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LINEARIZATION OF DIFFERENTIAL INCLUSIONS*

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Communicated by V. Veliov

Dedicated to our mentor and friend Asen Dontchev

ABSTRACT. In this paper we extend the approach of Dubovickii and Miljutin for linearization of the dynamics of smooth control systems to a non-smooth setting. We consider dynamics governed by a differential inclusion and we study the Clarke tangent cone to the set of all admissible trajectories starting from a fixed point. Our approach is based on the classical Filippovs theorem and on the important property “subtransversality” of two closed sets.

1. Introduction. The calculus of variations is the oldest branch of optimization, dating from over three hundred years ago (cf. Rockafellar [28]). The development of functional analysis made it possible to consider the basic problem of calculus of variations as well as optimal control problems as constrained optimization problems in an infinite-dimensional space. This approach proves its

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efficiency during the years in the seminal works of, e.g., P. D. Loewen and R.-T. Rockafellar ([24] and [25]), R.-T. Rockafellar (cf. [27]), A. Ioffe and V. Tihomirov (cf. [18]), F. Clarke ([8]) and etc.

As early as in 1965, Dubovickii and Miljutin applied systematically functional analysis based approach (cf. [14]). They realized the importance of convex approximations of closed sets for obtaining necessary conditions for nonlinear problems in optimization, in particular in dynamic optimization. They consider the dynamics of the system as a constraint as well. More precisely, they study the integral operator $P : C([t_0, t_1], \mathbb{R}^n) \times L^1([t_0, t_1], U) \rightarrow C([t_0, t_1], \mathbb{R}^n)$, where U is a closed subset of $\mathbb{R}^m$, defined by

$$P(x, u)(t) := x(t) - x_0 - \int_{t_0}^t f(x(\tau), u(\tau), \tau) d\tau, t \in [t_0, t_1].$$

Here, $(x, u, t) \mapsto f(x, u, t)$ is assumed to be measurable with respect to t for each (x, u) and continuously differentiable with respect to (x, u) for each $t \in [t_0, t_1]$ with bounded partial derivatives.

Then the constraint Q corresponding to the dynamics can be written as follows:

$$Q := \{(x, u) \in C([t_0, t_1], \mathbb{R}^n) \times L^1([t_0, t_1], U) : P(x, u) = \mathbf{0}\}.$$

Let $(\bar{x}, \bar{u}) \in Q$. Applying the Lyusternik theorem, they obtain that the tangent space $\hat{T}_Q(\bar{x}, \bar{u})$ to the set Q at the point $(\bar{x}, \bar{u})$ is the set of all $(x, u) \in C([t_0, t_1], \mathbb{R}^n) \times L^1([t_0, t_1], U)$ satisfying for almost all $t \in [t_0, t_1]$ the linearized system

$$\dot{x}(t) = f'_x(\bar{x}(t), \bar{u}(t), t)x(t) + f'_u(\bar{x}(t), \bar{u}(t), t)u(t) \quad \text{with } x(0) = \mathbf{0}.$$

This approach is extended to a non-smooth setting by Aubin and Frankowska (cf. [4], [16] and [17]). They provide estimates of the contingent, adjacent and circatangent derivatives of the solution map associated with an differential inclusion. Our contribution is very close to their results, but in different function spaces.

In this paper we consider dynamics governed by a differential inclusion instead of a smooth control system. Let us denote by Q the set of all

$$(x, y) \in L^\infty([0, T], \mathbb{R}^n) \times L^1([0, T], \mathbb{R}^n)$$

such that

$$x(t) = x_0 + \int_0^t y(\tau) d\tau \quad \text{and} \quad y(t) \in F(x(t)) \quad \text{a.e on } [0, T],$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is Lipschitz continuous.

Remark 1.1. For simplicity, we consider here the most simple but non trivial case. The obtained results can be generalized also to the non autonomous case. But this will be a part of a future work.

Let $(\bar{x}, \bar{y}) \in Q$. Then we try to approximate the set Q by identifying the Clarke tangent cone $\hat{T}_Q(\bar{x}, \bar{y})$ to the set Q at the point $(\bar{x}, \bar{y}) \in Q$. Note that we follow the same idea of Dubovickii and Miljutin to approximate the dynamics but in a different space: instead of the space of (trajectory, control) we use the space of (trajectory, velocity). This simplifies significantly the proofs and allows to relax the smoothness assumptions.

