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ON SEQUENCES WHICH ARE NOT UNIFORMLY CONVERGING ON ANY OPEN SUBSET*

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Dedicated to the memory of Asen L. Dontchev

ABSTRACT. We consider the property of nonuniform convergence to 0 of a sequence of functions on any open subset of a metric space. We consider three examples with respect to three different characteristics. Next we show that the three characteristics cannot be present simultaneously. For this purpose we introduce the so-called *height* function, which we use to quantify how far is a sequence of functions from satisfying any of the third characteristic. Moreover, we study properties of the height function and its relation to uniform convergence. Finally, we show that this quantification is precise.

1. Introduction. In this note we provide three examples of sequences of functions pointwise converging almost everywhere on $[0, 1]$ to the constant zero function, but not converging uniformly on any subinterval of $[0, 1]$. The first

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example consists of a sequence which converges pointwise everywhere, but the functions are not continuous (each function has jump discontinuity at one point). The second example consists of continuous functions, however the convergence is almost everywhere. The third, and least trivial example, consists of continuous functions and the convergence is everywhere. This example is moreover modified so that nor the sequence, nor any of its subsequences converge uniformly on any subinterval. Finding such sequence has been proposed as a problem 4.112 of a book by Biler and Witkowski, [1]. The solution in the book proposes another sequence of functions, which in our opinion is less constructive than the one we present. Recall that according to Egorov's theorem there exist sets with measure arbitrarily close to 1 on which the sequence converges uniformly. However, these sets need not be union of intervals, and in fact may not contain any interval.

We observe that the first and second example satisfy a strong form of the negation of uniform convergence on some subinterval – which we call property $\mathcal{SN}$. We go on to show that continuity, pointwise convergence everywhere and the property $\mathcal{SN}$ could not be satisfied simultaneously. In order to do this we introduce a so called height function associated with a sequence. We formulate results in terms of the upper level sets of the height function, measuring how far is a sequence of functions which consists of continuous functions and which converges pointwise everywhere, from satisfying the third property – $\mathcal{SN}$. More concretely, we show that the upper level sets should be nowhere dense. Finally we show that this result is precise, in the sense that for every nowhere dense set, there exists a sequence, whose height function has this set as upper level set.

2. Notation and preliminary facts. We will denote the set of positive integers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. When we use the term “almost all” and “almost everywhere” we will refer to sets with full Lebesgue measure. We will denote Lebesgue measure with λ . The support of a function f , denoted by $\text{supp}(f)$, stands for the set where the function is nonzero, i.e. $\text{supp}(f) = \{x \mid f(x) \neq 0\}$. The closure of a set A is denoted by $\overline{A}$, and its interior by $\text{int } A$. When we talk about metric spaces, the open ball centered at x with radius r is denoted $\mathbf{B}_r(x)$. The closed ball is denoted by $\overline{\mathbf{B}}_r(x)$. First we state the famous theorem of Egorov, which in some sense, inspired our investigations.

Theorem 2.1 (Egorov's theorem). *Let $\{f_n\}_{n \geq 1}$ be a sequence of measurable functions defined on $[0, 1]$ such that for almost all $x \in [0, 1]$ holds that $f_n(x) \rightarrow 0$. Then for every $\varepsilon > 0$ there exists a measurable set $B \subset [0, 1]$ such*

that $\lambda(B) > 1 - \varepsilon$ and the sequence of functions converges uniformly to 0 on B .

Next we state the famous Borel-Cantelli Lemma.

Theorem 2.2 (Theorem 1.41, [3]). *Let $\{A_n\}_{n \geq 1}$ be a sequence of measurable sets in some measure space (X, μ) . Let*

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty.$$

Then the set

$$\{x \in X \mid x \in A_n \text{ for infinitely many } n\}$$

has measure 0.

We will also need the following notorious theorem from mathematical analysis and topology.

Theorem 2.3 (Baire category theorem, [2], Chapter 6, Theorem 34). *Let $(\mathbf{X}, \rho)$ be a complete metric space. If $\{U_n\}_{n \geq 1}$ is a sequence of dense, open subsets of $\mathbf{X}$, then $\bigcap_{n \geq 1} U_n$ is dense in $\mathbf{X}$.*

Remark 2.4. Recall that countable intersections of open sets are called G_δ sets. One could show using Baire theorem, that any dense and G_δ subset of complete metric space is uncountable and even has the cardinality of the continuum. Dense and G_δ sets, also called comeagre, are considered to be large in a topological sense. For example, countable intersection of dense and G_δ sets is not only nonempty, but dense and G_δ as well.

3. Three examples. In this section we provide three examples of sequences of functions defined on $[0, 1]$ which converge pointwise almost everywhere to 0, yet they do not converge uniformly on any subinterval of $[0, 1]$. We also consider some additional properties of each sequence. In each example we will construct sets with almost full measure on which the convergence is uniform (whose existence is asserted by the theorem of Egorov).

3.1. First example. Let the sequence $\{r_n\}_{n \geq 1}$ stand for an enumeration of the rational numbers contained in $[0, 1]$ and define the functions

$$(1) \quad \delta_n(x) = \begin{cases} 1, & x = r_n \\ 0, & \text{otherwise} \end{cases}$$

Theorem 3.1. *The sequence $\{\delta_n\}_{n \geq 1}$ defined by (1) converges pointwise everywhere on $[0, 1]$ to the constant zero function. However, the convergence is not uniform on any subinterval of $[0, 1]$.*

Proof. For every fixed $x \in [0, 1]$, the sequence of numbers $\{\delta_n(x)\}_{n \geq 1}$ may contain at most one nonzero element. Thus, $\lim_{n \rightarrow \infty} \delta_n(x) = 0$ for any $x \in [0, 1]$. On the other hand, the sequence does not converge uniformly on any interval. Indeed, let $0 \leq p < q \leq 1$. Then there exists a subsequence $\{r_{n_k}\}_{k \geq 1}$ such that $r_{n_k} \in [p, q]$ for all k . Thus, $\sup_{[p, q]} \delta_{n_k} = 1$ for all k so the subsequence $\{\delta_{n_k}\}_{k \geq 1}$ does not converge uniformly to 0 on $[p, q]$. Hence the whole sequence does not converge uniformly on $[p, q]$. $\square$

Remark 3.2. Observe that removing the null set $\{r_n\}_{n \geq 1}$ (which is even countable), the sequence of functions $\{\delta_n\}_{n \geq 1}$ converges uniformly on the complement $C := [0, 1] \setminus \{r_n\}_{n \geq 1}$. The existence of such sets does not follow by Theorem 2.1. From it follows the uniform convergence on a set B with measure arbitrarily close to 1, while in Example 3.1, we have uniform convergence on the set C with measure which is exactly 1. We shall see later that such set need not exist in general.

