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ASYMPTOTIC STABILITY OF SINGULAR SOLUTIONS FOR THE HYPERELASTIC ROD EQUATION

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Abstract. We consider a hyperelastic rod equation that includes the integrable Camassa-Holm equation. We construct singular wave solutions and study their asymptotic stability near the blow-up time. To this end, we introduce suitable transformations and reformulate the problem as a dissipative quasilinear evolution equation. By employing rigorous energy estimates, commutator bounds, and semigroup theory, we demonstrate how the stability properties strongly depend on the parameter ranges of the equation.

1. Introduction and main results. In this paper, we consider the following hyperelastic rod equation

$$u_t - u_{txx} + c_0 u_x + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{1.1}$$

Equation (1.1) was derived as a model for nonlinear waves in cylindrical hyperelastic rods in [8]. The Cauchy problem for the equation (1.1) was considered in [1, 12, 19, 20].

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When $\gamma = 1$, equation (1.1) reduces to the integrable Camassa-Holm equation, which was derived as a shallow water model [2]. The Camassa-Holm equation is integrable in the sense of an infinite-dimensional Hamiltonian system [4]. The Cauchy problem for the Camassa-Holm equation is well studied [3, 5, 6, 7, 14].

Recently, for the Dullin-Gottwald-Holm (DGH) equation

$$u_t - \alpha^2 u_{txx} + c_0 u_x + 3uu_x + \Gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.2}$$

in [15] self-similar solutions were constructed and their asymptotic stability was investigated.

Equation (1.2) was derived in [9] as a model for shallow-water waves. The Cauchy problem for the Dullin-Gottwald-Holm equation has been investigated in both the periodic and non-periodic cases in [16, 18, 19]. For the DGH equation, the existence of singular solutions and their asymptotic stability were considered in [11].

The aim of this paper is to construct singular solutions to equation (1.1) and investigate their asymptotic stability. We show that the asymptotic stability of singular solutions depends on the parameter ranges of the equation. Our investigations closely follow the methods in [11, 15].

The first result provides an explicit singular solution of equation (1.1).

Theorem 1.1. *Let $T > 0$ be the maximal existence time of the solution. Then equation (1.1) has the following explicit singular solution:*

$$\overline{u}(t,x) = -\frac{1}{3}\left(c_0 + \frac{x}{T-t} + \frac{1}{T-t}\right). \tag{1.3}$$

Here, $T > 0$ denotes the maximal existence time (the blow-up time) at which the solution breaks down, meaning that $\partial_x u \to -\infty$ as $t \to T^-$. The dynamics and the finite-time blow-up behavior of this singular solution are governed by the balance between the nonlinear terms and the underlying parameters of the equation (such as c_0 and γ), upon which the asymptotic stability properties ultimately depend.

The second result demonstrates the stability of the explicit solution constructed in Theorem 1.1 for the hyperelastic rod equation (1.1).

Theorem 1.2. *Let $s > 2$, $\gamma \leq 2$, and δ be sufficiently small. Then the explicit solution* (1.3) *to equation* (1.1) *is asymptotically stable, i.e., if the initial condition $u_0(x)$ satisfies*

$$\|u_0(\cdot) - \overline{u}(0,\cdot)\|_{\mathbb{H}^{s+1}(\mathbb{R})} \leq \delta,$$

then there exists a solution $u(t,x)$ *of* (1.1)*, satisfying*

$$\lim_{t\to T^-} \|u(t,\cdot)-\overline{u}(t,\cdot)\|_{\mathbb{H}^s(\mathbb{R})}=0, \quad (t,x)\in(0,T)\times\mathbb{R}.$$

Remark 1.1. The explicit solution given in Theorem 1.1 satisfies

$$\partial_x u=-\frac{1}{3(T-t)}\to-\infty, \quad \text{as} \quad t\to T^-.$$

Notations: By $\|\cdot\|_{\mathbb{L}^2}$ and $\|\cdot\|_{\mathbb{H}^s}$ we denote the norms of spaces $\mathbb{L}^2(\mathbb{R})$ and $\mathbb{H}^s(\mathbb{R})$, respectively. The spaces $\mathbb{L}^2(\mathbb{R})$ and $\mathbb{H}^s(\mathbb{R})$ are abbreviated as $\mathbb{L}^2$ and $\mathbb{H}^s$, respectively. $\star$ denotes convolution. $[A,B]$ stands for the commutator of two linear operators A and B.

The rest of this article is organized as follows. In Section 2, we construct a singular solution for the equation (1.1). In Section 3, we study the asymptotic stability of the singular solution.

2. Construction of explicit singular solution. In this section, we seek a solution of equation (1.1) in the form

$$u(t,x)=\sum_{i=0}^{n} A_i(t)x^i. \tag{2.1}$$

Substituting (2.1) into (1.1), we obtain the following system of ordinary differential equations for A_0 and A_1:

$$\begin{cases} A_{0,t}+c_0A_1+3A_0A_1=0,\\ \\ A_{1,t}+3A_1^2=0. \end{cases}$$

Solving the above system, we get

$$\begin{cases} A_0=\dfrac{1}{3t+c_1}e^{-c_2}(c+c_3-c_0e^{c_2}t),\\ \\ A_1=\dfrac{1}{3t+c_1}, \end{cases}$$

where c, c_1, c_2, and c_3 are integration constants. Thus, (1.1) admits an explicit solution

$$u=\frac{1}{3t+c_1}\left[e^{-c_2}(c+c_3-c_0e^{c_2}t)+x\right]. \tag{2.2}$$

We are interested in singular solutions of (1.1). For $c_4 = e^{-c_2}(c + c_3)$, we have $u = \dfrac{1}{3t + c_1}(c_4 - c_0 t + x)$. Let $c_1 = -3T$ and $c_4 = c_0 T + \widehat{c}$, thus the solution (2.2) is written in the form

$$u = -\frac{1}{3}\left(c_0 + \frac{x}{T-t} + \frac{\widehat{c}}{T-t}\right),$$

where $\widehat{c}$ is a constant. We choose $\widehat{c} = 1$ and obtain an explicit singular solution of the equation from Theorem 1.1.

