Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **49** (2023), 97–106

DOI: 10.55630/serdica.2023.49.97-106

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

HADAMARD INVERSE FUNCTION THEOREM PROVED BY VARIATIONAL ANALYSIS*

Milen Ivanov, Nadia Zlateva

Communicated by V. Veliov

Dedicated to A. L. Dontchev

Abstract. We present a proof of Hadamard Inverse Function Theorem by the methods of Variational Analysis, adapting an idea of I. Ekeland and E. Séré.

1. Introduction. The classical example $(x, y) \to e^x(\cos y, \sin y)$ shows that – except in dimension one – the derivative may be everywhere invertible while the function itself is invertible only locally. Probably the historically first sufficient condition for global invertibility is given by J. S. Hadamard, see (1) in Theorem 1 below.

2020 *Mathematics Subject Classification:* 47J07, 47J30.

Key words: inverse function, variational analysis.

*The support of the Scientific Fund of Sofia University under grant 80-10-62/25.04.2023 is gratefully acknowledged.

An excellent overview – both from research and educational perspective – of this topic is given in [7].

Here we present a new proof based on a recent idea by I. Ekeland and E. Séré we found in a beamer file of a draft for conference talk [4]. This idea allows obtaining a continuous right inverse to f on any compact, see Proposition 1. The key Proposition 1 is proved by methods of Variational Analysis in the flavour of the monographs of A. Dontchev [2], A. Dontchev and T. Rockafellar [3], A. Ioffe [5], and J.-P. Penot [6].

We work in a Banach space $(X, \|\cdot\|)$ and denote its closed unit ball by B_X. Recall that the function

$$f : X \to X$$

is called *Fréchet differentiable at* $x \in X$ if there is a bounded linear operator $f'(x) : X \to X$ such that

$$\lim_{\|h\| \to 0} \frac{f(x+h) - f(x) - f'(x)h}{\|h\|} = 0.$$

The function f is called *smooth*, denoted $f \in C^1$, if the function

$$x \to f'(x)$$

is norm-to-norm continuous.

We present a modern proof to the following classical

Theorem 1 (Hadamard)**.** *Let $f \in C^1$, $f'(x)$ be invertible for all x and satisfying*

$$\|[f'(x)]^{-1}\| \le M, \quad \forall x \in X, \tag{1}$$

for some $M > 0$.

Then f is C^1 invertible on X.

In other words, there is $g \in C^1$ such that

$$g(f(x)) = f(g(x)) = x, \quad \forall x \in X.$$

The work is organised as follows. In the next Section 2 we give the necessary preliminary known facts. In Section 3 we prove the key Proposition 1 and in the final Section 4 we complete the proof of Hadamard Theorem 1.

2. Preliminaries. We start with recalling the classical (local) Inverse Function Theorem.

Theorem 2. *Let $f \in C^1$ and let $f'(x_0)$ be invertible. Then there are $\varepsilon, \delta > 0$ such that for each y such that*

$$\|y - f(x_0)\| < \varepsilon$$

there is unique $x =: g(y)$ such that $\|x - x_0\| < \delta$ and

$$f(x) = y.$$

Moreover, $g \in C^1$ and

$$g'(f(x_0)) = [f'(x_0)]^{-1}. \tag{2}$$

The following statements are also well-known.

Lemma 1. *Let f be C^1. Let $K \subset X$ be compact and let $r > 0$. Then*

$$f(x + th) = f(x) + tf'(x)h + o(t) \text{ uniformly on } x \in K \text{ and } h \in rB_X.$$

More precisely, there is $\alpha(t) \to 0$ as $t \to 0$ such that

$$\sup\{\|f(x + th) - f(x) - tf'(x)h\| : x \in K,\ h \in rB_X\} \le \alpha(t)t.$$

Lemma 2. *Let X be a Banach space and let $A(x)$ be bounded linear operator for each $x \in X$. Let the function $x \to A(x)$ be norm-to-norm continuous at x_0. If $A(x_0)$ is invertible then*

$$x \to A^{-1}(x)$$

is continuous at x_0.