3.2. Second example. Here we provide an example of a sequence of continuous functions.

Let us define the sequence of functions $\{g_n\}_{n \geq 1}$ as

$$(2) \quad g_n(x) = \begin{cases} 1 + n^2(x - r_n), & r_n - \frac{1}{n^2} \leq x < r_n \\ 1 - n^2(x - r_n), & r_n \leq x \leq r_n + \frac{1}{n^2} \\ 0, & \text{otherwise} \end{cases}.$$

Theorem 3.3. *The sequence of continuous functions $\{g_n\}_{n \geq 1}$ defined by (2) converges pointwise almost everywhere on $[0, 1]$ to the constant zero function. The convergence is not uniform on any subinterval of $[0, 1]$.*

Proof. Continuity follows directly from inspecting the left and right limits of g_n at $r_n - \frac{1}{n^2}, r_n$ and $r_n + \frac{1}{n^2}$. Clearly

$$\text{supp}(g_n) = \left(r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2} \right).$$

Set $A_n = \text{supp}(g_n) \cap [0, 1]$. Consider the set

$$(3) \quad \begin{aligned} D &:= \{x \in [0, 1] \mid x \in A_n \text{ for finitely many } n\} \\ &= \left\{x \in [0, 1] \mid |x - r_n| < \frac{1}{n^2} \text{ for finitely many } n\right\} \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} \lambda(A_n) \leq \sum_{n=1}^{\infty} \frac{2}{n^2} < \infty,$$

Theorem 2.2 implies that $\lambda([0, 1] \setminus D) = 0$. Hence $\lambda(D) = 1$.

For any $x \in D$ it holds that $\lim_{n \rightarrow \infty} g_n(x) = 0$ since $g_n(x) \neq 0$ for at most finitely many n . Thus the sequence $\{g_n\}_{n \geq 1}$ converges pointwise almost everywhere to 0.

Observe that $g_n(r_n) = 1$ for all n and analogously to the proof of Theorem 3.1 we obtain that the sequence does not converge uniformly on any subinterval of $[0, 1]$. $\square$

Observe that Lemma 4.8 implies that although D has measure 1, $[0, 1] \setminus D \neq \emptyset$. Arguing as in Theorem 4.9 implies that $\{g_n\}_{n \geq 1}$ converges pointwise almost everywhere but not everywhere.

Now we construct sets with almost full measure on which the convergence of $\{g_n\}_{n \geq 1}$ is uniform.

Proposition 3.4. *Let $\{g_n\}_{n \geq 1}$ be defined by (2) and $D \subseteq [0, 1]$ be as in (3). For $d \in \mathbb{N}$ define*

$$B_d = \left\{x \in D \mid |x - r_n| \geq \frac{1}{n^2} \forall n \geq d\right\}.$$

Then $\lim_{d \rightarrow \infty} \lambda(B_d) = 1$ and $\{g_n\}_{n \geq 1}$ converges uniformly on B_d for all $d \in \mathbb{N}$.

Proof. Clearly $B_d \subseteq B_{d+1}$ and $\bigcup_{d=1}^{\infty} B_d = D$. Using the continuity properties of Lebesgue measure we obtain that for every $\varepsilon > 0$ there exists d such that $\lambda(B_d) > 1 - \varepsilon$. Note that $B_d = D \setminus \bigcup_{n \geq d} \text{supp}(g_n)$. Hence for every $n \geq d$ and every $x \in B_d$ one has $g_n(x) = 0$. Thus we obtain that $\{g_n\}_{n \geq 1}$ converges to 0 uniformly on B_d . $\square$

It is natural to ask whether there exists a set $\mathfrak{D} \subset [0, 1]$ such that $\lambda(\mathfrak{D}) = 1$ and the sequence $\{g_n\}_{n \geq 1}$ converges uniformly on $\mathfrak{D}$ – see Remark 3.2. It turns out that such set does not exist. The latter follows from

Proposition 3.5. *Let $\{f_n\}_{n \geq 1}$ be a sequence of continuous functions which does not converge uniformly on $[0, 1]$ to the constant zero function. Let G be a dense subset of $[0, 1]$. Then the sequence does not converge uniformly to 0 on G either.*

Proof. Since $\{f_n\}_{n \geq 1}$ does not converge uniformly to 0, there exist $\varepsilon > 0$ and a subsequence $\{f_{n_k}\}_{k \geq 1}$ such that $\sup_{[0,1]} |f_{n_k}| > \varepsilon$ for every k . Observe that since G is dense in $[0, 1]$ and since each of f_{n_k} is continuous, we have

$$\sup_G |f_{n_k}| = \sup_{[0,1]} |f_{n_k}| > \varepsilon$$

Indeed, by continuity of the functions and the Weierstrass theorem, there exists a point at which the supremum is attained. This point could be approximated with elements of G , since G is dense. Using the sequential characterization of continuity, this means that the supremum over G is the same as the supremum over the whole interval $[0, 1]$. This shows that the whole sequence does not converge uniformly to 0 on G as well. $\square$

It remains to observe that every set of measure 1 in $[0, 1]$ is dense in $[0, 1]$ and apply the preceding proposition to $\{g_n\}_{n \geq 1}$.

3.3. Third example. The function $\gcd(\cdot, \cdot)$ applied to positive integers will denote the greatest common divisors of the arguments. For $k, n \in \mathbb{N}$, let us denote $b_{k,n} = \frac{1}{2^n} + \frac{k}{2^n}$ and define

$$\Lambda_{k,n}(x) = \begin{cases} \frac{\delta(2^n, k)}{2^n} (1 + 4^n (x - b_{k,n})), & b_{k,n} - \frac{1}{4^n} \leq x \leq b_{k,n} \\ \frac{\delta(2^n, k)}{2^n} (1 - 4^n (x - b_{k,n})), & b_{k,n} < x \leq b_{k,n} + \frac{1}{4^n} \\ 0, & \text{else} \end{cases}$$

Now for $n \in \mathbb{N}$ define

$$(4) \quad h_n(x) := \sum_{k=1}^{2^n-1} \Lambda_{k,n}(x).$$

Theorem 3.6. *The sequence $\{h_n\}_{n \geq 1}$ consists of continuous functions defined on $[0, 1]$ converging pointwise everywhere to 0. Neither the sequence, nor any of its subsequences converges uniformly to 0 on some subinterval of $[0, 1]$.*