3. Asymptotic stability of singular solutions. In this section, we consider the stability of the explicit solution constructed in the previous section. Consider the perturbation of the form

$$u(t,x) = v(t,x) + \overline{u}(t,x), \tag{3.1}$$

where $\overline{u}(t,x) = -\dfrac{1}{3}(c_0 + \dfrac{x}{T-t} + \dfrac{1}{T-t})$.

Substituting (3.1) into (1.1), we get the following dissipative quasilinear equation:

$$\begin{aligned} & v_t - v_{txx} + \frac{\gamma}{3}\left(c_0 + \frac{x}{T-t} + \frac{1}{T-t}\right) v_{xxx} + \frac{2\gamma}{3(T-t)} v_{xx} \\ & - \left(\frac{x}{T-t} + \frac{1}{T-t}\right) v_x - \frac{1}{T-t} v + 3 v v_x \\ = \;& \gamma(2 v_x v_{xx} + v v_{xxx}), \quad \forall (t,x) \in (0,T) \times \mathbb{R} \end{aligned} \tag{3.2}$$

with the initial condition

$$v(0,x) = v_0(x) = u_0(x) + \frac{1}{3}\left(c_0 + \frac{x}{T} + \frac{1}{T}\right), \quad \forall x \in \mathbb{R}.$$

We next introduce the new coordinates

$$\tau = -ln(T-t), \quad \rho = \frac{x}{T-t} \tag{3.3}$$

the solution v is written in the form $v(t,x) = \psi(\tau, \rho)$. By direct computation, we obtain

$$v_x(t,x) = e^{\tau}\psi_\rho, \quad v_t(t,x) = e^{\tau}(\psi_\tau + \rho\psi_\rho),$$

$$v_{xx}(t,x) = e^{2\tau}\psi_{\rho\rho}, \quad v_{tx}(t,x) = e^{2\tau}(\psi_{\tau\rho} + \psi_\rho + \rho\psi_{\rho\rho}),$$

$$v_{xxx}(t,x) = e^{3\tau}\psi_{\rho\rho\rho}, \quad v_{txx}(t,x) = e^{3\tau}(\psi_{\tau\rho\rho} + 2\psi_{\rho\rho} + \rho\psi_{\rho\rho\rho}).$$

Thus, (3.2) reduces to

$$(\psi - e^{2\tau}\psi_{\rho\rho})_\tau + \frac{2(\gamma-3)}{3}e^{2\tau}\psi_{\rho\rho} + e^{2\tau}\left(\frac{\gamma c_0 + (\gamma-3)\rho}{3} + \frac{\gamma}{3}e^{\tau}\right)\psi_{\rho\rho\rho}$$
$$-\psi + 3\psi\psi_\rho - e^{\tau}\psi_\rho = \gamma e^{2\tau}(2\psi_\rho\psi_{\rho\rho} + \psi\psi_{\rho\rho\rho}). \tag{3.4}$$

Let $k = e^{-\tau}\rho$ and $\overline{v}(\tau,k) = e^{-\tau}\psi(\tau,\rho)$. Then

$$\psi_\tau = e^{\tau}\overline{v} + e^{\tau}\overline{v}_\tau - \rho\overline{v}_k,\ \ \psi_\rho = \overline{v}_k,\ \ \psi_{\rho\rho} = e^{-\tau}\overline{v}_{kk},\ \ \psi_{\rho\rho\rho} = e^{-2\tau}\overline{v}_{kkk},$$

$$\psi_{\tau\rho} = \overline{v}_{\tau k} - k\overline{v}_{kk},\ \ \psi_{\tau\rho\rho} = e^{-\tau}(\overline{v}_{\tau kk} - \overline{v}_{kk} - k\overline{v}_{kkk}).$$

Now, (3.4) can be written in the form

$$\overline{v}_\tau - \overline{v}_{\tau kk} + \frac{2\gamma-3}{3}\overline{v}_{kk} + e^{-\tau}\frac{\gamma}{3}(c_0 + ke^{\tau} + e^{\tau})\overline{v}_{kkk}$$
$$-(k+1)\overline{v}_k + 3\overline{v}\overline{v}_k = \gamma(2\overline{v}_k\overline{v}_{kk} + \overline{v}\overline{v}_{kkk}).$$

Define the operator $\Lambda = (1-\partial_{kk})^{\frac{1}{2}}$. We have $\Lambda^{-2}\overline{v} = p(k)\star\overline{v}$ for each $\overline{v}\in\mathbb{L}^2$, where $p(k) = \dfrac{1}{2}e^{-|k|}$. Let $w(\tau,k) = \overline{v}(\tau,k) - \overline{v}_{kk}(\tau,k)$, then $\overline{v}(\tau,k) = p\star w$, where $k\in\mathbb{R}$. Moreover

$$(p\star w)_{kk} = p\star w - w,\ \ \overline{v}_k = (p\star w)_k,\ \ \overline{v}_{kkk} = (p\star w)_k - w_k.$$