Our main result is the following

Theorem 1.2. *Let the absolutely continuous function $\bar{x}$ satisfy $(\bar{x}, \dot{\bar{x}}) \in Q$. Then it holds true that*

$$\begin{aligned} \hat{T}_Q(\bar{x}, \dot{\bar{x}}) \supset \{ (v, \dot{v}) \in X \times Y : v \text{ is absolutely continuous on } [0, T] \text{ and} \\ (v(t), \dot{v}(t)) \in \hat{T}_{Gr F}(\bar{x}(t), \dot{\bar{x}}(t)) \text{ a.e. on } [0, T] \}. \end{aligned}$$

Our approach is based on the classical Filippov's theorem and on the important property "subtransversality" of two closed sets. This term is recently introduced in [13] in relation to proving linear convergence of the alternating projections algorithm. However, this property has been around for more than 20 years as well, but under different names – see Remark 4 in [23] and the references therein. The definition below is taken from the recent book [20].

Definition 1.3. *Let A and B be closed subsets of the Banach space X . A and B are said to be subtransversal at $x_0 \in A \cap B$, if there exist $\delta > 0$ and $K > 0$, such that*

$$d(x, A \cap B) \leq K(d(x, A) + d(x, B))$$

for all x in a δ -neighborhood of x_0 .

Here the distance $d(y, C)$ between the point y and the set C is defined as

$$d(y, C) := \inf \{ \|y - z\| : z \in C \}.$$

A characterization of subtransversality is metric subregularity of some associated map (cf., for example, [19], [21], [5], [9], [10], [2]). Therefore, in fact, the present paper is based on the property "metric subregularity". This concept

is widely studied in the book of Asen L. Dontchev and R. Tyrrell Rockafellar [12], where many interesting and nontrivial applications are presented.

The paper is organized as follows. The next section contains statement of the problem and some preliminaries as well as a simple subtransversality criterion. The third section is devoted to the study of the Clarke tangent cone to a set of functions on an interval such that their values almost everywhere belong to the graph of a Lipschitz continuous multi-valued map. Linearization of dynamics is obtained in the last section.

2. Statement of the problem and some preliminaries. Throughout the paper if Z is a Banach space, $\bar{\mathbf{B}}_r(z)$ will denote the closed ball with radius r centered at z . We denote by $\bar{\mathbf{B}}_Z$ ($\bar{\mathbf{B}}$ if there is no ambiguity about the space) the closed unit ball centered at the origin.

Let us remind the definition of the Pompeiu-Hausdorff distance $d(C, D)$ between two subsets C and D of Z (cf., for example, [12], page 144):

$$d(C, D) = \inf \{ \tau \geq 0 : C \subset D + \tau \bar{\mathbf{B}}_Z, D \subset C + \tau \bar{\mathbf{B}}_Z \}.$$

Let the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be Lipschitz continuous with respect to the Pompeiu-Hausdorff distance with Lipschitz constant L , i.e.

$$d(F(x), F(y)) \leq L \|x - y\| \text{ for each } x \text{ and } y \text{ from } \mathbb{R}^n.$$

Let us consider the autonomous differential inclusion

$$(2.1) \quad \dot{x}(t) \in F(x(t)) \text{ a.e. on } [0, T], \quad x(0) = x_0.$$

Without loss of generality we may think that $x_0 = \mathbf{0}$. We denote by $X := L^\infty([0, T], \mathbb{R}^n)$ and $Y := L^1([0, T], \mathbb{R}^n)$, and define the sets $Q := A \cap B$, where

$$A := \left\{ (x, y) \in X \times Y : x(t) = \int_0^t y(s) ds \text{ for each } t \in [0, T] \right\}$$

and

$$B := \{ (x, y) \in X \times Y : (x(t), y(t)) \in \text{Gr } F \text{ a.e. on } [0, T] \}.$$

We consider the Banach space $X \times Y$ equipped with the norm

$$\|(x, y)\| = LT\|x\|_\infty + \|y\|_1.$$

It holds true that an absolutely continuous function $\bar{x}$ is a solution of (2.1) if and only if $(\bar{x}, \dot{\bar{x}}) \in Q = A \cap B$.