Proof. Clearly $\Lambda_{k,n}$ is continuous and attains nonnegative values only. Next we show the following claims:

- (a) if $k \neq l$ with $1 \leq k \leq 2^n - 1$ and $1 \leq l \leq 2^n - 1$ then $\text{supp}(\Lambda_{k,n}) \cap \text{supp}(\Lambda_{l,n}) = \emptyset$;
- (b) if $m \neq n$, $m \geq 2$, $n \geq 2$, $1 \leq k \leq 2^n - 1$, $1 \leq l \leq 2^m - 1$ and $\frac{\delta(2^n, k)}{2^n} = \frac{\delta(2^m, l)}{2^m}$ then $\text{supp}(\Lambda_{k,n}) \cap \text{supp}(\Lambda_{l,m}) = \emptyset$.
- (c) if $1 \leq k \leq 2^n - 1$ then $\sup_{x \in [0,1]} \Lambda_{k,n}(x) = \frac{\delta(2^n, k)}{2^n}$.

The first claim holds because of the inequality $b_{k+1,n} - \frac{1}{4^n} > b_{k,n} + \frac{1}{4^n}$.

For the second claim, since $k < 2^n$ and $l < 2^m$ we may assume that $\frac{\delta(2^n, k)}{2^n} = \frac{\delta(2^m, l)}{2^m} = \frac{1}{2^s}$ for some $s \in [1, \min(n, m)]$. Thus $k = 2^{n-s}k_0$ and $l = 2^{m-s}l_0$, with k_0 and l_0 odd integers. Assume without loss of generality that $b_{k,n} \geq b_{l,m}$. This is equivalent to

$$(5) \quad (k_0 - l_0)2^{m+n} + 2^{m+s} - 2^{n+s} \geq 0,$$

which combined with $n \geq s$ implies

$$k_0 - l_0 \geq \frac{1}{2^{m-s}} - \frac{1}{2^{n-s}} \geq -1.$$

Since k_0 and l_0 are odd integers, this implies $k_0 \geq l_0$.

The fact that $\text{supp}(\Lambda_{k,n}) \cap \text{supp}(\Lambda_{l,m}) = \emptyset$ would follow from the inequality

$$(6) \quad b_{k,n} - \frac{1}{4^n} > b_{l,m} + \frac{1}{4^m}.$$

Consider the case $k_0 = l_0$. From (5) we obtain $m \geq n$ and since $m \neq n$, we see that $n \leq m - 1$. The inequality (6) in this case reduces to

$$2^{2m+n} > 2^{m+2n} + 2^{2m} + 2^{2n}.$$

This is equivalent to

$$\frac{2^{2m}}{2^m + 1} > \frac{2^{2n}}{2^n - 1}$$

and since the right hand side is increasing it suffices to check the inequality only for $n = m - 1$, in which case it reduces to $2^m > 5$ which is true since $m > n \geq 2$.

Now consider the case $k_0 > l_0$. Since both are odd integers, we have $k_0 - l_0 \geq 2$. The inequality (6) takes the form

$$k_0 2^{2m+2n-s} - l_0 2^{2m+2n-s} + 2^{2m+n} - 2^{m+2n} - 2^{2m} - 2^{2n} > 0.$$

In view of $k_0 - l_0 \geq 2$ and $s \leq m$, this follows from

$$\begin{aligned} & 2 \cdot 2^{2m+2n-s} + 2^{2m+n} - 2^{m+2n} - 2^{2m} - 2^{2n} = \\ & 2^{2m}(2^n - 1) + 2^{2n}(2^m(2^{m-s} - 1) + 2^{2m-s} - 1) > 0 \end{aligned}$$

The third claim is clear, once one observes that the maximal value of $\Lambda_{k,n}$ is achieved at $b_{k,n}$.

Now we turn to the properties of the sequence $\{h_n\}_{n \geq 1}$. Clearly all the functions are continuous and nonnegative, since each $\Lambda_{k,n}$ is continuous and non-negative. Next we show that for every $x \in [0, 1]$, $\lim_{n \rightarrow \infty} h_n(x) = 0$. To this end fix $x_0 \in [0, 1]$. If $x_0 \in \text{supp}(h_n)$ for finitely many n , we are done. Otherwise, suppose that for some subsequence $\{h_{n_j}\}_{j \geq 1}$ we have $x_0 \in \bigcap_{j \geq 1} \text{supp}(h_{n_j})$. According to

(a) this implies that for each j there is unique k_j such that $x_0 \in \text{supp}(\Lambda_{k_j, n_j})$ and $h_{n_j}(x_0) = \Lambda_{k_j, n_j}(x_0)$. By (b) we obtain that for $i \neq j$

$$\frac{\delta(2^{n_j}, k_j)}{2^{n_j}} \neq \frac{\delta(2^{n_i}, k_i)}{2^{n_i}},$$

hence $\left\{ \frac{\delta(2^{n_j}, k_j)}{2^{n_j}} \right\}_{j \geq 1}$ is a sequence of distinct negative powers of 2, hence is convergent to 0. Using that and applying (c) we obtain

$$\limsup_{n \rightarrow \infty} h_n(x_0) = \limsup_{j \rightarrow \infty} h_{n_j}(x_0) = \limsup_{j \rightarrow \infty} \Lambda_{k_j, n_j}(x_0) \leq \lim_{j \rightarrow \infty} \frac{\delta(2^{n_j}, k_j)}{2^{n_j}} = 0.$$

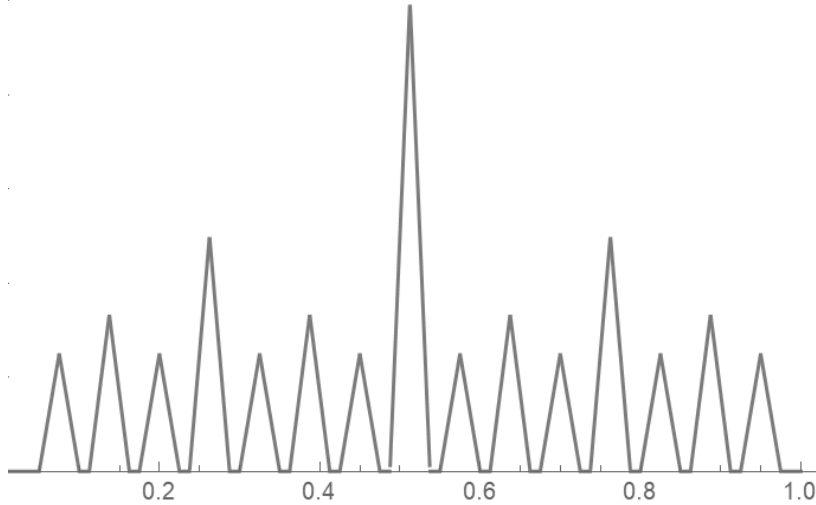
Now we show that for any subinterval of $[0, 1]$, no subsequence of $\{h_n\}_{n \geq 1}$ converges uniformly to 0 on this subinterval. To this end, fix a subinterval