Thus, (3.4) can be written in the form

$$w_\tau - \frac{2\gamma-3}{3}w - e^{-\tau}\frac{\gamma}{3}(c_0 + e^{\tau}k + e^{\tau})w_k + \frac{2\gamma-3}{3}(p\star w)$$
$$+\left\{e^{-\tau}\frac{\gamma}{3}(c_0 + e^{\tau}k + e^{\tau}) - (k+1)\right\}(p\star w)_k + 3(p\star w)(p\star w)_k \tag{3.5}$$
$$= \ 2\gamma(p\star w)_k(p\star w - w) + \gamma(p\star w)[(p\star w)_k - w_k]$$

with the initial condition

$$w(0,k) = w_0(k) \ = \psi_0(k) - v_{kk}(0,k)$$
$$= u_0(x) - u_0''(x) + \frac{1}{3}\left(\frac{x}{T} + \frac{1}{T} + c_0\right) \tag{3.6}$$

and the boundary conditions

$$\lim_{|k|\to+\infty} w(\tau,k)=0, \quad \lim_{|k|\to+\infty} w_k(\tau,k)=0. \tag{3.7}$$

The following commutator estimate, established in [13], will be useful in the estimates.

Lemma 3.1. *Let $s>0$. Then it holds*

$$\|[\Lambda^s,u]v\|_{\mathbb{L}^2}\le C\left(\|\partial_x u\|_{\mathbb{L}^\infty}\|\Lambda^{s-1}v\|_{\mathbb{L}^2}+\|\Lambda^s u\|_{\mathbb{L}^2}\|v\|_{\mathbb{L}^\infty}\right), \tag{3.8}$$

where C is a positive constant depending on s.

We now estimate the solutions of equation (3.5). Let $s>0$. Applying Λ^s to both sides of (3.5), we obtain

(3.9)

$$\begin{aligned}
&(\Lambda^s w)_\tau-\frac{2\gamma-3}{3}\Lambda^s w-e^{-\tau}\Lambda^s\left[\frac{\gamma}{3}(c_0+e^\tau k+e^\tau)w_k\right]+\frac{2\gamma-3}{3}\Lambda^s(p\star w)\\
&+\Lambda^s\left\{\left[e^{-\tau}\frac{\gamma}{3}(c_0+e^\tau k+e^\tau)-(k+1)\right](p\star w)_k\right\}\\
&+3\Lambda^s[(p\star w)(p\star w)_k]\\
=\;&2\gamma\Lambda^s[(p\star w)_k(p\star w-w)]+\gamma\Lambda^s[(p\star w)((p\star w)_k-w_k)].
\end{aligned}$$

Lemma 3.2. *Let $s>2$, $\gamma\le 2$, and w be a solution of equation (3.5), then*

$$\|w\|_{\mathbb{H}^s}\le\frac{1}{\|w_0\|_{\mathbb{H}^s}^{-1}-C\tau},$$

where C is a positive constant depending on s.

P r o o f. Multiplying both sides of (3.9) by $\Lambda^s w$ and integrating, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial\tau}\|w\|^2_{\mathbb{H}^s} - \frac{2\gamma-3}{3}\|w\|^2_{\mathbb{H}^s} + \frac{2\gamma-3}{3}\int_{\mathbb{R}} \Lambda^s w\Lambda^s(p\star w)dk \\
&-e^{-\tau}\int_{\mathbb{R}} \Lambda^s w\Lambda^s\left[\frac{\gamma}{3}(c_0+e^\tau k+e^\tau))w_k\right]dk \\
&+\int_{\mathbb{R}} \Lambda^s w\Lambda^s\left\{\left[e^{-\tau}\frac{\gamma}{3}(c_0+e^\tau k+e^\tau)-(k+1)\right](p\star w)_k\right\}dk \\
&+3\int_{\mathbb{R}} \Lambda^s w\Lambda^s[(p\star w)(p\star w)_k]dk \\
=\ & 2\gamma\int_{\mathbb{R}} \Lambda^s w\Lambda^s[(p\star w)_k(p\star w-w)]dk \\
&+\gamma\int_{\mathbb{R}} \Lambda^s w\Lambda^s[(p\star w)((p\star w)_k-w_k)]dk.
\end{aligned} \tag{3.10}$$

We now estimate all the terms in (3.10). Integrating by parts, we obtain

$$\frac{2\gamma-3}{3}\int_{\mathbb{R}} \Lambda^s w\Lambda^s(p\star w)dk = \frac{2\gamma-3}{3}\|w\|^2_{\mathbb{H}^{s-1}}. \tag{3.11}$$

For the second term, integration by parts yields

$$\begin{aligned}
&-e^{-\tau}\int_{\mathbb{R}} \Lambda^s w\Lambda^s\left[\frac{\gamma}{3}(c_0+e^\tau k+e^\tau)w_k\right]dk \\
=\ & -e^{-\tau}\int_{\mathbb{R}}\left[\frac{\gamma}{3}(c_0+e^\tau k+e^\tau)w_k\right]\Lambda^{2s}wdk \\
=\ & e^{-\tau}\int_{\mathbb{R}}\left[\frac{\gamma}{3}(c_0+e^\tau k+e^\tau)\Lambda^{2s}w\right]_k wdk \\
=\ & \frac{\gamma}{3}\int_{\mathbb{R}}\Lambda^{2s}w.wdk + e^{-\tau}\int_{\mathbb{R}}\frac{\gamma}{3}(c_0+e^\tau k+e^\tau)\Lambda^{2s}w_k wdk \\
=\ & \frac{\gamma}{3}\int_{\mathbb{R}}\Lambda^{2s}w.wdk - \frac{\gamma}{6}\int_{\mathbb{R}}(\Lambda^s w)^2dk \\
=\ & \frac{\gamma}{6}\|w\|^2_{\mathbb{H}^s}.
\end{aligned} \tag{3.12}$$