We fix a point $(\bar{x}, \bar{y}) \in Q$ and we are interested in the Clarke tangent cone $\hat{T}_Q(\bar{x}, \bar{y})$ to the set Q at the point $(\bar{x}, \bar{y}) \in Q$. Here is the respective definition:

Definition 2.1. *Let S be a closed subset of the Banach space Z and $z_0 \in S$. We say that the vector v belongs to the Clarke tangent cone $\hat{T}_S(z_0)$ to the set S at the point z_0 if for each $\varepsilon > 0$ there exist $\delta > 0$ and $\lambda > 0$ such that for each point $z \in S \cap (z_0 + \delta \bar{B})$ the set $S \cap (z + t(v + \varepsilon \bar{B}))$ is non empty for each $t \in [0, \lambda]$.*

We shall use the following characterization of the Clarke tangent cone taken from [29] (cf. Lemma 2.1):

Lemma 2.2. *In order for v to be in the complement of $\hat{T}_S(z_0)$ to the set S at the point z_0 it is necessary and sufficient that there be a neighbourhood V of v such that for each neighborhood Ω of z_0 there exists a point $z \in \Omega \cap S$ and a positive real λ such that*

$$(z + (0, \lambda)V) \cap S = \emptyset.$$

As we expressed the set Q representing the dynamics as an intersection of two relatively simpler sets, we need some relation between the Clarke tangent cone to the intersection and the intersection of the Clarke tangent cones to both sets at the reference point. More precisely, we need

Definition 2.3. *Let A and B be closed subsets of the Banach space Z and let z_0 belong to $A \cap B$. We say that A and B have tangential intersection property at z_0 with respect to the Clarke tangent cones $\hat{T}_A(z_0)$ and $\hat{T}_B(z_0)$ if*

$$\hat{T}_{A \cap B}(z_0) \supset \hat{T}_A(z_0) \cap \hat{T}_B(z_0).$$

The above definition is taken from [6]. In the same paper the following assertion is proved

Proposition 2.4. *Let A and B be closed subsets of the Banach space Z and let A and B be subtransversal at $z_0 \in A \cap B$. Then the sets A and B have tangential intersection property at z_0 with respect to the Clarke tangent cones $\hat{T}_A(z_0)$ and $\hat{T}_B(z_0)$.*

To apply the above written proposition, we need to verify that the sets A and B are subtransversal. The next assertion is useful in our particular case.

It is the same as Theorem 1 (iii) in [23] (in finite dimensions) or Proposition 1 in [22] (in normed spaces). Here, we provide an easy proof for readers' convenience.

Proposition 2.5. *Let A and B be closed subsets of the metric space X . A and B are subtransversal at $\bar{x} \in A \cap B$, if there exist $K > 0$ and $\delta > 0$ such that*

$$d(x, A \cap B) \leq K(d(x, B))$$

for all $x \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$.

Proof. Let y in $\bar{\mathbf{B}}_{\frac{\delta}{3}}(\bar{x})$ be arbitrary. We choose a positive real $\mu \in (0, \delta/3)$ and a point $x \in A$ such that $d(y, x) \leq d(y, A) + \mu$. Using this inequality and the inclusion $\bar{x} \in A$, we obtain that

$$d(x, \bar{x}) \leq d(\bar{x}, y) + d(y, x) \leq \frac{\delta}{3} + d(y, A) + \mu \leq d(\bar{x}, y) + \frac{2\delta}{3} \leq \delta.$$

Thus, $x \in \bar{\mathbf{B}}_\delta(\bar{x})$ and, using the assumption, we obtain the estimate

$$\begin{aligned} d(y, A \cap B) &\leq d(y, x) + d(x, A \cap B) \leq d(y, x) + Kd(x, B) \\ &\leq d(y, A) + \mu + Kd(y, B) \leq (K + 1)(d(y, A) + d(y, B)) + \mu. \end{aligned}$$

By letting $\mu \downarrow 0$ we obtain

$$d(y, A \cap B) \leq (K + 1)(d(y, A) + d(y, B)). \quad \square$$

3. Clarke tangent cones.

Let us remind that

$$B = \{(x, y) \in X \times Y : (x(t), y(t)) \in \text{Gr } F \text{ a.e. on } [0, T]\},$$

where the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant $L > 0$.