(p, q) , $p < q$. Since the rationals of the form $\frac{k}{2^n}$ are dense in $[0, 1]$, there exists n_0 and an odd integer k_0 such that $\frac{k_0}{2^{n_0}} \in (p, q)$. Consider $\gamma_l := \frac{1 + 2^l k_0}{2^{n_0+l}}$. Since $\gamma_l \xrightarrow{l \rightarrow \infty} \frac{k_0}{2^{n_0}}$ there exists l_0 , such that for $l \geq l_0$ it holds that $\gamma_l \in (a, b)$. Observe that

$$h_{n_0+l}(\gamma_l) = \Lambda_{2^l k_0, n_0+l}(\gamma_l) = \frac{1}{2^{n_0}}.$$

Thus for $n \geq n_0 + l_0$ it holds that $\sup_{x \in (p, q)} h_n(x) \geq \frac{1}{2^{n_0}}$, hence no subsequence of $\{h_n\}_{n \geq 1}$ converges uniformly to 0 on (p, q) . $\square$

Here is the graph of h_4 (scaled appropriately, so that the graph is better visualised)



Next we show how to construct sets with almost full measure on which the convergence of $\{h_n\}_{n \geq 1}$ is uniform.

Proposition 3.7. *For $\varepsilon \in (0, 1)$ define*

$$B_\varepsilon = [0, 1] \setminus \left(\bigcup_{n \geq 1} \bigcup_{1 \leq k \leq 2^n - 1} \left(b_{k,n} - \frac{\varepsilon}{3^{n+1}}, b_{k,n} + \frac{\varepsilon}{3^{n+1}} \right) \right).$$

Then $\lambda(B_\varepsilon) \geq 1 - \varepsilon$ and the sequence $\{h_n\}_{n \geq 1}$ converges uniformly on B_ε .

Proof. Observe that

$$\lambda([0, 1] \setminus B_\varepsilon) \leq \sum_{n=1}^{\infty} (2^n - 1) \frac{2\varepsilon}{3^{n+1}} = \varepsilon.$$

Thus $\lambda(B_\varepsilon) \geq 1 - \varepsilon$. Now fix n_0 such that for $n \geq n_0$ it holds that $\frac{\varepsilon}{3^{n+1}} > \frac{1}{4^n}$. Thus for $n \geq n_0$ and $k < 2^n - 1$ we obtain that $\text{supp}(\Lambda_{k,n}) \cap B_\varepsilon = \emptyset$, and consequently $\text{supp}(h_n) \cap B_\varepsilon = \emptyset$. Hence for $n \geq n_0$ the function h_n vanishes on B_ε and so the sequence $\{h_n\}_{n \geq 1}$ converges uniformly to 0 on B_ε . $\square$

4. The height function and $\mathcal{SN}$ property. In this section $\mathbf{X}$ will be a metric space with a metric ρ . We will consider real-valued functions defined on a subset J of $\mathbf{X}$.

In the previous section we constructed three sequences of functions pointwise converging almost everywhere to 0 but not converging uniformly on any subinterval. More precisely, we showed the negation of uniform convergence on any subinterval:

for any $[p, q] \subseteq [0, 1]$ there exists $\varepsilon > 0$ such that for any n_0 there exists $n \geq n_0$ and $x \in [p, q]$ such that $|f_n(x)| \geq \varepsilon$.

However, inspecting the first two examples, we observe that the sequences actually satisfy a stronger property than this negation. We state it in the next

Definition 4.1. *We say that the sequence of functions $\{f_n\}_{n \geq 1}$ defined on a subset J of the metric space $\mathbf{X}$, with J having nonempty interior, satisfies property $\mathcal{SN}$ (strongly nonuniform), if there exists $\varepsilon > 0$ such that for any open subset $U \subseteq J$ and any n_0 there exists $n \geq n_0$ and $x \in U$ such that $|f_n(x)| \geq \varepsilon$.*

Additional interesting properties are continuity of the sequence of functions, and pointwise convergence everywhere (in contrast with merely almost everywhere). Here is a table showing which properties of interest are true for the sequences from the three examples.

Property \ Sequence	$\{\delta_n\}_{n \geq 1}$	$\{g_n\}_{n \geq 1}$	$\{h_n\}_{n \geq 1}$
Continuity of the functions	No	Yes	Yes
Pointwise convergence everywhere	Yes	No	Yes
Property $\mathcal{SN}$	Yes	Yes	No

It is natural to ask – is it possible to construct a sequence satisfying all three listed properties?

This turns out to be impossible. We will show that in the presence of $\mathcal{SN}$ and continuity of the functions, one may not preserve their pointwise convergence everywhere.

We begin with a definition of a function associated with a sequence of functions. We will be interested in its upper level sets. They help us to quantify, how far is a sequence of functions from satisfying the property $\mathcal{SN}$. In the case of a sequence of continuous functions converging everywhere to 0 we find an upper "bound" and after that we show that it is precise.

Definition 4.2. *Given a sequence of functions $\{f_n\}_{n \geq 1}$ defined on some subset J of the metric space $\mathbf{X}$, we define the height function $\mathfrak{h}$ (with respect to the sequence $\{f_n\}_{n \geq 1}$) at $x_0 \in J$ as*

$$\mathfrak{h}(x_0) = \inf_{\delta > 0} \limsup_{n \rightarrow \infty} \sup_{x \in \mathbf{B}_\delta(x_0) \cap J} |f_n(x)|.$$

The function could attain value $+\infty$. As $\limsup_{n \rightarrow \infty} \sup_{x \in \mathbf{B}_\delta(x_0) \cap J} |f_n(x)|$ is increasing with respect to δ , the infimum in the definition of $\mathfrak{h}$ could be replaced with limit as $\delta \rightarrow 0$. For each $m \in \mathbb{N}$ define the set

$$T_m := \left\{ x \in J \mid \mathfrak{h}(x) \geq \frac{1}{m} \right\}.$$

Clearly $T_k \subseteq T_m$ whenever $k \leq m$. Basic property of these sets is contained in the following