Next is a precursor to Ekeland Variational Principle, see [1, Chapter 5, Section 1]. Of course, it easily follows from Ekeland Variational Principle itself, see e.g. [5, Basic Lemma]. See also the comments concerning the "Basic Lemma" on [5, p. 93]. Here we present a proof based on what is called in these comments "simple iteration".

Lemma 3. *Let X be a Banach space and let $\mu : X \to \mathbb{R}^+ \cup \{\infty\}$ be lower semicontinuous and such that for some $r > 0$*

$$\forall x:\ 0 < \mu(x) < \infty \Rightarrow \exists y:\ \mu(y) < \mu(x) - r\|y - x\|.$$

Then for each $x \in \operatorname{dom} \mu$ *there is* $y \in X$ *such that*

$$\mu(y) = 0 \text{ and } r\|y - x\| \leq \mu(x).$$

Proof. Fix $x_0 \in \operatorname{dom} \mu$ such that $\mu(x_0) > 0$. Let $x_1, x_2, \ldots, x_n$ be already chosen in the following way.

Set

$$\nu_n = \sup\{\|x - x_n\| : \ \mu(x) < \mu(x_n) - r\|x - x_n\|\}. \tag{3}$$

We are given that the set in the right hand side is nonempty, so $\nu_n > 0$. Also, since $\mu \geq 0$, we have that $\nu_n \leq \mu(x_n)/r < \infty$.

Choose a x_{n+1} such that

$$\mu(x_{n+1}) < \mu(x_n) - r\|x_{n+1} - x_n\| \text{ and } \|x_{n+1} - x_n\| > \nu_n/2. \tag{4}$$

Note that

$$\|x_{n+1} - x_0\| \leq \sum_{i=0}^{n} \|x_{i+1} - x_i\| \leq \sum_{i=0}^{n} (\mu(x_i) - \mu(x_{i+1}))/r \leq \mu(x_0)/r.$$

If $\mu(x_{n+1}) = 0$, we are done. If not, we continue by induction.

If we would end up with an infinite sequence $(x_n)_0^\infty$, then from the above inequality $\sum_{i=0}^{\infty} \|x_{i+1} - x_i\| \leq \mu(x_0)/r$, so $x_n \to \bar{x}$ as $n \to \infty$ and $\|\bar{x} - x_0\| \leq \mu(x_0)/r$. From (4) it follows that $\nu_n \to 0$.

If $\mu(\bar{x}) > 0$ then we can find $\bar{y}$ such that

$$\mu(\bar{y}) < \mu(\bar{x}) - r\|\bar{y} - \bar{x}\|. \tag{5}$$

Since μ is lower semicontinuous, we will have for all n large enough $\mu(\bar{y}) < \mu(x_n) - r\|\bar{y} - x_n\|$. Hence, see (3), $\nu_n \geq \|\bar{y} - x_n\|$ for all n large enough. Since $\nu_n \to 0$, we get that $\bar{y} = \bar{x}$ which contradicts (5).

So, $\mu(\bar{x}) = 0$ and we are done. □

3. Right inverse à la Ekeland & Séré.

3. Right inverse à la Ekeland & Séré. The following is what distinguishes our proof of Hadamard Theorem.

Proposition 1. *Let $f \in C^1$, $f'(x)$ be invertible for all x and let f satisfy* (1)*. Let $K \subset X$ be compact. Then f has a continuous right inverse on K, that is, there is a continuous $g : K \to X$ such that*

$$f(g(x)) = x, \quad \forall x \in K.$$

Moreover, if $f(0) = 0 \in K$ then there is a continuous right inverse g of f on K that satisfies

$$g(0) = 0.$$

Proof. Let $C(K, X)$ be the space of all continuous functions from K to X. It is clear that when equipped with the norm

$$\|g\|_\infty := \max_{y \in K} \|g(y)\|$$

it is a Banach space.

