \in K\} \\ + \sup\left\{ \frac{\|(A - \bar{A})(x'') - (A - \bar{A})(x')\|}{\|x'' - x'\|} : x', x'' \in K \right\}$$

where, by (11), the first term is bounded by L. Now we compute

$$\|(A - \bar{A})(x'') - (A - \bar{A})(x')\| \\ \leq \|A(x'') - A(x')\| + (\alpha + 2)\| \, \|x''\|D(x'') - \|x'\|D(x')\| \\ \leq \|A(x'') - A(x')\| \\ + (\alpha + 2)\big[| \, \|x''\| - \|x'\| \, | \, \|D(x''\| + \|x'\| \, \|D(x'') - D(x')\|\big].$$

Dividing by $\|x'' - x'\|$ and remembering (7), we get

$$\begin{aligned}
&\sup\left\{\frac{\|(A-\bar{A})(x'')-(A-\bar{A})(x')\|}{\|x''-x'\|} : x', x'' \in K\right\} \\
&\leq \sup\left\{\frac{\|A(x'')-A(x')\|}{\|x''-x'\|} : x', x'' \in K\right\} \\
&\quad + (\alpha+2)\sup\left\{\frac{\big|\,\|x''\|-\|x'\|\,\big|}{\|x''-x'\|}\|D(x'')\| : x', x'' \in K\right\} \\
&\quad + (\alpha+2)\sup\left\{\|x'\|\frac{\|D(x'')-D(x')\|}{\|x''-x'\|} : x', x'' \in K\right\} \\
&\leq A_1 + (\alpha+2)K_0^\alpha + (\alpha+2)K_0 D_1 = M \ \text{(by (12))}
\end{aligned}$$

since

$$\|D(x)\| = \|x\|^\alpha \|j(x)\| \leq K_0^\alpha$$

for every $x \in K$. So (20), hence Theorem 4, is proved. □

The proof of Theorem 8 requires the following known necessary condition for the Aubin property of a set-valued mapping [6, Theorem 1.41]. Let $S : E \to F$ be a set-valued mapping acting between the Banach spaces E, F. Let $\bar{x}, \bar{y}$ be given such that $\bar{x} \in S(\bar{y})$, and consider

$$\rho(x, y) = \text{dist}[x, S(y)], x \in F, y \in E.$$

See [6, Definition 1.40 (ii)] for the definition of the Aubin property.

Theorem 16. *If S has the Aubin property at $\bar{y}$ for $\bar{x}$, then ρ is Lipschitz with respect to y, uniformly in x, near $(\bar{y}, \bar{x})$.*

We present a proof of Theorem 16 which is different from that of [6], and similar to the one of [5, Theorem 3E.5] for the finite-dimensional setting.

P r o o f o f T h e o r e m 16. Let $\text{lip}(S; \bar{y}|\bar{x})$ be the infimum of those constants $k \geq 0$ such that for some neighborhoods U of $\bar{x}, V$ of $\bar{y}$, we have

$$S(y) \cap U \subset S(y') + k\|y - y'\|B \text{ for every } y, y' \in V; \tag{21}$$

we denote by $B(u, r)$ the closed ball of center u and radius r, and let $B = B(0, 1)$. Fix $k > \text{lip}(S; \bar{y}|\bar{x})$. Then by (21) there exist positive numbers a, b such that

$$S(y) \cap B(\bar{x}, a) \subset S(y') + k\|y' - y\|B \text{ for every } y, y' \in B(\bar{y}, b). \tag{22}$$

Moreover $S(y) \cap B(\bar{x}, a) \neq \emptyset$ for each y near $\bar{y}$, see [5, Proposition 3E.1]. We can assume that $a/4k < b$. Let $y \in B(\bar{y}, a/4k), x \in B(\bar{x}, a/4)$. Given $\varepsilon > 0$ there exists $u \in S(y)$ such that

$$\|u - x\| \leq \varepsilon + \operatorname{dist}[x, S(y)]. \tag{23}$$

By the Lipschitz property of dist $[\cdot, S(y)]$, (23) and (22) with $y = \bar{y}$ we get, denoting by e the excess,

$$\begin{aligned} \|x - u\| &\leq \varepsilon + \text{ dist } [x, S(y)] \leq \varepsilon + \|x - \bar{x}\| + \operatorname{dist}[\bar{x}, S(y)] \\ &\leq \varepsilon + \|x - \bar{x}\| + e[S(\bar{y}) \cap B(\bar{x}, a), S(y)] \leq \varepsilon + \frac{a}{4} + k\|\bar{y} - y\| \leq \varepsilon + \frac{a}{2} \end{aligned}$$

hence

$$\|u - \bar{x}\| \leq \|u - x\| + \|x - \bar{x}\| \leq \varepsilon + \frac{a}{2} + \frac{a}{4} < a$$

provided $\varepsilon < a/4$. Then by (23)

$$\operatorname{dist}[x, S(y) \cap B(\bar{x}, a)] \leq \|x - u\| \leq \varepsilon + \operatorname{dist}[x, S(y)]. \tag{24}$$