Proposition 4.3. *For every $m \in \mathbb{N}$ the set T_m is relatively closed. In particular it is closed subset of $\mathbf{X}$ if J is closed itself.*

Proof. Let $\{x_n\}_{n \geq 1} \subset T_m$ be a sequence converging to some $\bar{x}$ in J and $\varepsilon > 0$ be arbitrary. Thus for every x_n we can find $k_n \geq n$ and $y_n \in J$ such that $\rho(y_n, x_n) < \frac{1}{n}$ and $|f_{k_n}(y_n)| \geq \frac{1}{m} - \varepsilon$. Thus $y_n \rightarrow \bar{x}$ and so for any $\delta > 0$ it holds that $y_n \in \mathbf{B}_\delta(\bar{x})$ for large enough n . Since $k_n \rightarrow \infty$ we obtain that for any $\delta > 0$ it holds that

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbf{B}_\delta(\bar{x}) \cap J} |f_n(x)| \geq \frac{1}{m} - \varepsilon.$$

Since ε was arbitrary, we obtain $\bar{x} \in T_m$ as well. $\square$

The statement of this proposition essentially shows that the height function $\mathfrak{h}$ is upper semicontinuous. The relation of $\mathfrak{h}$ to the uniform convergence of the sequence $\{f_n\}_{n \geq 1}$ is established in the following

Proposition 4.4. *Let the sequence $\{f_n\}_{n \geq 1}$ be defined on $\mathbf{X}$. The sequence $\{f_n\}_{n \geq 1}$ converges uniformly to 0 on any compact subset of $\mathbf{X}$ if and only if $\mathfrak{h}(x) = 0$ for all $x \in \mathbf{X}$.*

Proof. Let $\mathfrak{h}(x_0) = \eta > 0$ for some $x_0 \in \mathbf{X}$. Then for any n there exists $k_n \geq n$ and $x_n \in \mathbf{B}_{1/n}(x_0)$, such that $|f_{k_n}(x_n)| \geq \frac{\eta}{2}$. Since $x_n \rightarrow x_0$, the set $K := \{x_j\}_{j \geq 0}$ is compact. However,

$$\limsup_{n \rightarrow \infty} \sup_{x \in K} |f_n(x)| \geq \limsup_{n \rightarrow \infty} \sup_{x \in K} |f_{k_n}(x)| \geq \frac{\eta}{2}.$$

Hence $\{f_n\}_{n \geq 1}$ does not converge uniformly on K .

Now let $\mathfrak{h}(z) = 0$ for all $z \in \mathbf{X}$ and let K be a compact subset of $\mathbf{X}$. Fix $\varepsilon > 0$. For a $z \in \mathbf{X}$, $\mathfrak{h}(z) = 0$ implies the existence of $\delta_z > 0$ and $n_z \in \mathbb{N}$ such that for all $n \geq n_z$ it holds that $\sup_{x \in \mathbf{B}_{\delta_z}(z)} |f_n(x)| < \varepsilon$. The family $\{\mathbf{B}_{\delta_z}(z) \mid z \in K\}$ is an open cover of K and since K is compact, there exist finitely many $z_1, z_2, \dots, z_k$ such that $K \subset \bigcup_{i=1}^k \mathbf{B}_{\delta_{z_i}}(z_i)$. Let $n_0 = \max\{n_{z_1}, \dots, n_{z_k}\}$. Thus for any $n \geq n_0$ and any $x \in K$ it holds that $|f_n(x)| < \varepsilon$, which proves uniform convergence on K . $\square$

In particular, if $\{f_n\}_{n \geq 1}$ converges uniformly on all of $\mathbf{X}$ then $\mathfrak{h}(x) = 0$ for all $x \in \mathbf{X}$.

Corollary 4.5. *If $\bigcup_{n \geq 1} T_n = \{x \mid \mathfrak{h}(x) > 0\}$ is dense in $\mathbf{X}$ then $\{f_n\}_{n \geq 1}$ does not converge uniformly to 0 on any open subset. The converse holds if $\mathbf{X}$ in addition is locally compact.*

Remark 4.6. The equivalence in Corollary 4.5 cannot be extended to metric spaces which are not locally compact, e.g. infinite-dimensional normed spaces. Consider the space of square-summable sequences – ℓ_2 and the functions f_n defined by $f_n(a) = a_n$ for any $a = \{a_i\}_{i \geq 1} \in \ell_2$. These functions are continuous and such that $\mathfrak{h}(x) = 0$ for all $x \in \ell_2$, in particular $\{f_n\}_{n \geq 1}$ converges uniformly to 0 on all compact sets. However, $\{f_n\}_{n \geq 1}$ does not converge uniformly on any open ball. This is related to the fact that the sequence of the standard orthonormal

basis in ℓ_2 is weakly convergent to 0, but not convergent in the norm topology.

Let us identify the sets T_n in the examples we presented in Section 3. We begin with the third example, i.e. with the sequence $\{h_n\}_{n \geq 1}$ where h_n is defined in (4). Clearly $\mathfrak{h}\left(\frac{k}{2^n}\right) = \frac{\gcd(k, 2^n)}{2^n}$.

Now let q be rational, whose denominator as an irreducible fraction is not a power of 2 or q is irrational. We will show that $\mathfrak{h}(q) = 0$. Assume that for some $m \in \mathbb{N}$, $\mathfrak{h}(q) > \frac{1}{2^{m-1}}$. Fix δ such that there does not exist a rational number with denominator 2^s for $s \leq m$ in $(q - \delta, q + \delta)$ – this is possible since there are finitely many rationals in $[0, 1]$ with denominator 2^s , $s \leq m$, and q is not any of them. Thus for any $n \in \mathbb{N}$ there exist $x_n \in (q - \delta/2, q + \delta/2)$ and $j_n \geq n$ such that $h_{j_n}(x_n) \geq \frac{1}{2^m}$. Thus for each n there exists k_n such that

$$(7) \quad \text{supp}(\Lambda_{k_n, j_n}) \cap \left(q - \frac{\delta}{2}, q + \frac{\delta}{2}\right) \neq \emptyset$$

and $\frac{\delta(2^{j_n}, k_n)}{2^{j_n}} \geq \frac{1}{2^m}$. The latter implies that $\frac{k_n}{2^{j_n}}$ is a number with denominator in lowest terms less than 2^m . This moreover implies that $\frac{k_n}{2^{j_n}} \notin (q - \delta, q + \delta)$ for all n by the choice of δ . However observe that

$$\text{supp}(\Lambda_{k_n, j_n}) \equiv \left(\frac{k_n}{2^{j_n}} + \frac{1}{2^{j_n}} - \frac{1}{4^{j_n}}, \frac{k_n}{2^{j_n}} + \frac{1}{2^{j_n}} + \frac{1}{4^{j_n}}\right).$$

In order for (7) to hold we must have $\frac{k_n}{2^{j_n}} \leq q - \delta$. However since $j_n \geq n$, for large enough n it holds that $\frac{1}{2^{j_n}} + \frac{1}{4^{j_n}} < \frac{\delta}{2}$. Thus

$$\frac{k_n}{2^{j_n}} + \frac{1}{2^{j_n}} + \frac{1}{4^{j_n}} < q - \delta + \frac{\delta}{2} = q - \frac{\delta}{2},$$

which contradicts (7).

