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VARIATIONAL PRINCIPLES FOR MONOTONE VARIATIONAL INEQUALITIES: THE SINGLE-VALUED CASE*

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Dedicated to the memory of Asen L. Dontchev

ABSTRACT. We consider a parameterized variational inequality (A, Y) in a Banach space E defined on a closed, convex and bounded subset Y of E by a monotone operator A depending on a parameter. We prove that under suitable conditions, there exists an arbitrarily small monotone perturbation of A such that the perturbed variational inequality has a solution which is a continuous function of the parameter, and is near to a given approximate solution. In the nonparametric case this can be considered as a variational principle for variational inequalities, an analogue of the Borwein-Preiss smooth variational principle.

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*Part of this work was done when I was with the Stevens Institute of Technology, Department of Mathematics, New Jersey, USA.

Some applications are given: an analogue of the Nash equilibrium problem, defined by a partially monotone operator, and a variant of the parametric Borwein-Preiss variational principle for Gâteaux differentiable convex functions under relaxed assumptions.

The tool for proving the main result is a useful lemma about existence of continuous ε -solutions of a variational inequality depending on a parameter. It has an independent interest and allows a direct proof of an analogue of Ky Fan's inequality for monotone operators, introduced here, which leads to a new proof of the Schauder fixed point theorem in Gâteaux smooth Banach spaces.

1. Preliminaries. The variational principles in optimization gained popularity after the appearance of the famous Ekeland's variational principle [13] in 1973 and its multiple applications in the non-linear analysis [3]. After that series of new variational principles appeared, as: Borwein-Preiss' smooth variational principle [5], Deville-Godefroy-Zizler's variational principle [8] and its variant [9]; and relatively recently, variational principles in topological spaces: [6], [18], [19], [17]. Another direction of development of the variational principles is their parametrization, i.e. when the solution depends in a continuous way on a parameter. Such results are presented in [14], [15], [24], [7].

Since the convex minimization problems on closed sets have extensions to variational inequalities defined by monotone operators, it is natural to ask for extensions of the variational principles to monotone variational inequalities. In this paper we present such extensions, also to parametric variational inequalities, meaning that the solution depends continuously on a parameter under suitable assumptions.

Recall the classical result ([20], Theorem 1.4, Ch. III) that the variational inequality (A, Y) (sometimes called the *Stampacchia variational inequality*), defined by

$$(1) \quad \text{find } y_0 \in Y \quad \text{such that} \quad \langle A(y_0), y_0 - y \rangle \leq 0 \quad \forall y \in Y$$

has a solution, if $A : Y \rightarrow E^*$ is a monotone operator, continuous on the finite dimensional subspaces and Y is a non-empty weakly compact convex subset of a reflexive Banach space E with dual E^* . The notion *continuous on the finite dimensional subspaces* means that $A : Y \cap M \rightarrow E^*$ is *weakly* continuous for every finite dimensional subspace M of E with $Y \cap M \neq \emptyset$ (see [20], Definition 1.2, Ch. III). This result (with the same proof) is valid in an arbitrary Banach space E , if we replace the notion *weakly continuous* with *weak*-continuous* in the above definition, and we will refer to this property as A being *weak*-continuous on the finite dimensional subspaces*.

A weaker form of (1) is the so called *Minty variational inequality*:

$$(2) \quad \text{find } y_0 \in Y \quad \text{such that} \quad \langle A(y), y_0 - y \rangle \leq 0 \quad \forall y \in Y.$$

Recall that an operator $A : Y \rightarrow E^*$ is called *monotone*, if

$$\langle A(x) - A(y), x - y \rangle \geq 0 \quad \forall x \in Y, y \in Y.$$

Note that the classical result cited above ([20], Theorem 1.4, Ch. III) implies that

$$(3) \quad \inf_{z \in Y} \sup_{y \in Y} \langle A(y), z - y \rangle = \inf_{z \in Y} \sup_{y \in Y} \langle A(z), z - y \rangle = 0,$$

and the infimum in (3) is attained.

In a Banach space with a Gâteaux differentiable norm, if the monotone operator is the gradient of a convex function g and depends in a suitable way on a parameter x , i.e. $A(x, y) = g'_y(x, y)$, then there is an arbitrarily small monotone perturbation of the monotone operator $g'_y(x, \cdot)$, such that the perturbed variational inequality has a solution depending continuously on the parameter. This follows under the conditions of the parametric Borwein-Preiss variational principle [14].

The aim of this paper is to prove similar (perturbed) result for a parametric variational inequality $(A(x, \cdot), Y)$, with a parameter $x \in X$, X being a paracompact space, $Y \subset E$ being a weakly compact and convex, when $A(x, \cdot) : E \rightarrow E^*$ is a monotone operator, weak*-continuous on the finite dimensional subspaces and

$$(4) \quad A(\cdot, y) : X \rightarrow E^* \quad \text{is a weak}^*\text{-continuous mapping for every } y \in Y.$$

Such a result, presented in Theorem 2 below can be considered, in the nonparametric case, as a variational principle for variational inequalities, an analogue of the Borwein-Preiss smooth variational principle. Note that a similar analogue for variational inequalities of the Ekeland variational principle [13] is not possible in this setting of single-valued monotone operators, because the norm is not differentiable at zero.