The main result in this section is the following

Theorem 3.1. *Let $(\bar{x}, \bar{y})$ be an element of B . Then, it holds true that*

$$\hat{T}_B(\bar{x}, \bar{y}) = \{(v, w) \in X \times Y : (v(t), w(t)) \in \hat{T}_{Gr F}(\bar{x}(t), \bar{y}(t)) \text{ a.e. on } [0, T]\}.$$

The proof of the theorem is split in the following two propositions. We want to point out that the first proposition holds true without the assumption of Lipschitz continuity of F .

Proposition 3.2. *Let $(\bar{x}, \bar{y})$ be an element of B . Then, it holds true that*

$$\hat{T}_B(\bar{x}, \bar{y}) \subset \{(v, w) \in X \times Y : (v(t), w(t)) \in \hat{T}_{\text{Gr } F}(\bar{x}(t), \bar{y}(t)) \text{ a.e. on } [0, T]\}.$$

Proof. Let (v, w) be an arbitrary element of $\hat{T}_B(\bar{x}, \bar{y})$. Let us assume the contrary: there exists a measurable subset E of $[a, b]$ with measure $\mu(E) > 0$ such that $(v(t), w(t)) \notin \hat{T}_{\text{Gr } F}(\bar{x}(t), \bar{y}(t))$ for all $t \in E$.

According to Treiman's Lemma 2.2, the vector $(v(t), w(t))$ does not belong to $\hat{T}_{\text{Gr } F}(\bar{x}(t), \bar{y}(t))$ if and only if there exists $\varepsilon_t > 0$ such that for each positive integer k there exist a point $(x_{t,k}, y_{t,k}) \in \text{Gr } F$ with

$$\|(x_{t,k}, y_{t,k}) - (\bar{x}(t), \bar{y}(t))\| < \frac{1}{k}$$

and a number $\lambda_{t,k} \in \left(0, \frac{1}{k}\right)$ such that

$$(3.1) \quad ((x_{t,k}, y_{t,k}) + (0, \lambda_{t,k})(v(t), w(t)) + \bar{\mathbf{B}}_{\varepsilon_t}(\mathbf{0})) \cap \text{Gr } F = \emptyset$$

holds true.

Because $\mu(E) > 0$ and we have that

$$E = \bigcup_{s=1}^{\infty} E_s, \quad \text{where } E_s = \left\{t \in E : \varepsilon_t > \frac{1}{s}\right\},$$

there exists a positive integer s such that $\mu(E_s) > 0$. We set

$$\varepsilon_0 := \inf\{\varepsilon_t : t \in E_s\} \geq \frac{1}{s} > 0.$$

Without loss of generality we may think that $\varepsilon_0 \in (0, 1)$. Next, we choose a compact subset E_0 of E_s such that $\mu(E_0) > 0$.

Let us fix an arbitrary positive integer k . Since the functions v , w , $\bar{x}$ and $\bar{y}$ are measurable on $[0, T]$, Lusin's theorem implies the existence of a compact subset $\bar{E}$ of E_0 with $\mu(\bar{E}) > 0$ such that the restrictions of v , w , $\bar{x}$ and $\bar{y}$ on the set $\bar{E}$ are continuous, hence uniformly continuous. Then, there exists $\delta_0 \in \left(0, \frac{1}{k}\right)$ such that

$$(3.2) \quad \|v(t_2) - v(t_1)\| < \frac{\varepsilon_0}{2}, \|w(t_2) - w(t_1)\| < \frac{\varepsilon_0}{2},$$

$$(3.3) \quad \|\bar{x}(t_2) - \bar{x}(t_1)\| < \frac{1}{k} \text{ and } \|\bar{y}(t_2) - \bar{y}(t_1)\| < \frac{1}{k},$$

whenever $t_1, t_2 \in \bar{E}$ with $|t_2 - t_1| < \delta_0$.