Similarly, for the third linear term, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \Lambda^s w \Lambda^s \left\{ \left[e^{-\tau} \frac{\gamma}{3} (c_0 + e^{\tau} k + e^{\tau}) - (k+1) \right] (p \star w)_k \right\} dk \\
= \; & \int_{\mathbb{R}} \Lambda^{2s} w \left\{ \left[e^{-\tau} \frac{\gamma}{3} (c_0 + e^{\tau} k + e^{\tau}) - (k+1) \right] (p \star w)_k \right\} dk \\
= \; & - \int_{\mathbb{R}} \left\{ \Lambda^{2s} w \left[e^{-\tau} \frac{\gamma}{3} (c_0 + e^{\tau} k + e^{\tau}) - (k+1) \right] \right\}_k (p \star w) dk \\
= \; & - \int_{\mathbb{R}} \Lambda^{2s} w_k \left[e^{-\tau} \frac{\gamma}{3} (c_0 + e^{\tau} k + e^{\tau}) - (k+1) \right] (p \star w) dk \\
& - \left(\frac{\gamma}{3} - 1 \right) \int_{\mathbb{R}} \Lambda^{2s} w (p \star w) dk \qquad (3.13) \\
= \; & - \frac{1}{2} \int_{\mathbb{R}} [(\Lambda^{s-1} w)^2]_k \left[e^{-\tau} \frac{\gamma}{3} (c_0 + e^{\tau} k + e^{\tau}) - (k+1) \right] dk \\
& - \frac{\gamma - 3}{3} \int_{\mathbb{R}} \Lambda^{2s} w (p \star w) dk \\
= \; & \frac{1}{2} \times \left(\frac{\gamma}{3} - 1 \right) \int_{\mathbb{R}} (\Lambda^{s-1} w)^2 dk - \frac{\gamma - 3}{3} \int_{\mathbb{R}} \Lambda^{2s} w (p \star w) dk \\
= \; & - \frac{\gamma - 3}{6} \| w \|^2_{\mathbb{H}^{s-1}}.
\end{aligned}
$$

Next, integration by parts and Sobolev embeddings yield

$$
\begin{aligned}
& 3 \int_{\mathbb{R}} \Lambda^s w \Lambda^s ((p \star w)(p \star w)_k) dk \\
= \; & \frac{3}{2} \int_{\mathbb{R}} \Lambda^s w \Lambda^s ((p \star w)^2)_k dk \\
= \; & - \frac{3}{2} \int_{\mathbb{R}} w_k \Lambda^{2s} ((p \star w)^2) dk \\
= \; & - \frac{3}{2} \int_{\mathbb{R}} w_k (\Lambda^{s-1} w)^2 dk \qquad (3.14) \\
\leq \; & \frac{3}{2} \| w_k \|_{\mathbb{L}^\infty} \| w \|^2_{\mathbb{H}^{s-1}} \\
\leq \; & \frac{3}{2} \| w \|^3_{\mathbb{H}^{s-1}}.
\end{aligned}
$$

To handle the first group of nonlinear terms involving commutators, we apply Kato's commutator estimate (3.8) and the Sobolev embedding $\mathbb{H}^{s-1}(\mathbb{R}) \subset \mathbb{L}^\infty(\mathbb{R})$ with $s > 2$, which yields

$$
\begin{aligned}
& 2|\gamma| \left| \int_{\mathbb{R}} \Lambda^s w \Lambda^s ((p \star w)_k (p \star w - w)) dk \right| \\
= \; & 2|\gamma| \left| \int_{\mathbb{R}} \{ [\Lambda^s, (p \star w - w)](p \star w)_k + (p \star w - w) \Lambda^s (p \star w)_k \} \Lambda^s w dk \right| \\
= \; & 2|\gamma| \left| \int_{\mathbb{R}} [\Lambda^s, (p \star w - w)](p \star w)_k \Lambda^s w dk \right| \\
& +2|\gamma| \left| \int_{\mathbb{R}} (p \star w - w) \Lambda^s (p \star w)_k \Lambda^s w dk \right| \qquad (3.15) \\
\leq \; & |\gamma| C (\| (p \star w - w)_k \|_{\mathbb{L}^\infty} \| \Lambda^{s-1} (p \star w)_k \|_{\mathbb{L}^2} \\
& \quad + \| \Lambda^s (p \star w - w) \|_{\mathbb{L}^2} \| (p \star w)_k \|_{\mathbb{L}^\infty}) \| w \|_{\mathbb{H}^s} \\
& +2|\gamma| (\| (p \star w - w) \|_{\mathbb{L}^\infty} + \| (p \star w - w)_k \|_{\mathbb{L}^\infty}) \| w \|_{\mathbb{H}^s}^2 \\
\leq \; & |\gamma| C \| w \|_{\mathbb{H}^s}^3 .
\end{aligned}
$$