Theorem 2 is also an extension of the result in [16], where instead of condition (4), the following much stronger condition was used:

$$(5) \quad \text{The family of mappings } \{A(\cdot, y) : X \rightarrow E^*, y \in Y\} \quad \text{is equi-continuous.}$$

The term “arbitrarily small monotone perturbation” makes sense if we consider the monotone operators here as elements of the normed space \mathcal{F} of all w^* -continuous bounded functions $F : Y \rightarrow E^*$ with the norm $\|F\| = \sup_{x \in Y} \|F(x)\|^*$.

See also Remark 3.

Note that different types of variational inequalities are parametrized also by other authors with purpose of investigating stability properties of the solution set as a function of the parameter, see [11], [12].

The basic tool in this paper is a useful lemma (Lemma 1) which gives existence of a continuous selection of the ε -solution set of a parametric Minty variational inequality.

As applications we obtain an analogue of the Nash equilibrium problem defined by a partially monotone operator, and a variant of the parametric Borwein-Preiss variational principle for Gâteaux differentiable convex functions under relaxed assumptions.

We introduce and prove an analogue of Ky Fan’s inequality for monotone operators, and show how it implies the Schauder fixed point theorem in Gâteaux smooth Banach spaces.

2. Continuous ε -solutions of a parametric Minty variational inequality.

Lemma 1. *Let X be a paracompact topological space, E be a Banach space, $Y \subset E$ is a non-empty, weakly compact and convex set, $A : X \times Y \rightarrow E^*$ has the properties:*

(a) *for every $x \in X$, $A(x, \cdot)$ is monotone and weak*-continuous on the finite dimensional subspaces,*

(b) *for every $y \in Y$ the mapping $A(\cdot, y)$ is weak*-continuous.*

Then: for every $\varepsilon > 0$ there exists a continuous mapping $y_\varepsilon : X \rightarrow Y$ such that

$$(6) \quad \langle A(x, y), y_\varepsilon(x) - y \rangle < \varepsilon \quad \forall x \in X, \quad \forall y \in Y.$$

Proof. For every $z \in Y$ define

$$S_z = \left\{ x \in X : \sup_{y \in Y} \langle A(x, z), z - y \rangle < \varepsilon \right\}.$$

We will prove that S_z is open. Let $x_0 \in S_z$. We will show that there exists $\beta > 0$ such that

$$\sup_{y \in Y} \langle A(x, z), z - y \rangle < \varepsilon \quad \forall x \in B(x_0, \beta).$$

Assume the contrary: for every natural n there exists $x_n \in B(x_0, 1/n)$ such that

$$\sup_{y \in Y} \langle A(x_n, z), z - y \rangle \geq \varepsilon.$$

Denote $\varepsilon_0 = \sup_{y \in Y} \langle A(x_0, z), z - y \rangle$. Then $\varepsilon_0 < \varepsilon$, by definition of S_z .

Choose $\gamma \in \left(0, \frac{\varepsilon - \varepsilon_0}{2}\right)$. Then we have, for large n ,

$$\begin{aligned} \sup_{y \in Y} \langle A(x_n, z), z - y \rangle &\leq \sup_{y \in Y} \langle A(x_n, z) - A(x_0, z), z - y \rangle + \sup_{y \in Y} \langle A(x_0, z), z - y \rangle \\ &\leq \langle A(x_n, z) - A(x_0, z), z - y_0 \rangle + \gamma + \varepsilon_0 \quad (\text{for some } y_0 \in Y) \\ &\leq 2\gamma + \varepsilon_0 \quad (\text{by weak}^*\text{-continuity of } A(\cdot, z) \text{ at } x_0) \\ &< \varepsilon, \end{aligned}$$

a contradiction.

So we proved that S_z is open for every $z \in X$.

By (3) applied for $A(x, \cdot)$ we have

$$\bigcup_{z \in X} S_z = X.$$

Since X is paracompact, there exists a locally finite refinement $\{U_j\}_{j \in J}$ of the cover $\{S_z\}_{z \in X}$. So, for each $j \in J$ there exists $z_j \in Y$ such that $U_j \subset S_{z_j}$. Let $\{p_j\}_{j \in J}$ be a continuous partition of unity, subordinated to $\{U_j\}_{j \in J}$. Then for any $x \in X$ there are finite number of indices $j \in J_x$ for which $p_j(x) \neq 0$ and

$$\text{supp}(p_j) \subset U_j \subset S_{z_j}, \quad p_j(x) \in [0, 1], \quad \sum_{j \in J} p_j(x) = 1 \quad \forall x \in X.$$

Define the continuous function $y_\varepsilon : X \rightarrow Y$ by

$$y_\varepsilon(x) = \sum_{j \in J} p_j(x) z_j.$$

For $j \in J_x$ we can write

$$\langle A(x, z_j), z_j - y \rangle < \varepsilon \quad \forall y \in Y.$$

By monotonicity we have

$$\langle A(x, y), z_j - y \rangle < \varepsilon \quad \forall y \in Y.$$

Multiplying both sides by $p_j(x)$ and summing, we obtain

$$\left\langle A(x, y), y_\varepsilon(x) - y \right\rangle < \varepsilon \quad \forall y \in Y, \quad \forall x \in X,$$

which is (6). \square

3. Main result. Now we present our main result stating that, under some assumptions, a parametric Stampacchia variational inequality after arbitrarily small monotone perturbation has a solution, which is a continuous function of the parameter. Also, it has an ε -approximate solution, which is again a continuous function of the parameter.