Consider the following measure

$$\mu : C(K, X) \to \mathbb{R}^+$$

of how much a given function g differs from a right inverse of f:

$$\mu(g) := \max_{y \in K} \|f(g(y)) - y\|.$$

It is clear that μ is lower semicontinuous. (It is easy to check that it is continuous but we do not need this.)

The claim is that there exists g such that $\mu(g) = 0$.

In order to check the condition of Lemma 3, fix $\hat{g} \in C(K, X)$ such that

$$\mu(\hat{g}) > 0.$$

Set $u : K \to X$ as

$$u(y) := y - f(\hat{g}(y)).$$

By definition,

$$\mu(\hat{g}) = \|u\|_\infty.$$

So, u is not identically equal to zero, because $\mu(\hat{g}) > 0$.

Put

$$w(y) := [f'(\hat{g}(y))]^{-1}u(y), \quad \forall y \in K.$$

Because $x \to f'(x)$ is continuous, from Lemma 2 it follows that

$$y \to [f'(\hat{g}(y))]^{-1}$$

is norm-to-norm continuous, so $w \in C(K, X)$.

Therefore, for $t > 0$

$$g_t := \hat{g} + tw \in C(K, X).$$

Note for future reference that from (1) it follows that $\|w\|_\infty \le M\|u\|_\infty$, that is

$$\|w\|_\infty \le M\mu(\hat{g}). \tag{6}$$

Our next aim is to estimate $\mu(g_t)$. By definition

$$\mu(g_t) := \max_{y \in K} \|f(g_t(y)) - y\|.$$

For $y \in K$ define $\varphi_y : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\varphi_y(t) := \|f(g_t(y)) - y\|,$$

hence

$$\mu(g_t) := \max_{y \in K} \varphi_y(t). \tag{7}$$

Because the set $\hat{g}(K)$ is compact, as well as the set $w(K)$, from Lemma 1 it follows that

$$\max_{y \in K} \|f(\hat{g}(y) + tw(y)) - f(\hat{g}(y)) - tf'(\hat{g}(y))w(y)\| = \alpha(t)t,$$

where $\alpha(t) \to 0$ as $t \to 0$. But

$$f'(\hat{g}(y))w(y) = f'(\hat{g}(y))[f'(\hat{g}(y))]^{-1}u(y) = u(y),$$

so

$$\|f(g_t(y)) - f(\hat{g}(y)) - tu(y)\|_\infty = \alpha(t)t.$$

Therefore, for any $y \in K$

$$\begin{aligned} \varphi_y(t) &= \|f(g_t(y)) - y\| \\ &\le \|f(\hat{g}(y)) + tu(y) - y\| + \|f(g_t(y)) - f(\hat{g}(y)) - tu(y)\| \\ &= \|(t-1)u(y)\| + \alpha(t)t. \end{aligned}$$

Since $\varphi_y(0) = \|u(y)\|$ we have that for small t

$$\varphi_y(t) \le (1-t)\varphi_y(0) + \alpha(t)t.$$

Taking a maximum over $y \in K$, see (7), we get

$$\mu(g_t) \le (1-t)\mu(g_0) + \alpha(t)t,$$

or, in other words,

$$\mu(\hat{g} + tw) \le \mu(\hat{g}) - t\mu(\hat{g}) + \alpha(t)t. \tag{8}$$

Since $\mu(\hat{g}) > 0$ and $\alpha(t) \to 0$, there exists $\delta > 0$ such that $|\alpha(t)| < \mu(\hat{g})/2$ for $t \in (0, \delta)$. So,

$$\mu(\hat{g} + tw) < \mu(\hat{g}) - (t/2)\mu(\hat{g}), \quad \forall t \in (0, \delta).$$

From (6) we have $\mu(\hat{g}) \ge (1/M)\|w\|_\infty$, whence we deduce

$$\mu(\hat{g} + tw) < \mu(\hat{g}) - (1/2M)\|tw\|_\infty, \quad \forall t \in (0, \delta),$$

and we can apply Lemma 3 with $r = 1/2M$, $x = \hat{g}$ and $y = \hat{g} + (\delta/2)w$, to conclude that μ vanishes somewhere.