Similarly, for the remaining nonlinear terms, we obtain the following estimate:

$$
\begin{aligned}
& |\gamma| \left| \int_{\mathbb{R}} \Lambda^s w \Lambda^s [(p \star w)((p \star w)_k - w_k)] dk \right| \\
\leq \; & |\gamma| \left| \int_{\mathbb{R}} [\Lambda^s, p \star w]((p \star w)_k - w_k) \Lambda^s w dk \right| \\
& +|\gamma| \left| \int_{\mathbb{R}} (p \star w) \Lambda^s ((p \star w)_k - w_k) \Lambda^s w dk \right| \qquad (3.16) \\
\leq \; & |\gamma| C (\| (p \star w)_k \|_{\mathbb{L}^\infty} \| \Lambda^{s-1} (p \star w - w_k) \|_{\mathbb{L}^2} \\
& \quad + \| \Lambda^s (p \star w) \|_{\mathbb{L}^2} \| (p \star w)_k - w_k \|_{\mathbb{L}^\infty}) \| w \|_{\mathbb{H}^s} + 2|\gamma| \| p \star w \|_{\mathbb{L}^\infty} \| w \|_{\mathbb{H}^2}^2 \\
\leq \; & |\gamma| C \| w \|_{\mathbb{H}^s}^3 ,
\end{aligned}
$$

where C is a positive constant depending on s.

Substituting (3.11)-(3.16) into (3.10), we obtain

$$\frac{1}{2}\frac{d}{d\tau}\|w\|^2_{\mathbb{H}^s} - \frac{2\gamma-3}{3}\|w\|^2_{\mathbb{H}^s} + \frac{2\gamma-3}{3}\|w\|^2_{\mathbb{H}^{s-1}} + \frac{\gamma}{6}\|w\|^2_{\mathbb{H}^s} - \frac{\gamma-3}{6}\|w\|^2_{\mathbb{H}^{s-1}}$$

$$\leq \quad |\gamma| C\|w\|^3_{\mathbb{H}^s} + \frac{3}{2}\|w\|^3_{\mathbb{H}^{s-1}}.$$

For $\gamma \leq 2$, we have

$$\frac{1}{2}\frac{d}{d\tau}\|w\|^2_{\mathbb{H}^s} \leq C\|w\|^3_{\mathbb{H}^s},$$

where C is a positive constant depending only on γ. Hence

$$-\frac{d}{d\tau}\|w\|^{-1}_{\mathbb{H}^s} \leq C.$$

Integrating the above inequality with respect to τ from 0 to τ, we get

$$\|w\|^{-1}_{\mathbb{H}^s} \geq \|w_0\|^{-1}_{\mathbb{H}^s} - C\tau,$$

$$\|w\|_{\mathbb{H}^s} \leq \frac{1}{\|w_0\|^{-1}_{\mathbb{H}^s} - C\tau}. \qquad \square$$

We now consider equation (3.5) with the initial data (3.6) and boundary conditions (3.7). Define the operator L by

$$\begin{aligned} L[w] = \quad & \frac{2\gamma-3}{3}w - \frac{2\gamma-3}{3}p\star w + e^{-\tau}\left[\frac{\gamma}{3}(c_0 + e^{\tau}k + e^{\tau})\right] w_k \\ & - \left[e^{-\tau}\frac{\gamma}{3}(c_0 + e^{\tau}k + e^{\tau}) - (k+1)\right](p\star w)_k. \end{aligned} \tag{3.17}$$

The equation (3.5) can be written in the form

$$w_\tau = L[w] + f(w), \tag{3.18}$$

where the nonlinear part f is

$$f(w) = -3(p\star w)(p\star w)_k + 2\gamma(p\star w)_k(p\star w - w) + \gamma(p\star w)[(p\star w)_k - w_k].$$

Lemma 3.3. *Let $s > 2$. Then*

- $L[w] \in \mathbb{H}^s, \quad \forall w \in \mathcal{D}(L)$.
- *L is a closed and densely defined linear operator in $\mathbb{H}^s$.*

P r o o f. It follows from the definition of L in (3.17). □

Lemma 3.4. *Let $s > 2$ and $\gamma \le 2$. Then L is a dissipative operator on $\mathbb{H}^s$, i.e. $(L[w], w)_s \le 0$.*

P r o o f. By direct calculations, we have

$$\begin{aligned}
&\int_{\mathbb{R}} (\Lambda^s L[w])\Lambda^s w dk \\
= \;& \frac{2\gamma-3}{3}\|w\|^2_{\mathbb{H}^s} - \frac{2\gamma-3}{3}\int_{\mathbb{R}} \Lambda^s w \Lambda^s (p \star w) dk \\
&+ \int_{\mathbb{R}} \Lambda^s w \Lambda^s \left[e^{-\tau}\frac{\gamma}{3}(c_0 + e^{\tau}k + e^{\tau}) w_k \right] dk \\
&- \int_{\mathbb{R}} \Lambda^s w \Lambda^s \left\{ \left[e^{-\tau}\frac{\gamma}{3}(c_0 + e^{\tau}k + e^{\tau}) - (k+1) \right] (p \star w)_k \right\} dk.
\end{aligned}$$

Combining this with (3.11)-(3.13) leads to

$$\begin{aligned}
&\int_{\mathbb{R}} (\Lambda^s L[w])\Lambda^s w dk \\
= \;& \frac{2\gamma-3}{3}\|w\|^2_{\mathbb{H}^s} - \frac{2\gamma-3}{3}\|w\|^2_{\mathbb{H}^{s-1}} - \frac{\gamma}{6}\|w\|^2_{\mathbb{H}^s} + \frac{\gamma-3}{6}\|w\|^2_{\mathbb{H}^{s-1}} \\
= \;& -\frac{2-\gamma}{2}\|w\|^2_{\mathbb{H}^s} - \frac{\gamma-1}{2}\|w\|^2_{\mathbb{H}^{s-1}} \le 0.
\end{aligned}$$ □