Theorem 2. *Suppose that X is a paracompact topological space, E is a Banach space with a Gâteaux differentiable norm $\|\cdot\|$, $Y \subset E$ is a non-empty, weakly compact and convex set, $A : X \times Y \rightarrow E^*$ has the properties:*

(a) *for every $x \in X$, $A(x, \cdot)$ is monotone and weak*-continuous on the finite dimensional subspaces,*

(b) *for every $y \in Y$ the mapping $A(\cdot, y)$ is weak*-continuous.*

Let $\varepsilon > 0$ and $y_0 : X \rightarrow Y$ be a continuous mapping such that

$$\left\langle A(x, y), y_0(x) - y \right\rangle < \varepsilon \quad \forall x \in X, \forall y \in Y.$$

Then for every $\lambda > 0, p > 1$ there exists a continuous mapping $v : X \rightarrow Y$ such that:

$$(7) \quad \left\langle \left(A + \frac{\varepsilon}{\lambda^p} \Gamma \right) (x, y), v(x) - y \right\rangle \leq 0, \quad \forall x \in X, \forall y \in Y$$

(the Minty perturbed parametric variational inequality),

$$(8) \quad \left\langle \left(A + \frac{\varepsilon}{\lambda^p} \Gamma \right) (x, v(x)), v(x) - y \right\rangle \leq 0, \quad \forall x \in X, \forall y \in Y,$$

(the Stampacchia perturbed parametric variational inequality),

$$(9) \quad \left\langle A(x, v(x)), v(x) - y \right\rangle \leq \frac{\varepsilon p}{\lambda} \text{diam}(Y), \quad \forall x \in X, \forall y \in Y,$$

(the Stampacchia ε -parametric variational inequality) and

$$(10) \quad \|v(x) - y_0(x)\| \leq \lambda, \quad \forall x \in X,$$

where

$$\Gamma(x, y) = \sum_{n=0}^{\infty} \mu_n \nabla_y \|y - y_n(x)\|^p,$$

∇_y means the gradient with respect to the variable y , $\sum_{n=0}^{\infty} \mu_n = 1$, and $y_n : X \rightarrow Y$ are continuous mappings converging uniformly on X to v .

Proof. Put

$$K(q, \nu) = \left(\frac{q^2(1-\nu) + \nu}{p\nu(1-q)(1-\nu)^p(1-q^{\frac{1}{p}})^p} \right)^{\frac{1}{p}}.$$

Since

$$\lim_{\nu \rightarrow 0_+} \left(\lim_{q \rightarrow 0_+} K(q, \nu) \right) = \frac{1}{p^{\frac{1}{p}}} < 1,$$

there exist $q, \nu \in (0, 1)$ such that $K(q, \nu) < 1$.

Put

$$\mu_n = (1-q)q^n, \varepsilon_n = \varepsilon q^{2n}.$$

By Lemma 1, define inductively, for $n \geq 0$:

$$(11) \quad A_{n+1}(x, y) = A_n(x, y) + \frac{\varepsilon}{\lambda^p} \mu_n \nabla_y \|y - y_n(x)\|^p, \quad A_0 = A, \varepsilon_0 = \varepsilon,$$

where $y_n : X \rightarrow Y$ is a continuous mapping such that

$$(12) \quad \langle A_n(x, y), y_n(x) - y \rangle < \varepsilon_n. \quad \forall x \in X, \forall y \in Y.$$

We will prove that

$$(13) \quad \|y_{n+1}(x) - y_n(x)\| < \lambda q^{\frac{n}{p}} \left(1 - q^{\frac{1}{p}}\right), \quad \forall x \in X.$$

Put $y = \nu y_n(x) + (1-\nu)y_{n+1}(x)$. Then $y \in Y$ and

$$(14) \quad y_{n+1}(x) - y_n(x) = \frac{1}{1-\nu} (y - y_n(x)) = \frac{1}{\nu} (y_{n+1}(x) - y).$$

Consider the case when $y_{n+1}(x) \neq y_n(x)$, otherwise (13) is satisfied. We write (11) in the following form (note that $y \neq y_n(x)$):

$$(15) \quad \frac{\varepsilon}{\lambda^p} \mu_n \nabla_y \left\| y - y_n(x) \right\|^p = \frac{\varepsilon}{\lambda^p} \mu_n p \left\| y - y_n(x) \right\|^{p-1} \nabla_y \left\| y - y_n(x) \right\| \\ = A_{n+1}(x, y) - A_n(x, y).$$