The compactness of the set $\bar{E}$ implies that there exist points

$$t_1 < t_2 < \dots < t_m \text{ in } \bar{E},$$

such that for each $t \in \bar{E}$ there exists $i \in \{1, 2, \dots, m\}$ with $|t - t_i| < \delta_0$. Using that $(v(t_i), w(t_i)) \notin \hat{T}_{\text{Gr } F}(\bar{x}(t_i), \bar{y}(t_i))$ and that $\varepsilon_0 \leq \varepsilon_{t_i}$ for each $i \in \{1, 2, \dots, m\}$, we obtain the existence of $(x_{t_i, k}, y_{t_i, k}) \in \text{Gr } F$ with $\|(x_{t_i, k}, y_{t_i, k}) - (\bar{x}(t_i), \bar{y}(t_i))\| < \frac{1}{k}$ and $\lambda_{t_i, k} \in \left(0, \frac{1}{k}\right)$ such that

$$(3.4) \quad ((x_{t_i, k}, y_{t_i, k}) + (0, \lambda_{t_i, k})((v(t_i), w(t_i)) + \bar{\mathbf{B}}_{\varepsilon_0}(\mathbf{0}))) \cap \text{Gr } F = \emptyset$$

according to (3.1). We set

$$\lambda_k = \min\{\lambda_{t_i, k} : i \in \{1, 2, \dots, m\}\} \in \left(0, \frac{1}{k}\right)$$

and

$$(x_k(t), y_k(t)) := \begin{cases} (x_{t_i, k}, y_{t_i, k}), & \text{if } t \in \bar{E} \text{ and} \\ & i = \min\{j \in \{1, 2, \dots, m\} : |t - t_j| < \delta_0\} \\ (\bar{x}(t), \bar{y}(t)), & \text{if } t \notin \bar{E} \end{cases}.$$

Clearly, we have $(x_k(t), y_k(t)) \in \text{Gr } F$ for each $t \in [0, T]$ and $(x_k, y_k) \in X \times Y$, hence $(x_k, y_k) \in B$. Using the triangle inequality, the definitions of $(x_k(t), y_k(t))$, $(x_{t_i, k}, y_{t_i, k})$ and (3.3), we obtain for $t \in \bar{E}$

$$\begin{aligned} \|(x_k(t), y_k(t)) - (\bar{x}(t), \bar{y}(t))\| &\leq \|(x_{t_i, k}, y_{t_i, k}) - (\bar{x}(t_i), \bar{y}(t_i))\| \\ &+ \|(\bar{x}(t_i), \bar{y}(t_i)) - (\bar{x}(t), \bar{y}(t))\| < \frac{1}{k} + \frac{1}{k} = \frac{2}{k}. \end{aligned}$$

Therefore, due to (3.2) and the choice of λ_k , for each $t \in \bar{E}$ it holds true

$$\begin{aligned} &(x_k(t), y_k(t)) + (0, \lambda_k) \left((v(t), w(t)) + \bar{\mathbf{B}}_{\frac{\varepsilon_0}{2}}(\mathbf{0}) \right) \\ &= (x_{t_i, k}, y_{t_i, k}) + (0, \lambda_k) \left((v(t_i), w(t_i)) + \bar{\mathbf{B}}_{\frac{\varepsilon_0}{2}}(\mathbf{0}) + (v(t), w(t)) - (v(t_i), w(t_i)) \right) \end{aligned}$$

$$\subset (x_{t_i,k}, y_{t_i,k}) + (0, \lambda_{t_i,k}) \left((v(t_i), w(t_i)) + \bar{\mathbf{B}}_{\varepsilon_0}(\mathbf{0}) \right).$$

Using (3.4), we obtain that

$$(3.5) \quad \left((x_k(t), y_k(t)) + (0, \lambda_k) \left((v(t), w(t)) + \bar{\mathbf{B}}_{\frac{\varepsilon_0}{2}}(\mathbf{0}) \right) \right) \cap \text{Gr } F = \emptyset$$

for $t \in \bar{E}$.

We have that $(v, w) \in \hat{T}_B(\bar{x}, \bar{y})$. Then for

$$(3.6) \quad \varepsilon := \frac{\varepsilon_0 \min\{1, \mu(\bar{E})\}}{4} < \frac{\varepsilon_0}{2}$$

there exists $\delta > 0$ such that for each $(x, y) \in ((\bar{x}, \bar{y}) + \bar{\mathbf{B}}_\delta(\mathbf{0})) \cap B$ and all $\lambda \in [0, \delta)$ it holds

$$(3.7) \quad ((x, y) + \lambda((v, w) + \bar{\mathbf{B}}_\varepsilon(\mathbf{0}))) \cap B \neq \emptyset.$$