Let $y' \in B(\bar{y}, a/4k)$, then by (22) and (24)

$$\operatorname{dist}[x, S(y') + k\|y' - y\|B] \leq \text{ dist } [x, S(y) \cap B(\bar{x}, a)] \leq \varepsilon + \operatorname{dist}[x, S(y)].$$

Since ε is arbitrarily small, this yields

$$\operatorname{dist}[x, S(y') + k\|y - y'\|B] \leq \operatorname{dist}[x, S(y)],$$

so that

$$\operatorname{dist}[x, S(y')] - k\|y - y'\| \leq \operatorname{dist}[x, S(y') + k\|y - y'\|B] \leq \operatorname{dist}[x, S(y)],$$

hence

$$\operatorname{dist}[x, S(y')] \leq k\|y - y'\| + \operatorname{dist}[x, S(y)].$$

The roles of y, y' are symmetric, hence

$$|\operatorname{dist}[x, S(y')] - \operatorname{dist}[x, S(y)]\,| \leq k\|y - y'\|$$

for every y, y' sufficiently close to $\bar{y}$ and x sufficiently close to $\bar{x}$, thus proving the theorem. □

P r o o f o f T h e o r e m 8. Since $Q \in WC$, its solution mapping S is Lipschitz on some open ball H of center 0 and radius δ_1. Then if $q', q'' \in H$, we have for some constant $T > 0$

$$S(q') \subset S(q'') + T\|q' - q''\|B, \tag{25}$$

where B denotes the closed unit ball of X. Let $p' \in H, u \in S(p')$. By (25), S has the Aubin property at p' for u. Write

$$\rho(u,p) = \mathrm{dist}[u, S(p)].$$

From Theorem 16 it follows that $\rho(u,\cdot)$ is Lipschitz in some open ball V of center p' and radius δ_2. By the results of [7] (remembering (6)), $\rho(u,\cdot)$ is Gateaux differentiable in some set D dense in V, and the mean value theorem holds. Thus if $q', q'' \in V$ we have

$$\rho(u,q'') - \rho(u,q') \le \|q'' - q'\| \sup\{\|\nabla\rho(u,p)\| : p \in D\}, \tag{26}$$

where $\nabla\rho$ denotes the Gateaux gradient of $\rho(u,\cdot)$. Let now $p \in D, 0 < t < 1, 0 < \varepsilon < 1$ and let $w \in S(p)$ be such that

$$\|u - w\| \le \mathrm{dist}[u, S(p)] + \varepsilon t. \tag{27}$$

We know that $S(p+th) \ne \emptyset$ if $\|h\| \le 1$ and $t > 0$ is sufficiently small since $Q \in WC$. For every such h we have

$$\frac{1}{t}[\rho(u,p+th) - \rho(u,p)] \to \langle\nabla\rho(u,p), h\rangle \text{ as } t \to 0+\,. \tag{28}$$

Then by (26), (27) and (14), if $\|h\| \le 1$,

$$\begin{aligned}\rho(u,p+th) - \rho(u,p) &= \mathrm{dist}[u, S(p+th)] - \mathrm{dist}[u, S(p)] \\ &\le \|u - w\| + \mathrm{dist}[w, S(p+th)] - \mathrm{dist}[u, S(p)] \\ &\le \varepsilon t + \mathrm{dist}[w, S(p+th)] \le \varepsilon t + \|w - z(t)\| \le (\varepsilon + C^*)t\end{aligned}$$

provided $t \ge 0$ is sufficiently small. By (28) we obtain

$$\langle\nabla\rho(u,p), h\rangle \le \varepsilon + C^*$$

hence $\|\nabla\rho(u,p)\| \le C^*$ for every $p \in D$. Let $\delta \le \min\{\delta_1, \delta_2/2\}$, let $p', p'' \in X^*$ be such that $\|p'\| < \delta, \|p''\| < \delta$. Then $p', p'' \in V$, hence by (26) with $q'' = p'', q' = p'$,

$$\rho(u,p'') - \rho(u,p') = \rho(u,p'') = \mathrm{dist}[u, S(p'')] \le C^*\|p' - p''\|,$$

whence the excess

$$e[S(p'), S(p'')] \leq C^* \|p' - p''\|$$

and similarly for $e[S(p''), S(p')]$. It follows that

$$\text{hausd}[S(p'), S(p'')] \leq C^* \|p' - p''\|$$

for every sufficiently small p', p'', thus (15) is proved. □

Proof of Theorem 11. The lower bound comes from [14, Theorem 4.1]. We consider $\bar{Q}$ of Example 14, then $\bar{Q} \in B_m \cap IC \cap Z^*$ by (19), so that by (20)

$$\text{dist}(Q, IC_m \cap Z_m^*) \leq L + M.$$

Then, remembering Theorem 8, we obtain the upper bound, completing the proof. □

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