Multiplying (15) by $y - y_n(x)$ and using (14) and the obvious identity

$$(16) \quad \left\langle \nabla \|z\|, z \right\rangle = \|z\| \quad \forall z \in E, z \neq 0,$$

we obtain

$$\frac{\varepsilon}{\lambda^p} p \mu_n (1 - \nu)^p \left\| y_{n+1}(x) - y_n(x) \right\|^p \\ \leq \frac{1 - \nu}{\nu} \left\langle A_{n+1}(x, y), y_{n+1}(x) - y \right\rangle + \left\langle A_n(x, y), y_n(x) - y \right\rangle \\ < \frac{1 - \nu}{\nu} \varepsilon_{n+1} + \varepsilon_n \\ = \varepsilon q^{2n} \left(q^2 \frac{1 - \nu}{\nu} + 1 \right).$$

Hence

$$\left\| y_{n+1}(x) - y_n(x) \right\| < \lambda K(q, \nu) q^{\frac{n}{p}} \left(1 - q^{\frac{1}{p}} \right) < \lambda q^{\frac{n}{p}} \left(1 - q^{\frac{1}{p}} \right),$$

(since $K(q, \nu) < 1$), which is (13). For $m > n$ we obtain

$$(17) \quad \left\| y_m(x) - y_n(x) \right\| < \lambda \left(q^{\frac{n}{p}} + \dots + q^{\frac{m-1}{p}} \right) \left(1 - q^{\frac{1}{p}} \right)$$

$$(18) \quad = \lambda q^{\frac{n}{p}} \left(1 - q^{\frac{m-n}{p}} \right).$$

When $m > n \rightarrow \infty$, the right hand side converges to zero, which shows that $\{y_n(x)\}$ is a fundamental sequence, converging uniformly on $x \in X$ to some $v(x)$, which implies that $v(\cdot)$ is continuous. When $n = 0$ and m tends to infinity, we obtain (10).

We shall prove (7), the Minty parametric variational inequality.

Assume the contrary: for some $x' \in X$ and $y' \in Y$ we have

$$\left\langle \left(A + \frac{\varepsilon}{\lambda^p} \Gamma \right) (x', y'), v(x') - y' \right\rangle > 0.$$

Take $0 < \beta < \left\langle \left(A + \frac{\varepsilon}{\lambda^p} \Gamma \right) (x', y'), v(x') - y' \right\rangle$ and an integer n such that

$$\left\| v(x') - y_n(x') \right\| < \frac{\beta}{3} \left\| \left(A + \frac{\varepsilon}{\lambda^p} \Gamma \right) (x', y') \right\|^{-1}, \varepsilon_n < \beta/3$$

and $\frac{p\varepsilon}{\lambda^p} \sum_{k=n}^{\infty} \mu_k < \frac{\beta}{3} (\text{diam } Y)^{-p}$.

Then we have:

$$\begin{aligned}
 & \left\langle \left(A + \frac{\varepsilon}{\lambda^p} \Gamma \right) (x', y'), v(x') - y' \right\rangle \\
 &= \left\langle \left(A + \frac{\varepsilon}{\lambda^p} \Gamma \right) (x', y'), v(x') - y_n(x') \right\rangle + \left\langle A_n(x', y'), y_n(x') - y' \right\rangle \\
 & \quad + \left\langle \left(A + \frac{\varepsilon}{\lambda^p} \Gamma - A_n \right) (x', y'), y_n(x') - y' \right\rangle \\
 & \leq \left\| \left(A + \frac{\varepsilon}{\lambda^p} \Gamma \right) (x', y') \right\| \|v(x') - y_n(x')\| + \varepsilon_n \\
 & \quad + \frac{p\varepsilon}{\lambda^p} \sum_{k=n}^{\infty} \mu_k \|y' - y_k(x')\|^{p-1} \left\langle \nabla_y \|y - y_k(x')\| \Big|_{y=y'}, y_n(x') - y' \right\rangle \\
 & < 2\beta/3 + \frac{p\varepsilon}{\lambda^p} (\text{diam } Y)^p \sum_{k=n}^{\infty} \mu_k < \beta,
 \end{aligned}$$

a contradiction and (7) is proved.

In order to prove (8), it is enough to show that for any $\gamma > 0$

$$(19) \quad \left\langle \left(A + \frac{\varepsilon}{\lambda^p} \Gamma \right) (x, v(x)), v(x) - y \right\rangle \leq \gamma, \quad \forall x \in X, \forall y \in Y.$$

Put

$$\Gamma_k(x, y) = \sum_{n=0}^k \mu_n \nabla_y \|y - y_n(x)\|^p.$$

Then

$$\Gamma(x, y) - \Gamma_k(x, y) = \sum_{n=k+1}^{\infty} \mu_n \nabla_y \|y - y_n(x)\|^p.$$