We fix an arbitrary positive $k > \frac{(L+1)T}{\delta}$ and obtain that for the corresponding element (x_k, y_k) the following inequality holds true

$$\|(x_k, y_k) - (\bar{x}, \bar{y})\| = LT\|x_k - \bar{x}\|_\infty + \|y_k - \bar{y}\|_1 \leq LT\frac{1}{k} + T\frac{1}{k} = \frac{(L+1)T}{k} < \delta.$$

Then (3.7) implies that for each $\lambda \in [0, \delta)$ there exists $(p, q) \in X \times Y$ with $\|(p, q) - (v, w)\| < \varepsilon$ such that

$$(x_k, y_k) + \lambda(p, q) \in B,$$

i.e.

$$(3.8) \quad (x_k(t), y_k(t)) + \lambda(p(t), q(t)) \in \text{Gr } F \text{ a.e. on } [0, T].$$

We set

$$C := \left\{ t \in \bar{E} : \|q(t) - w(t)\| \geq \frac{\varepsilon_0}{2} \right\}.$$

Then

$$\begin{aligned} \varepsilon &\geq \|q - w\|_1 = \int_0^T \|q(t) - w(t)\| dt \geq \int_{\bar{E}} \|q(t) - w(t)\| dt \\ &\geq \int_C \|q(t) - w(t)\| dt \geq \mu(C) \frac{\varepsilon_0}{2}, \end{aligned}$$

and hence

$$\mu(C) \leq \frac{2\varepsilon}{\varepsilon_0} \leq \frac{2\varepsilon_0\mu(\bar{E})}{4\varepsilon_0} = \frac{\mu(\bar{E})}{2}.$$

Therefore, $\mu(\bar{E} \setminus C) \geq \frac{\mu(\bar{E})}{2} > 0$. The definition of the set C implies that

$$q(t) \in w(t) + \bar{\mathbf{B}}_{\frac{\varepsilon_0}{2}}(\mathbf{0}) \text{ for each } t \in \bar{E} \setminus C.$$

Also, the inequalities

$$\varepsilon > \|p - v\|_\infty \geq \|p(t) - v(t)\| \text{ for almost each } t \in [0, T]$$

imply that

$$p(t) \in v(t) + \bar{\mathbf{B}}_\varepsilon(\mathbf{0}) \text{ for each } t \in \bar{E} \setminus C.$$

According to (3.6) we have that $0 < \varepsilon < \frac{\varepsilon_0}{2}$. So, the last two inclusions and (3.8) imply that

$$(3.9) \quad ((x_k(t), y_k(t)) + \lambda((v(t), w(t)) + \bar{\mathbf{B}}_\varepsilon(\mathbf{0}))) \cap \text{Gr } F \neq \emptyset$$

for each $t \in \bar{E} \setminus C$. Because $\bar{E} \setminus C \subset \bar{E}$ and $\mu(\bar{E} \setminus C) > 0$, the last relation contradicts to (3.5). This completes the proof. $\square$

Proposition 3.3. *Let $(\bar{x}, \bar{y})$ be an element of B . Then, it holds true that*

$$\hat{T}_B(\bar{x}, \bar{y}) \supset \{(v, w) \in X \times Y : (v(t), w(t)) \in \hat{T}_{\text{Gr } F}(\bar{x}(t), \bar{y}(t)) \text{ a.e. on } [0, T]\}.$$

Proof. Without loss of generality, we may assume that $L \geq 1$. Let (v, w) be an arbitrary element of the right-hand side of the above inclusion. Let us fix an arbitrary $\varepsilon \in (0, 1)$.

The assumption $(v(t), w(t)) \in \hat{T}_{\text{Gr } F}(\bar{x}(t), \bar{y}(t))$ a.e. on $[0, T]$ directly yields that for almost all $t \in [0, T]$ there exists $\delta_t > 0$ such that for all $(x', y') \in \text{Gr } F$ with $\|x' - \bar{x}(t)\| < \delta_t$ and $\|y' - \bar{y}(t)\| < \delta_t$ and for all $\lambda \in (0, \delta_t)$ it holds

$$(3.10) \quad \left((x', y') + \lambda \left((v(t), w(t)) + \frac{\varepsilon}{4(L+1)T} \bar{\mathbf{B}} \right) \right) \cap \text{Gr } F \neq \emptyset.$$

Since $w \in Y$, there exists $\nu \in (0, \varepsilon)$ such that $\int_E \|w(t)\| dt < \frac{\varepsilon}{8}$ for each set $E \subset [0, T]$ with $\mu(E) < \nu$.