This shows that for every n

$$T_{2^n} \equiv \left\{ \frac{1}{2^n}, \frac{2}{2^n}, \dots, \frac{2^n - 1}{2^n} \right\}.$$

To find T_n for the first and second example we establish the relation between the property $\mathcal{SN}$ and the upper level sets of the height function.

Proposition 4.7. *Let $\{f_n\}_{n \geq 1}$ be a sequence of functions defined on some closed set $J \subset \mathbf{X}$ with nonempty interior. Then $\{f_n\}_{n \geq 1}$ satisfies property $\mathcal{SN}$ on J if and only if there exists m such that $T_m \equiv J$.*

Proof. Let $\mathcal{SN}$ hold with $\varepsilon > 0$. Fix $m \geq \frac{1}{\varepsilon}$ and $x_0 \in J$. Then $\mathcal{SN}$ implies that for any k there exists $n_k \geq k$ and $x_k \in \mathbf{B}_{1/k}(x_0)$ with $|f_{n_k}(x_k)| \geq \varepsilon$. Fix k and observe that, by construction, $x_l \in \mathbf{B}_{1/k}(x_0)$ for all $l \geq k$, hence

$$\sup_{x \in \mathbf{B}_{1/k}(x_0)} |f_{n_l}(x)| \geq \varepsilon,$$

for all $l \geq k$. This implies that

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbf{B}_{1/k}(x_0)} |f_n(x)| \geq \varepsilon \geq \frac{1}{m}.$$

Since this holds for all k , we obtain $\mathfrak{h}(x_0) \geq \frac{1}{m}$. Since x_0 was arbitrary in J , we obtain $T_m \equiv J$.

Now assume that for some m it holds $T_m \equiv J$. Fix an open subset $U \subseteq J$ and $\mathbf{B}_\delta(x_0) \subset U$ for some x_0 and $\delta > 0$. Then

$$\mathfrak{h}(x_0) \leq \limsup_{n \rightarrow \infty} \sup_{x \in \mathbf{B}_\delta(x_0)} |f_n(x)| \leq \limsup_{n \rightarrow \infty} \sup_{x \in U} |f_n(x)|.$$

Hence for every n_0 there exists $n \geq n_0$ and $x \in U$ such that

$$|f_n(x)| \geq \frac{\mathfrak{h}(x_0)}{2} \geq \frac{1}{2m}.$$

Since U was arbitrary, we obtain that $\mathcal{SN}$ holds with $\varepsilon = \frac{1}{2m}$. $\square$

Since $\mathcal{SN}$ holds in both the first and second example with $\varepsilon = 1$ we obtain that $T_1 = [0, 1]$ in both examples.

So there is a significant difference between examples 1 and 2 from one side, and example 3 on the other side. In examples 1 and 2 we have $T_1 = [0, 1]$ – i.e. the whole interval; in example 3, T_n is a finite set for all n . In order to show to what extent this difference could be made lesser, we need the following basic consequence of Baire Category Theorem.

Lemma 4.8. *Let $\mathbf{X}$ be a separable complete metric space and $U \subset \mathbf{X}$ be open set. Let $\{x_n\}_{n \geq 1}$ be a dense sequence in U and $\{\delta_n\}_{n \geq 1}$ be a sequence of*

positive real numbers. Then the set of all $z \in U$ such that for infinitely many n it holds that

$$z \in \mathbf{B}_{\delta_n}(x_n)$$

contains a dense and G_δ subset of $\overline{U}$.

Proof. For $n \in \mathbb{N}$, let $D_n = \bigcup_{k \geq n} \mathbf{B}_{\delta_k}(x_k) \cap U$. These sets are clearly open. Moreover, for any n the set D_n is a dense subset of U . Indeed, removing finitely many elements from a dense subset of an open set still results in a dense subset. Applying Theorem 2.3, to the set $G := \bigcap_{n \geq 1} D_n$ (which is by definition G_δ) considered as a subset of the complete metric space $\overline{U}$, we obtain that G is dense in $\overline{U}$. Let us note that any $z \in G$ satisfies $z \in \mathbf{B}_{\delta_n}(x_n)$ for infinitely many n . $\square$

Theorem 4.9. Let $\mathbf{X}$ be a separable complete metric space and $\{f_n\}_{n \geq 1}$ be a sequence of continuous functions defined on $\mathbf{X}$, such that $\text{int } T_m \neq \emptyset$ for some $m \in \mathbb{N}$. Then the set H of all $z \in \mathbf{X}$ such that $\lim_{n \rightarrow \infty} f_n(z) \neq 0$ contains a dense and G_δ subset of $\overline{\text{int } T_m}$.

Proof. According to Proposition 4.7, property $\mathcal{SN}$ with $\varepsilon = 1/(2m)$ holds for the sequence $\{f_n\}_{n \geq 1}$ when restricted to $\overline{\text{int } T_m}$. Since $\mathbf{X}$ is separable, it is second countable, so let $\{U_n\}_{n \geq 1}$ consist of those sets in the countable base for the topology in $\mathbf{X}$, which are subsets of $\text{int } T_m$. According to $\mathcal{SN}$ for each n we can find $k_n \geq n$ and $x_n \in U_n$ such that $|f_{k_n}(x_n)| \geq \varepsilon$. Since f_{k_n} is continuous, we obtain $\delta_n > 0$ such that for any x with $\rho(x, x_n) < \delta_n$ it holds that $|f_{k_n}(x)| > \varepsilon/2$. Since any open subset of $\text{int } T_m$ contains U_n for some n , we obtain that x_n is dense in $\text{int } T_m$. According to Lemma 4.8 there exists a dense and G_δ subset G of $\overline{\text{int } T_m}$, such that for all $z \in G$ it holds that $\rho(z, x_n) < \delta_n$ for infinitely many n . Thus, if $z \in G$ it holds that $|f_{k_n}(z)| > \varepsilon/2$ for infinitely many n . Since $k_n \xrightarrow{n \rightarrow \infty} \infty$ we obtain $\lim_{n \rightarrow \infty} f_n(z) \neq 0$ for any $z \in G$. $\square$

Corollary 4.10. Let $\{f_n\}_{n \geq 1}$ be a sequence of continuous functions defined on the metric space $\mathbf{X}$, satisfying property $\mathcal{SN}$. Then the sequence $\{f_n\}_{n \geq 1}$ does not converge pointwise everywhere to 0. Moreover, the set H of points on which the sequence does not converge to 0 is uncountable and dense in $\mathbf{X}$.