Let now $x \in X$ and $y \in Y$ be fixed. Set

$$(20) \quad y_t = ty + (1-t)v(x) \in Y, t \in (0, 1], y_t = ty + (1-t)v(x),$$

put it in (7) and divide by $t > 0$. So

$$\left\langle \left(A + \frac{\varepsilon}{\lambda^p} \Gamma \right) (x, y_t), v(x) - y \right\rangle \leq 0, \quad \forall t \in (0, 1].$$

Take k sufficiently large such that

$$(21) \quad \left\langle \frac{\varepsilon}{\lambda^p} (\Gamma - \Gamma_k) (x, v(x)), v(x) - y \right\rangle < \gamma/3.$$

Since the function $\|\cdot\|^p$ is convex and Gâteaux differentiable, its derivative is weak*-continuous, so $\Gamma_k(x, \cdot)$ is weak*-continuous for every $x \in X$. Since $y_t \rightarrow v(x)$ as $t \rightarrow 0_+$, we can take $t > 0$ sufficiently small such that

$$(22) \quad \left\langle \frac{\varepsilon}{\lambda^p} \Gamma_k(x, v(x)), v(x) - y \right\rangle < \left\langle \frac{\varepsilon}{\lambda^p} \Gamma_k(x, y_t), v(x) - y \right\rangle + \gamma/3$$

and

$$(23) \quad \left\langle A(x, v(x)), v(x) - y \right\rangle < \left\langle A(x, y_t), v(x) - y \right\rangle + \gamma/3,$$

(since $A(x, \cdot)$ is weak*-continuous on the finite dimensional subspaces). Adding (21), (22) and (23), we obtain (19) and proved (8).

In order to prove (9), we evaluate its left hand side:

$$\begin{aligned} \left\langle A(x, v(x)), v(x) - y \right\rangle &\leq -\frac{\varepsilon}{\lambda^p} \left\langle \Gamma(x, v(x)), v(x) - y \right\rangle \\ &= -\frac{\varepsilon p}{\lambda^p} \sum_{n=0}^{\infty} \mu_n \|v(x) - y_n(x)\|^{p-1} \left\langle \nabla_z \|z - y_n(x)\| \Big|_{z=v(x)}, v(x) - y \right\rangle \\ &\leq \frac{\varepsilon p}{\lambda^p} \sum_{n=0}^{\infty} \mu_n \lambda^{p-1} \left\| \nabla_z \|z - y_n(x)\| \Big|_{z=v(x)} \right\| \|v(x) - y\| \\ &\quad (\text{here we used (17) and (18) when } m \rightarrow \infty) \\ &\leq \frac{\varepsilon p}{\lambda} \text{diam}(Y), \end{aligned}$$

and the theorem is proved. \square

Remark 3. It is clear that the operator $\Gamma(x, \cdot)$ is monotone and w^* -continuous for every $x \in X$, and the norm of the perturbation satisfies

$$\left\| \frac{\varepsilon}{\lambda^p} \Gamma(x, \cdot) \right\| \leq \frac{\varepsilon p}{\lambda^p} \text{diam}(Y)^{p-1},$$

so it could be arbitrarily small. This justifies the term “arbitrarily small monotone perturbation” in the abstract.

4. Application to Nash equilibrium problems for a system of variational inequalities. Variational inequalities over product spaces with different notions of monotonicity are considered in several papers – see for instance [1, 2, 21] and mainly in the recent comprehensive paper [4] and references therein.

One of the difficulties of working in product spaces is that the monotonicity properties, like pseudo and quasi-monotonicity imposed on the component operators are not preserved on the product operator and the authors are forced to impose stronger regularity properties, or to assume some hypothesis of generalized monotonicity on the product operator, rather than on the component operators. In this paper we work with the classical notion of monotone operator, however we relax some assumptions as joint continuity and compactness of one set.

The next theorem gives existence of an equilibrium for a system of variational inequalities, which we call *Nash equilibrium for variational inequalities*. In particular, we obtain solutions of a variational inequality defined by a partially monotone operator. It resembles the classical Nash equilibrium theorem for partially convex functions (see [3], Theorem 13, Ch.6]). Here not all domain sets are strong compacts: one of them is allowed to be weakly compact, but nevertheless, there exists an equilibrium solution for the perturbed variational inequality.

Denote

$$X = X_1 \times \cdots \times X_n, \quad x = (x_1, \dots, x_n),$$

$$x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

$$X_{-i} = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n, i = 1, \dots, n.$$

and adopt the notation $(x_{-i}, z) = (x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n)$

Theorem 4. *Let $E_i, i = 1, v, n$ be Gâteaux smooth Banach spaces, X_i be non-empty, convex and compact subset of E_i for $i = 2, \dots, n$, X_1 be a non-empty, convex and weakly compact subset of E_1 .*

The operators $A_i : X \rightarrow E_i^, i = 1, \dots, n$ satisfy the conditions:*

- (a) *the mapping $X_i \ni z \mapsto A_i(x_{-i}, z)$ is a monotone operator, weak*-continuous on finite dimensional subspaces of E_i for every $x_{-i} \in X_{-i}$,*
- (b) *the mapping $X_{-i} \ni x_{-i} \mapsto A_i(x_{-i}, z)$ is weak*-continuous for every $z \in X_i$.*