Lemma 3.5. *Let $s > 2$ and $\gamma \le 2$. Then the operator L is invertible on $\mathbb{H}^s$ and generates a C_0-semigroup $(\boldsymbol{S}(\tau))_{\tau \ge 0}$ in $\mathbb{H}^s$.*

P r o o f. First, we show that L is injective and surjective. Let $w \in \mathcal{D}(L)$ such that $L[w] = 0$. We have

$$\begin{aligned}
&\frac{2\gamma-3}{3}\Lambda^s w - \frac{2\gamma-3}{3}\Lambda^s (p \star w) + \Lambda^s \left[e^{-\tau}\frac{\gamma}{3}(c_0 + e^{\tau}k + e^{\tau}) w_k \right] \\
&-\Lambda^s \left\{ \left[e^{-\tau}\frac{\gamma}{3}(c_0 + e^{\tau}k + e^{\tau}) - (k+1) \right] (p \star w)_k \right\} = 0.
\end{aligned} \tag{3.19}$$

Multiplying (3.19) by $\Lambda^s w$ and integrating over $\mathbb{R}$, we get

$$\int_{\mathbb{R}} \Lambda^s L[w] \Lambda^s w dk = -\frac{2-\gamma}{2}\|w\|^2_{\mathbb{H}^s} - \frac{\gamma-1}{2}\|w\|^2_{\mathbb{H}^{s-1}} = 0.$$

The boundary conditions (3.7) imply that $w = 0$. So the operator L is injective. Next, $\forall g \in \mathbb{H}^1$ consider the equation

$$L[w] = g. \tag{3.20}$$

Applying Λ^s to (3.20) and multiplying the result by $\Lambda^s w$, and then integrating over $\mathbb{R}$, we get

$$-\frac{2-\gamma}{2}\|w\|^2_{\mathbb{H}^s} - \frac{\gamma-1}{2}\|w\|^2_{\mathbb{H}^{s-1}} = \int_{\mathbb{R}} \Lambda^s g \Lambda^s w dk.$$

For $1 < \gamma \leq 2$, we have

$$-\frac{1}{2}\|w\|^2_{\mathbb{H}^s} \leq \int_{\mathbb{R}} \Lambda^s g \Lambda^s w dk$$

and for $\gamma \leq 1$, we have

$$-\frac{2-\gamma}{2}\|w\|^2_{\mathbb{H}^s} \leq \int_{\mathbb{R}} \Lambda^s g \Lambda^s w dk.$$

It follows from Young's inequality that

$$\|w\|_{\mathbb{H}^s} \leq C(\epsilon)\|g\|_{\mathbb{H}^s}.$$

For $s > 2$, the standard theory of elliptic-type partial differential equations [10] yields a unique weak solution $w \in \mathbb{H}^1$, moreover, if $g \in \mathbb{H}^s$, then $w \in \mathbb{H}^{s+1}$. Hence, the operator L is surjective.

By the Lumer-Phillips theorem [17], the operator L generates a C_0-semigroup $(\mathbf{S}(\tau))_{\tau \geq 0}$ in $\mathbb{H}^s$. □

As a consequence, the results of Lemmas 3.3-3.5 give the following result.

Proposition 3.1. *Let $s > 2$. Then the Cauchy problem*

$$\begin{cases} \dfrac{d}{d\tau} w = Lw, \\ \\ w(0) = w_0 \end{cases}$$

with vanishing boundary has a unique solution $w(\tau) = \boldsymbol{S}(\tau)w_0$, where w_0 is the initial datum defined in (3.6).

The above Cauchy problem can be rewritten as the following integral equation:

$$w(\tau) = \mathbf{S}(\tau + \ln T)w_0 + \int_{-\ln T}^{\tau} \mathbf{S}(\tau - s)f(w(s))ds \ \text{ for } \ \tau \geq -\ln T.$$

To prove the existence of a solution to this integral equation, we define a closed ball in $\mathbb{H}^s$ $(s > 2)$:

$$B_\delta = \{w \in \mathbb{H}^s : \|w\|_{\mathbb{H}^s} < \delta \ll 1\}$$

and a map $\mathcal{T}$:

$$\mathcal{T}w(\tau) = \mathbf{S}(\tau)w_0 + \int_0^{\tau} \mathbf{S}(\tau - s)f(w(s))ds.$$

We use Banach's fixed-point theorem to prove that $\mathcal{T}$ has a fixed point in the space B_δ for $\delta < 1$. For this, we need the following auxiliary lemma.

Lemma 3.6 ([13])**.** *Let $s > 2$. Then $\mathbb{H}^s \cap \mathbb{L}^\infty$ is an algebra, and*

$$\|uv\|_{\mathbb{H}^s} \leq C(\|u\|_{\mathbb{L}^\infty}\|v\|_{\mathbb{H}^s} + \|u\|_{\mathbb{H}^s}\|v\|_{\mathbb{L}^\infty}),$$

where C is a positive constant depending on s.