Proof. According to Proposition 4.7, $\mathbf{X} \equiv T_m$ for some m . Then Theorem 4.9 implies that the set H contains a dense and G_δ subset. That H is uncountable follows from Remark 2.4. $\square$

On the other hand, with respect to the Lebesgue measure, the set H may not be large. Recall that in the second example we have a sequence of continuous functions for which $\mathcal{SN}$ holds and the corresponding set H (which in this example is a subset of $[0, 1] \setminus D$) has measure 0.

Corollary 4.11. *Let $\{f_n\}_{n \geq 1}$ be a sequence of continuous functions defined on the metric space $\mathbf{X}$, pointwise converging everywhere to 0. Then for every n the set T_n is nowhere dense and the sequence does not satisfy $\mathcal{SN}$ on any nonempty open subset of $\mathbf{X}$.*

Proof. Recall that for any $n \in \mathbb{N}$ the set T_n is closed according to Proposition 4.3. Theorem 4.9 implies that $\text{int } T_n$ is empty, hence T_n is nowhere dense. If the sequence satisfies $\mathcal{SN}$ on some open subset of $\mathbf{X}$, then according to Proposition 4.7, for some n , T_n contains this subset, thus having nonempty interior which leads to a contradiction. $\square$

The latter means that in a topological sense the sets T_n are small for everywhere converging sequences of continuous functions. T_n being small is better for continuity, but worse for nonuniformity of the convergence. T_n being large is better for nonuniformity, but worse for continuity. Still, it is natural to ask how big (in some sense) could T_n be. In the third example, T_n is finite for all n . But the class of nowhere dense sets is much more rich than just finite sets. For example there are nowhere dense closed subsets of $[0, 1]$ with positive Lebesgue measure, in particular there are sets whose cardinality is the continuum. We are going to prove that every nowhere dense set could be T_1 for some sequence of continuous functions converging everywhere to 0. We will need the following

Definition 4.12. *A subset D of a metric space $\mathbf{X}$ is called discrete, if for every $x \in D$, there exists $\varepsilon > 0$ such that $\mathbf{B}_\varepsilon(x) \cap D = \{x\}$.*

The key observation is presented in the next result from [4, Exercise 4G, 4., page 37]

Proposition 4.13. *Let $\mathbf{X}$ be a separable metric space and C be a nowhere dense closed subset of $\mathbf{X}$. Then there exists a discrete set D such that $\overline{D} \setminus D = C$.*

The proposition is stated for general metric spaces (not necessarily separable) in the book [4], and the author hints to the use of the Axiom of Choice for the proof. A sketch of a proof is presented by Brian Scott at an online forum. Here we present a more constructive proof avoiding this axiom, which is possible since the underlying space is separable. We will first prove the following

Lemma 4.14. *Let U be a subset of a separable metric space $\mathbf{X}$ and let $\varepsilon > 0$. Then there exists a set $D \subset U$, such that for all $x \in D$, $y \in D$ with $x \neq y$ it holds that $\rho(x, y) \geq \varepsilon/2$ and $U \subset \bigcup_{x \in D} \mathbf{B}_\varepsilon(x)$.*

Proof. We will construct the set D by induction. Since $\mathbf{X}$ is separable, U is separable as well, so let $\{a_n\}_{n \geq 1}$ be a dense subset of U . Set $x_1 = a_1$. Assume that $\{x_i\}_{i=1}^m$ is constructed for some $m \geq 1$. If $\{a_n\}_{n \geq 1} \subset \bigcup_{i=1}^m \mathbf{B}_{\varepsilon/2}(x_i)$, we are done. Otherwise, let

$$\ell := \min \left\{ k \mid a_k \notin \bigcup_{i=1}^m \mathbf{B}_{\varepsilon/2}(x_i) \right\},$$

and set $x_{m+1} = a_\ell$ observing that

$$(8) \quad a_j \in \bigcup_{i=1}^{m+1} \mathbf{B}_{\varepsilon/2}(x_i) \text{ for all } j = 1, 2, \dots, \ell.$$

Thus we constructed a (possibly finite) subsequence $\{x_i\}_{i \in I}$ of $\{a_n\}_{n \geq 1}$, where either $I = \{1, 2, \dots, m\}$ for some m , or $I = \mathbb{N}$. We claim that $D = \{x_i\}_{i \in I}$ is the desired set. Indeed, by the construction, when $k \in I$, $l \in I$ with $k < l$ one has $x_l \notin \mathbf{B}_{\varepsilon/2}(x_k)$, i.e. $\rho(x_l, x_k) \geq \varepsilon/2$. Now we will show that $\{a_n\}_{n \geq 1} \subset \bigcup_{i \in I} \mathbf{B}_{\varepsilon/2}(x_i)$. If I is finite, this holds by the construction. Otherwise, consider a_k for some k . For some large enough n , $x_n = a_s$ where $s > k$. From (8), it holds that

$$a_j \in \bigcup_{i=1}^n \mathbf{B}_{\varepsilon/2}(x_i)$$

for all $j = 1, 2, \dots, s$. In particular,

$$(9) \quad a_k \in \bigcup_{i=1}^n \mathbf{B}_{\varepsilon/2}(x_i) \subset \bigcup_{i \in I} \mathbf{B}_{\varepsilon/2}(x_i).$$

Now let $z \in U$ and let a_k be such that $\rho(a_k, z) < \varepsilon/2$. From (9), there exists x_j such that $\rho(x_j, a_k) < \varepsilon/2$, which yields that $\rho(z, x_j) < \varepsilon$. Thus $U \subset \bigcup_{x \in D} \mathbf{B}_{\varepsilon}(x)$. $\square$