Then: for every $\varepsilon > 0$ there exists $x^\varepsilon \in X$ such that

$$(24) \quad \left\langle \left(A_i + \varepsilon \Gamma_i \right) (x^\varepsilon), x_i^\varepsilon - y_i \right\rangle \leq 0, \quad \forall y_i \in X_i, \quad \forall i = 1, \dots, n.$$

where

$$(25) \quad \Gamma_i(x^\varepsilon) = \sum_{k=0}^{\infty} \mu_k \nabla_{x_i} \left\| x_i - y_{k,i}(x_{-i}^\varepsilon) \right\|^p,$$

∇_{x_i} means the gradient with respect to the variable x_i , $\sum_{k=0}^{\infty} \mu_k = 1$, and $y_{k,i} : X_{-i} \rightarrow X_i$ are continuous mappings converging uniformly on X_{-i} to a continuous mapping $v_i : X_{-i} \rightarrow X_i$ as $k \rightarrow \infty$.

If X_1 is also compact, then there exists $x^0 \in X$ such that

$$(26) \quad \left\langle A_i(x^0), x_i^0 - y_i \right\rangle \leq 0, \quad \forall y_i \in X_i, \quad \forall i = 1, \dots, n.$$

Proof. We apply Theorem 2 to A_i for every $i = 1, \dots, n$, considering x_{-i} as a parameter. So, there exists a continuous mapping $v_i : X_{-i} \rightarrow X_i$ such that

$$(27) \quad \left\langle (A_i + \varepsilon \Gamma_i)(x_{-i}, v_i(x_{-i})), v_i(x_{-i}) - y_i \right\rangle \leq 0, \quad \forall y_i \in X_i, \quad \forall x_{-i} \in X_{-i}.$$

where

$$\Gamma_i(x_{-i}, v_i(x_{-i})) = \sum_{k=0}^{\infty} \mu_k \nabla_y \|y - y_{k,i}(x_{-i})\|^p \Big|_{y=v_i(x_{-i})},$$

∇_y means the gradient with respect to the variable y , $\sum_{k=0}^{\infty} \mu_k = 1$, and $y_{k,i} : X_{-i} \rightarrow X_i$ are continuous mappings converging uniformly to v_i as $k \rightarrow \infty$.

Define the mappings $\phi_i : X_{-1} \rightarrow X_{-i}$, $i = 2, \dots, n$, by

$$\begin{aligned} \phi_i(x_{-1}) &= (v_1(x_{-1}), x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \quad i = 3, \dots, n-1, \\ \phi_2(x_{-1}) &= (v_1(x_{-1}), x_3, v, x_n), \\ \phi_n(x_{-1}) &= (v_1(x_{-1}), x_2, \dots, x_{n-1}). \end{aligned}$$

The composition

$$X_2 \times \dots \times X_n \ni x_{-1} \mapsto (v_2(\phi_2(x_{-1})), \dots, v_n(\phi_n(x_{-1}))) \in X_2 \times \dots \times X_n$$

is a continuous mapping from the compact set $X_2 \times \dots \times X_n$ to itself, and from Schauder's fixed point theorem it has a fixed point $x_{-1}^\varepsilon = (x_2^\varepsilon, \dots, x_n^\varepsilon)$, which means, component-wise,

$$x_i^\varepsilon = v_i(\phi_i(x_{-1}^\varepsilon)), \quad i = 2, \dots, n.$$

Denote $x_1^\varepsilon = v_1(x_{-1}^\varepsilon)$ and putting $\phi_i(x_{-1}^\varepsilon)$ in the place of x_{-i} in (27), we obtain (24).

Now we will prove (26), assuming that X_1 is also compact. Using monotonicity, from (24) we can write

$$(28) \quad \left\langle \left(A_i + \varepsilon \Gamma_i \right) (x_{-i}^\varepsilon, y_i), x_i^\varepsilon - y_i \right\rangle \leq 0, \quad \forall y_i \in X_i, \quad \forall i = 1, \dots, n.$$

Taking a convergent subsequence from $\{x^\varepsilon\}$ for $\varepsilon = 1/m$, denoting its limit by x^0 when $m \rightarrow \infty$, and passing to limits in (28), we obtain

$$(29) \quad \left\langle A_i(x_{-i}^0, y_i), x_i^0 - y_i \right\rangle \leq 0, \quad \forall y_i \in X_i, \quad \forall i = 1, \dots, n.$$

Now we apply the Minty lemma (see [20], Lemma 1.5, Ch.III), (which is valid also in non-reflexive spaces) and obtain (26). \square

The mappings A_i in the previous theorem can be considered as a component mappings of a partially monotone operator $A : X \rightarrow E_1^* \times \dots \times E_n^*$, for which we obtain an equilibrium after perturbation.