Lemma 3.7. *Let $s > 2$, $\gamma \leq 2$, and $\|w_0\|_{\mathbb{H}^{s+1}} < \delta$ for sufficiently small $\delta > 0$. Then $\mathcal{T}$ is a contraction mapping.*

P r o o f. By Lemma 3.6, we have

$$\begin{aligned}
\|f(w)\|_{\mathbb{H}^s} &\leq 3\|(p \star w)(p \star w)_k\|_{\mathbb{H}^s} \\
&\quad +2\gamma\|(p \star w)_k(p \star w - w)\|_{\mathbb{H}^s} + \gamma\|(p \star w)((p \star w)_k - w_k)\|_{\mathbb{H}^s} \\
&\leq C_1(\|p \star w\|_{\mathbb{H}^s}\|(p \star w)_k\|_{\mathbb{L}^\infty} \\
&\quad +\|(p \star w)_k\|_{\mathbb{L}^\infty}\|(p \star w - w)\|_{\mathbb{H}^s} + \|p \star w\|_{\mathbb{H}^s}\|(p \star w)_k - w_k\|_{\mathbb{L}^\infty}),
\end{aligned}$$

where C_1 is a positive constant.

Since $\mathbb{H}^s \subset \mathbb{L}^\infty$ and $w = \Lambda^2(p(k) \star \overline{v})$, Lemma 3.2 yields

$$\|f(w)\|_{\mathbb{H}^s} \leq C_1\|w\|_{\mathbb{H}^s}^2 < \frac{C_1}{\|w_0\|_{\mathbb{H}^s}^{-1} - C_\tau} < \frac{C_1}{\delta^{-1} - C_\tau} < \delta$$

for sufficiently small δ. Thus, $\mathcal{T}$ is a self-mapping on B_δ.

To show that $\mathcal{T}$ is a contraction mapping, let $w, \overline{w} \in B_\delta$, by Lemma 3.6 and a direct calculation shows that

$$\begin{aligned}
& \|f(w) - f(\overline{w})\|_{\mathbb{H}^s} \\
= \; & \| -3(p\star w)(p\star w)_k + 2\gamma(p\star w)_k(p\star w) \\
& -2\gamma(p\star w)_k w + \gamma(p\star w)((p\star w)_k - w_k) \\
& +3(p\star \overline{w})(p\star \overline{w})_k - 2\gamma(p\star \overline{w})_k(p\star \overline{w}) \\
& +2\gamma(p\star \overline{w})_k \overline{w} - \gamma(p\star \overline{w})((p\star \overline{w})_k - \overline{w}_k)\|_{\mathbb{H}^s} \\
\leq \; & 3\|(p\star w)(p\star w)_k - (p\star \overline{w})(p\star \overline{w})_k\|_{\mathbb{H}^s} \\
& +2|\gamma|\|(p\star w)_k(p\star w - w) - (p\star \overline{w})_k(p\star \overline{w} - \overline{w})\|_{\mathbb{H}^s} \\
& +|\gamma|\|(p\star w)((p\star w)_k - w_k) - (p\star \overline{w})((p\star \overline{w})_k - \overline{w}_k)\|_{\mathbb{H}^s} \\
\leq \; & 3\|(p\star (w-\overline{w}))(p\star w)_k\|_{\mathbb{H}^s} + 3\|(p\star \overline{w})(p\star (w-\overline{w}))_k\|_{\mathbb{H}^s} \\
& +2|\gamma|\|((p\star (w-\overline{w}))_k(p\star w - w)\|_{\mathbb{H}^s} \\
& +2|\gamma|\|(p\star \overline{w})_k(p\star (w-\overline{w}) - (w-\overline{w}))\|_{\mathbb{H}^s} \\
& +|\gamma|\|(p\star (w-\overline{w}))((p\star w)_k - w_k)\|_{\mathbb{H}^s} \\
& +|\gamma|\|(p\star \overline{w})((p\star (w-\overline{w}))_k - (w-\overline{w})_k)\|_{\mathbb{H}^s} \\
\leq \; & C(b)\delta\|w - \overline{w}\|_{\mathbb{H}^s}.
\end{aligned}$$

Thus,

$$\|\mathcal{T}w(\tau) - \mathcal{T}\overline{w}(\tau)\|_{\mathbb{H}^s} \leq C\delta\|w - \overline{w}\|_{\mathbb{H}^s}.$$

Since $\delta > 0$ is sufficiently small, $\mathcal{T}$ is a contraction mapping. □

Thus, we have the following existence results.

Proposition 3.2. *Let $s > 2$ be a fixed integer, $\gamma \leq 2$, and $\delta > 0$ be sufficiently small. Then*

- *if $\|w_0\|_{\mathbb{H}^{s+1}} < \delta$, there exists a unique solution $w \in B_\delta$ to (3.5) satisfying*

the initial data (3.6) and the boundary condition (3.7).

• *there exists a global solution* $\psi(\tau,\rho)\in\mathbb{H}^s$ *to* (3.4) *satisfying the initial data* (3.6) *and the boundary condition* (3.7). *Moreover, if the initial data* ψ_0 *satisfies* $\|\psi_0\|_{\mathbb{H}^{s+1}}<\delta$, *then*

$$\|\psi\|_{\mathbb{H}^s}\le\frac{\overline{C}\delta}{e^\tau(1-C\delta\tau)}.$$

Here C *and* $\overline{C}$ *are two positive constants depending on* s.