If $f(0) = 0 \in K$ then we can modify the above by considering instead of $C(K, X)$ the Banach space of continuous $g : K \to X$ such that $g(0) = 0$. It is clear that in this case $u(0) = w(0) = 0$ and everything else works in the same way. □

4. Proof of Theorem 1. It is enough to show that f is bijective.

Let $y \in X$ be arbitrary and set $K = \{y\}$. From Proposition 1 it follows that there is $x = g(y)$ such that $f(x) = y$. So, f is surjective, i.e. $f(X) = X$.

Let $a, b \in X$ be such that $f(a) = f(b)$. By considering instead of f the function

$$x \to f(b - x) - f(b)$$

we can assume without loss of generality that

$$b = 0 \text{ and } f(0) = 0.$$

Then

$$f(a) = 0.$$

Set

$$K := f([0, a]).$$

Since f is continuous, K is compact. From Proposition 1 there is a continuous

$$g : K \to X, \text{ such that } g(0) = 0 \text{ and } f(g(y)) = y, \quad \forall y \in K.$$

Consider

$$I := \{t \in [0, 1] : \ g(f(ta)) = ta\}.$$

Obviously, $0 \in I$, because $g(0) = 0$. Due to the continuity of g and f the set I is closed and, therefore, compact. Let

$$\bar{t} := \max\{t : t \in I\}.$$

Assume that $\bar{t} < 1$. By the local Inverse Function Theorem, see Theorem 2, there are $\delta, \varepsilon > 0$ such that for each $y \in X$ such that

$$\|y - f(\bar{t}a)\| < \varepsilon$$

there is unique $x \in X$ such that $\|x - \bar{t}a\| < \delta$ and

$$f(x) = y.$$

From the continuity of f there is $\beta > 0$ such that for all $t \in (\bar{t}, \bar{t} + \beta) \subset (0, 1)$ we have $\|ta - \bar{t}a\| < \delta$, $\|f(ta) - f(\bar{t}a)\| < \varepsilon$.

Hence, for $t \in (\bar{t}, \bar{t} + \beta)$, ta is the only solution of $f(\cdot) = f(ta)$.

Since $f(g(y)) = y$ for all $y \in K$ and since $f(ta) \in K$, we have that $f(g(f(ta))) = f(ta)$.

From the uniqueness of the solution to $f(\cdot) = f(ta)$ we obtain that $g(f(ta)) = ta$ for all $t \in (\bar{t}, \bar{t} + \beta)$ which contradicts the definition of $\bar{t}$.

So, $\bar{t} = 1$ meaning that $g(f(a)) = a$. But $f(a) = 0$. Since $g(0) = 0$, it follows that $a = 0$.

We have proved that if $f(a) = f(b)$ then $a = b$, so f is injective.

In Memoriam. Asen L. Dontchev has many significant contributions in Mathematics, and in particular in the Variational Analysis, Optimization and Optimal Control. At the same time, he considered the spread of the knowledge in these areas as important as the generation of new results and their further applications. He had a natural taste to good Mathematics and was capable to transform the theoretical progress into practical impact with impressive lightness and elegance. Asen always had a spring of mathematical ideas and attracted others in researching tasks he considered important and promising. With all this, and not only, he will stay in our harts and memories.

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Milen Ivanov
Radiant Life Technologies Ltd.
Nicosia, Cyprus
e-mail: `milen@radiant-life-technologies.com`

Nadia Zlateva
Sofia University
Faculty of Mathematics and Informatics
5 James Bourchier Blvd
1164 Sofia, Bulgaria
e-mail: `zlateva@fmi.uni-sofia.bg`

Received October 29, 2022
Accepted May 5, 2023