Corollary 5. *Assume that the conditions of Theorem 4 are satisfied. Define the operator*

$$A : X \rightarrow E_1^* \times \dots \times E_n^*, \quad \text{as} \quad A(x) = (A_1(x), \dots, A_n(x)).$$

Then:

(a) for every $\varepsilon > 0$ there exists $x^\varepsilon \in X$ such that

$$(30) \quad \left\langle \left(A + \varepsilon \Gamma \right) (x^\varepsilon), x^\varepsilon - y \right\rangle \leq 0, \quad \forall y \in X.$$

where

$$(31) \quad \Gamma(x) = \left(\Gamma_1(x), \dots, \Gamma_n(x) \right)$$

and Γ_i are defined by (25).

(b) for every $\varepsilon > 0$ there exists $z^\varepsilon \in X$ such that

$$(32) \quad \left\langle A(z^\varepsilon), z^\varepsilon - y \right\rangle < \varepsilon, \quad \forall y \in X.$$

If X_1 is also compact, then there exists $x^0 \in X$ such that

$$(33) \quad \left\langle A(x^0), x^0 - y \right\rangle \leq 0, \quad \forall y \in X.$$

Proof. Adding the inequalities in (24) for $i = 1, v, n$ we obtain (30). Similarly, adding the inequalities in (26) for $i = 1, \dots, n$ we obtain (33).

The inequality (32) follows from (30) by the same way as we proved (9) from (8). \square

Remark 6. The result in (b) for $n = 2$ can be considered as an analogue of Sion's minimax theorem [23], in which also one on the domain set is allowed to be non-compact and there is a saddle point after arbitrarily small function perturbations.

Now we present a variant of the parametric Borwein-Preiss smooth variational principle for smooth convex functions.

Theorem 7. *Suppose that X is a paracompact topological space, $(E, \|\cdot\|)$ is a Gâteaux smooth Banach space, $Y \subset E$ is non-empty, convex closed and bounded subset, the function*

$f : X \times Y \rightarrow \mathbb{R}$ has the properties:

(a) for every $x \in X$, $f(x, \cdot)$ is convex and Gâteaux differentiable on Y ,

(b) for every $y \in Y$ the mapping $x \mapsto f'_y(x, y)$ is weak-continuous.*

Let $\varepsilon > 0$ and $y_0 : X \rightarrow Y$ be a continuous mapping such that

$$f(x, y_0(x)) < f(x, y) + \varepsilon \quad \forall x \in X, \forall y \in Y.$$

Then for every $\lambda > 0, p > 1$ there exists a continuous mapping $v : X \rightarrow Y$ such that

$$\left(f + \frac{\varepsilon}{\lambda^p} \Gamma\right)(x, v(x)) \leq \left(f + \frac{\varepsilon}{\lambda^p} \Gamma\right)(x, y) \quad \forall x \in X, \forall y \in Y.$$

and

$$(34) \quad \|v(x) - y_0(x)\| \leq \lambda, \quad \forall x \in X,$$

where

$$\Gamma(x, y) = \sum_{n=0}^{\infty} \mu_n \|y - y_n(x)\|^p,$$

$$\mu_n \geq 0, \quad \sum_{n=0}^{\infty} \mu_n = 1,$$

and $y_n : X \rightarrow Y$ are continuous mappings converging uniformly on X to v .

Proof. By the Borwein-Preiss variational principle [5] we have

$$(35) \quad \inf_{z \in Y} \sup_{y \in Y} \left\langle f'_z(x, z), z - y \right\rangle = 0, \quad \forall x \in X,$$

as the infimum may not be attained.

The proof works as the proof of Theorem 2 applied to the monotone operator $f'(x, \cdot)$. The only difference is that, in order to justify (22) and (23), the choice of y_t is taken from the mean-value theorem (since the w^* -continuity of $f'_y(x, \cdot)$ is not guaranteed on the boundary of Y), applied for the interval $[v(x), v(x) + t(y - v(x))]$: for any $\gamma > 0$ there exists $t_0 > 0$ such that for any $t \in (0, t_0]$ (22) is satisfied with $y_t = ty + (1 - t)v(x)$ and there exists $c_t \in (v(x), v(x) + t(y - v(x)))$ such that

$$\begin{aligned} \langle f'_y(x, v(x)), v(x) - y \rangle &= -\langle f'_y(x, v(x)), y - v(x) \rangle \\ &< -\frac{f(x, v(x) + t(y - v(x))) - f(x, v(x))}{t} + \gamma/3 \\ &= -\langle f'_y(x, c_t), y - v(x) \rangle + \gamma/3 \\ &= \langle f'_y(x, c_t), v(x) - y \rangle + \gamma/3. \end{aligned}$$

So, $c_t = y_{t'}$ for some $t' \in (0, t_0)$ and (22) and (23) are satisfied for $y_{t'}$. \square

Remark 8. The parametric Borwein-Preiss variational principle in [15] requires equi-lower semi-continuity of the functions $\{f(x, y) : y \in Y_0\}$ for every bounded set $Y_0 \subset Y$, and the convex functions $f(x, \cdot)$ are not required to be Gâteaux differentiable.

If [7] the authors constructed counterexamples stating that a parametric variational principle is not valid without some equi-lower semi-continuity conditions, however the functions $f(x, \cdot)$ constructed there are not Gâteaux differentiable.