P r o o f. By Lemma 3.7 and Banach's fixed point theorem, the map $\mathcal{T}$ has a fixed point in B_δ, which is a solution to equation (3.5). Thus, there exists a global solution of (3.4) as

$$\psi(\tau,\rho)=e^\tau\overline{v}(\tau,e^{-\tau}\rho)=e^\tau((p\star w)(\tau,e^{-\tau}\rho)).$$

Furthermore, we have

$$v_{\rho\rho}=\psi_{\rho\rho}=(p\star w)_{kk}e^{-\tau}=e^{-\tau}(p\star w-w).$$

Thus, by Lemma 3.2, we get

$$\begin{aligned}\|\psi_{\rho\rho}\|_{\mathbb{H}^{s-2}}&\le e^{-\tau}\|p\star w-w\|_{\mathbb{H}^{s-2}}\le\overline{C}e^{-\tau}\|w\|_{\mathbb{H}^{s-2}}\\&\le\frac{\overline{C}}{e^\tau(\|w_0\|_{\mathbb{H}^s}^{-1}-C\tau)}\\&\le\frac{\overline{C}\delta}{e^\tau(1-C\delta\tau)},\end{aligned}$$

where we have used $\|w_0\|_{\mathbb{H}^s}\le\delta$ ($\delta>0$ is a sufficiently small constant) in the last inequality. □

In order to translate the result of Proposition 3.2 back to the (t,x)-coordinates, we introduce the following proposition.

Proposition 3.3. *Let* $s>2$ *be an integer,* $\gamma\le 2$, *and* $\delta>0$ *be a sufficiently small constant. Then*

(1) $u(t,x)$ *is a solution of Eq.* (1.1) *satisfying* $\lim\limits_{t\to T^-}\|u(t,\cdot)-\overline{u}(t,\cdot)\|_{\mathbb{H}^s}=0$ *for all* $\|u_0(\cdot)-\overline{u}(0,\cdot)\|_{\mathbb{H}^{s+1}}\le\delta$, *if and only if* $v(t,x)$ *is a solution of Eq.* (3.2) *satisfying* $\lim\limits_{t\to T^-}\|v(t,\cdot)\|_{\mathbb{H}^s}=0$ *for all* $\|v_0(\cdot)\|_{\mathbb{H}^{s+1}}\le\delta$.

(2) *if* $\psi(\tau,\rho)$ *is a solution of Eq.* (3.4) *satisfying the asymptotic behavior:* $\lim\limits_{\tau\to+\infty}\|\psi(\tau,\cdot)\|_{\mathbb{H}^s}=0$ *for all* $\|\psi_0\|_{\mathbb{H}^{s+1}}<\delta$. *Then* $v(t,x)$ *is a solution of Eq.* (3.2) *satisfying* $\lim\limits_{t\to T^-}\|v(t,\cdot)\|_{\mathbb{H}^s}=0$ *for all* $\|v_0(\cdot)\|_{\mathbb{H}^{s+1}}\le\delta$.

Proof. **(1)** On the one hand, let $u(t,x)$ be a solution of Eq. (1.1) with initial data $u_0(x)$. Using (3.1), we obtain that $v(t,x)$ solves (3.2) with initial data $v_0(x)=u_0(x)+\dfrac{1}{3}(c_0+\dfrac{x}{T}+\dfrac{1}{T})$. On the other hand, let $v(t,x)$ be a solution of Eq. (3.2). On substituting $v(t,x)=u(t,x)-\overline{u}(t,x)$ (defined by (3.1)) into Eq. (3.2), we find that $u(t,x)$ is a solution of Eq. (1.1) with initial data $u(0,x)=u_0(x)$. Furthermore, if $u(t,x)$ satisfies

$$\lim_{t\to T^-}\|u(t,\cdot)-\overline{u}(t,\cdot)\|_{\mathbb{H}^s}=0 \tag{3.21}$$

for $\|u_0(\cdot)-\overline{u}(0,\cdot)_{\mathbb{H}^s}\|\le\delta$, i.e., $\overline{u}(t,x)$ is asymptotically stable for Eq. (1.1). By using (3.1), we get the equivalent form of (3.21) as follows:

$$\lim_{t\to T^-}\|v(t,\cdot)\|_{\mathbb{H}^s}=0$$

for $\|v(0,\cdot)\|_{\mathbb{H}^s}\le\delta$. This completes the proof of (**1**).

(2) Recall the coordinate transformation (3.3) and the corresponding variable transformation (3.21), namely, $t=T(1-e^{-\tau})$, $x=\rho Te^{-\tau}$. Under this transformation, $(t,x)\in[0,T)\times\mathbb{R}^+$ is mapped to $(\tau,\rho)\in[0,+\infty)\times\mathbb{R}^+$. Thus, if $\psi(\tau,\rho)$ is a solution of Eq. (3.4) satisfying the asymptotic behavior: $\lim\limits_{\tau\to+\infty}\|\psi(\tau,\cdot)\|_{\mathbb{H}^s}=0$ for all $\|\psi_0\|_{\mathbb{H}^{s+1}}<\delta$. On substituting $\psi(\tau,\rho)=v(t,x)$ into (3.4), we obtain that $v(t,x)$ is a solution of Eq. (3.2) satisfying $\lim\limits_{t\to T^-}\|v(t,\cdot)\|_{\mathbb{H}^s}=0$ for all $\|v_0(\cdot)\|_{\mathbb{H}^{s+1}}\le\delta$. This completes the proof. □

By Propositions 3.2-3.3, we obtain the well-posedness of the initial value problem (3.2) and, moreover, the asymptotic stability of the explicit singular solution (1.3) of equation (1.1). This completes the proof of Theorem 1.2.

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