Proof of Proposition 4.13. Let $V = \mathbf{X} \setminus C$. For $n \in \mathbb{N}$, consider the sets

$$U_n = \left\{ x \in \mathbf{X} \mid d(x, C) \in \left[\frac{1}{n+1}, \frac{1}{n} \right) \right\}, \quad n \in \mathbb{N},$$

where $d(\cdot, \cdot)$ denotes the distance function from a point to set. These sets are disjoint subsets of V . Applying Lemma 4.14, for $U = U_n$ and $\varepsilon = \frac{1}{2n}$ we get a set $D_n \subset U_n$ such that $\rho(y', y'') \geq \frac{1}{2n}$ for distinct $y' \in D_n$, $y'' \in D_n$ and $U_n \subset \bigcup_{x \in D_n} \mathbf{B}_{1/n}(x)$. Let $D := \bigcup_{n \geq 1} D_n$. Clearly $D \subset V$. We claim that D is discrete set. Indeed, let $z \in D$. Then $z \in D_n$ for some n . If $y \in D_m$ for $|m - n| \geq 2$, then

$$\rho(y, z) \geq |d(y, C) - d(z, C)| \geq \frac{1}{(n+1)(n+2)}.$$

If $y \in D_n$, then clearly $\rho(y, z) \geq \frac{1}{2n}$ by the construction of D_n . Assume now that there are y_1, y_2 such that $y_i \in D_{n-1}$ and $y_i \in \mathbf{B}_{1/(4n)}(z)$ for $i = 1, 2$. The triangle inequality implies $\rho(y_1, y_2) < \frac{1}{2n}$, which shows that $y_1 = y_2$, by the construction of D_{n-1} . Similarly for D_{n+1} . This shows that there exists at most one point $y' \in D_{n-1}$ such that $y' \in \mathbf{B}_{1/(4n)}(z)$ and at most one point $y'' \in D_{n+1}$ such that $y'' \in \mathbf{B}_{1/(4n)}(z)$. Let

$$\delta = \min \left\{ \frac{1}{(n+1)(n+2)}, \frac{1}{2n}, \rho(z, y'), \rho(z, y'') \right\}.$$

The above considerations yield that $\rho(z, y) \geq \delta$ for all $y \in D$.

Now fix $x \in C$ and $\varepsilon > 0$. Since C is nowhere dense and closed, the boundary of V is C , so there exists $v \in V$ such that $\rho(x, v) < \varepsilon/2$. Thus $v \in U_n$ for some $n > \frac{2}{\varepsilon}$. Let $z \in D_n$ be such that $v \in \mathbf{B}_{1/n}(z)$. Hence $\rho(x, z) \leq$

$\rho(x, v) + \rho(v, z) \leq \frac{\varepsilon}{2} + \frac{1}{n} < \varepsilon$, which shows that $C \subset \overline{D}$. On the other hand, if $z \in D$, then $z \in D_n \subset U_n$ for some n , which shows by the definition of U_n , that $d(z, C) \geq \frac{1}{n+1} > 0$, i.e. $z \notin C$. Thus $C \subseteq \overline{D} \setminus D$.

It remains to prove that $\overline{D} \setminus D \subseteq C$. To this end fix an element $z \in \overline{D} \setminus D$. Thus there exists a sequence $\{\beta_i\}_{i \geq 1} \subset D$ such that $\beta_i \rightarrow z$. Since each D_n is discrete, there is no infinite subsequence belonging to one particular D_n . Thus, we may assume that there is at most one element of the sequence $\{\beta_i\}_{i \geq 1}$ in each D_n . Let $\beta_i \in D_{n_i}$ for each i . Since $\{n_i\}_{i \geq 1}$ consists of distinct integers, it tends to infinity. On the other hand, by the construction of D_n we have

$$d(\beta_i, C) < \frac{1}{n_i} \xrightarrow{i \rightarrow \infty} 0.$$

Thus $d(z, C) = 0$ and since C is closed we obtain $z \in C$. $\square$

Theorem 4.15. *Let A be a nowhere dense closed subset of a separable metric space $\mathbf{X}$. Then there exists a sequence of continuous functions defined on $\mathbf{X}$, pointwise converging everywhere to 0, for which $T_1 = A$.*

Proof. Applying Proposition 4.13 for the set A , we obtain a discrete set D such that $\overline{D} \setminus D = A$. Recalling that every discrete subset of a separable space is countable, thus we may assume that $D = \{d_n\}_{n \geq 1}$. Now we construct a sequence of disjoint closed balls $\{B_n\}_{n \geq 1}$ such that for all n , $d_n \in B_n$. Since D is discrete, there exists $\delta_1 > 0$ such that $\bar{B}_{\delta_1}(d_1) \cap D = \{d_1\}$. Set $B_1 = \bar{B}_{\delta_1}(d_1)$. Assume that for some m the sequence $\{B_i\}_{i=1}^m$ is constructed, so that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $B_i \cap D = \{d_i\}$. The set $K = \bigcup_{i=1}^m B_i$ is closed and $d_{m+1} \in \mathbf{X} \setminus K$, thus there exists δ' such that $\bar{B}_{\delta'}(d_{m+1}) \cap K = \emptyset$. Moreover, since D is discrete, there exists δ'' , such that $\bar{B}_{\delta''}(d_{m+1}) \cap D = \{d_{m+1}\}$. Thus for $\delta_{m+1} := \min\{\delta', \delta''\}$ we have $\bar{B}_{\delta_{m+1}}(d_{m+1}) \cap K = \emptyset$ and $\bar{B}_{\delta_{m+1}}(d_{m+1}) \cap D = \{d_{m+1}\}$. Hence, we set $B_{m+1} = \bar{B}_{\delta_{m+1}}(d_{m+1})$. Now, for every n , consider the disjoint closed sets $\{d_n\}$ and $\mathbf{X} \setminus \text{int } B_n$. The Urysohn's lemma ([2]) applied to these sets implies the existence of a continuous function f_n such that $f_n(d_n) = 1$ and $\text{supp}(f_n) \subseteq \text{int } B_n$. Consider the sequence $\{f_n\}_{n \geq 1}$. It is easy to observe that $\mathfrak{h}(x) = 1$ for all $x \in A$ and $\mathfrak{h}(x) = 0$ for $x \in \mathbf{X} \setminus A$. Moreover, $\text{supp}(f_n) \cap \text{supp}(f_m) = \emptyset$ for $n \neq m$, hence $\lim_{n \rightarrow \infty} f_n(x) = 0$ for all $x \in \mathbf{X}$. $\square$

Recall that a meagre set, also called set of first category, is a set which is a countable union of nowhere dense set. Thus Theorem 4.9 implies that the support of a height function is always meagre.

One further question one may investigate is whether for every meagre set M there exists a sequence of continuous functions, converging everywhere to 0, such that the support of its height function contains M .

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