5. Ky Fan's type inequality for monotone operators. In this section we introduce a statement which resembles the famous Ky Fan inequality [22]:

Theorem 9 (Ky Fan). *Suppose that X is a convex, compact and non-empty subset of a topological vector space E and $f : X \times X \rightarrow \mathbb{R}$ is a function with the properties:*

- (a) $f(\cdot, y)$ is lower semi-continuous for every $y \in X$,
- (b) $f(x, \cdot)$ is concave for every $x \in X$.

Then there exists $x_0 \in X$ such that

$$(36) \quad \sup_{y \in X} f(x_0, y) \leq \sup_{y \in X} f(y, y).$$

Theorem 10 (Analogue of Ky Fan's inequality for monotone operators).
 Suppose that X is a convex, compact and nonempty subset of a Banach space E , and $A : X \times X \rightarrow E^*$ has the properties:

(a) for every $x \in X$, $A(x, \cdot)$ is monotone, weak*-continuous on the finite dimensional subspaces,

(b) for every $y \in X$ the mapping $A(\cdot, y)$ is weak*-continuous.

Then there exists $x_0 \in X$ such that

$$(37) \quad \langle A(x_0, x_0), x_0 - y \rangle \leq 0 \quad \forall y \in X.$$

Proof. Step 1. Let X be a convex hull of finitely many points.

By Lemma 1, for every $\varepsilon > 0$ there exists a continuous function $v_\varepsilon : X \rightarrow X$ such that

$$\langle A(x, y), v_\varepsilon(x) - y \rangle \leq \varepsilon \quad \forall x, \forall y \in X.$$

Applying the Brouwer fixed point theorem for v_ε , we obtain a fixed point $x_\varepsilon = v_\varepsilon(x_\varepsilon)$, therefore

$$(38) \quad \langle A(x_\varepsilon, y), x_\varepsilon - y \rangle \leq \varepsilon \quad \forall y \in X.$$

Take a convergent subsequence from $\{x_{1/n}\}_{n=1}^\infty$, and denote its limit by x_0 . Then, passing to limits in (38) (using (b)), we obtain

$$(39) \quad \langle A(x_0, y), x_0 - y \rangle \leq 0 \quad \forall y \in X.$$

Step 2. Let now X be a nonempty, convex and compact subset of E .

Define, for any $y \in Y$, the set

$$S_y = \left\{ x \in X : \langle A(x, y), x - y \rangle \leq 0 \right\},$$

which is obviously closed.

By Step 1, $\bigcap_{y \in Y} S_y \neq \emptyset$ for any finite subset Y of X . By the finite intersection property (since X is compact), we obtain

$$\bigcap_{y \in X} S_y \neq \emptyset.$$

For $x_0 \in \bigcap_{y \in X} S_y$ we have

$$\langle A(x_0, y), x_0 - y \rangle \leq 0 \quad \forall y \in X.$$

Now we apply the Minty lemma and obtain (37). \square

Remark 11. If $A(x, y) = -f'_y(x, y)$ for some function $f : X \times X \rightarrow \mathbb{R}$, which is concave and Gâteaux differentiable with respect to the second variable, then (37) implies (36).

Next, we show that Theorem 10 (Ky Fan's analog for monotone operators), Brouwer's fixed point theorem and Schauder's fixed point theorem are equivalent in Gâteaux smooth Banach spaces.

Proposition 12. *Theorem 10 implies Schauder's fixed point theorem in Gâteaux smooth Banach spaces.*

Proof. Let $X \subset E$ be a compact, convex and non-empty subset of a Banach space E , and $F : X \rightarrow X$ be a continuous mapping.

Define the operator

$$A : X \times X \rightarrow E^* \quad \text{as} \quad A(x, y) = \nabla_y \|y - F(x)\|^2,$$

which is monotone and weak*-continuous in y and weak*-continuous in x .

By Theorem 10, there exists $x_0 \in X$ such that

$$\langle A(x_0, x_0), x_0 - y \rangle \leq 0, \quad \forall y \in X.$$

For $y = F(x_0)$, if $F(x_0) \neq x_0$, we obtain

$$\begin{aligned} 0 &\geq \left\langle \nabla_y \left\| y - F(x_0) \right\|^2 \Big|_{y=x_0}, x_0 - F(x_0) \right\rangle \\ &= 2 \left\| x_0 - F(x_0) \right\| \left\langle \nabla_y \left\| y - F(x_0) \right\|^2 \Big|_{y=x_0}, x_0 - F(x_0) \right\rangle \\ &= 2 \left\| x_0 - F(x_0) \right\|^2 \quad \left(\text{by the identity (16)} \right), \end{aligned}$$

a contradiction. Therefore, $F(x_0) = x_0$. \square

We will finish with an open (to the author) question.

Open question. *Is there a closed convex, bounded and non-empty set $Y \in E$, which is not weakly compact, and a w^* -continuous monotone operator*

$A : Y \rightarrow E^*$, which is not a gradient of a convex function such that (3) is satisfied?

If yes, then the variational principles for variational inequalities presented here will be valid also for the variational inequality (A, Y) and its parametrization.

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