Let $\bar{\varepsilon} := \min \left\{ \nu, \frac{\varepsilon}{8L(1 + \|v\|_\infty)} \right\} \in (0, \varepsilon)$. Using Lusin's theorem we obtain a compact set $\bar{E} \subset [0, T]$ with $\mu(\bar{E}) > T - \frac{\bar{\varepsilon}}{2}$ such that the measurable functions $\bar{x}, \bar{y}, v$ and w are continuous on $\bar{E}$. Then, for each $t \in \bar{E}$ there exists $\eta_t \in (0, \delta_t)$ such that

$$(3.11) \quad \|v(t) - v(\tau)\| < \frac{\varepsilon}{4(L+1)T}, \quad \|w(t) - w(\tau)\| < \frac{\varepsilon}{4(L+1)T},$$

$$(3.12) \quad \|\bar{x}(t) - \bar{x}(\tau)\| < \frac{\delta_t}{2} \text{ and } \|\bar{y}(t) - \bar{y}(\tau)\| < \frac{\delta_t}{2}$$

for all $\tau \in \bar{E}$ with $|\tau - t| < \eta_t$.

Because $\{(t - \eta_t, t + \eta_t)\}_{t \in [0, T]}$ is an open cover of the compact set $\bar{E}$, there exists a finite set $\{t_i\}_{i=1}^k$ such that $\bar{E} \subset \bigcup_{i=1}^k (t_i - \eta_{t_i}, t_i + \eta_{t_i})$.

Let us set $\eta := \frac{1}{2} \min\{LT, 1\} \cdot \min\{\eta_{t_1}, \eta_{t_2}, \dots, \eta_{t_k}\} > 0$ and choose an arbitrary $\delta \in \left(0, \frac{\eta\bar{\varepsilon}}{4}\right)$. Next, we fix an arbitrary point $(x, y) \in B$ with $\|(x, y) - (\bar{x}, \bar{y})\| < \delta$ and an arbitrary real $\lambda \in (0, \delta)$. Due to the chosen norm in $X \times Y$, $\|(x, y) - (\bar{x}, \bar{y})\| < \delta$ implies that

$$(3.13) \quad \|x(t) - \bar{x}(t)\| < \frac{\delta}{LT} \text{ a.e. on } [0, T] \text{ and } \int_0^T \|y(t) - \bar{y}(t)\| dt < \delta.$$

Let us set

$$E_0 := \left\{ t \in [0, T] : \|y(t) - \bar{y}(t)\| > \frac{2\delta}{\bar{\varepsilon}} \right\} \subset [0, T].$$

Then we have that

$$\delta > \int_0^T \|y(t) - \bar{y}(t)\| dt \geq \int_{E_0} \|y(t) - \bar{y}(t)\| dt > \mu(E_0) \frac{2\delta}{\bar{\varepsilon}},$$

hence $\mu(E_0) < \frac{\bar{\varepsilon}}{2}$.

For every $t \in \bar{E} \setminus E_0$ there exists an index $i \in \{1, 2, \dots, k\}$ for which $t \in (t_i - \eta_{t_i}, t_i + \eta_{t_i})$. Using (3.12), (3.13) and the definition of E_0 , we obtain

$$\|x(t) - \bar{x}(t_i)\| \leq \|x(t) - \bar{x}(t)\| + \|\bar{x}(t) - \bar{x}(t_i)\| < \frac{\delta}{LT} + \frac{\delta_{t_i}}{2} < \delta_{t_i}$$

and

$$\|y(t) - \bar{y}(t_i)\| \leq \|y(t) - \bar{y}(t)\| + \|\bar{y}(t) - \bar{y}(t_i)\| < \frac{2\delta}{\varepsilon} + \frac{\delta_{t_i}}{2} < \delta_{t_i}$$

because $\delta < \frac{\eta\bar{\varepsilon}}{4} \leq \min\{LT, 1\} \frac{\eta_{t_i}\bar{\varepsilon}}{4} < \min\{LT, 1\} \frac{\delta_{t_i}\bar{\varepsilon}}{4} < \frac{\delta_{t_i}\varepsilon}{4}$. Moreover, we have that

$$\|v(t) - v(t_i)\| < \frac{\varepsilon}{4(L+1)T} \text{ and } \|w(t) - w(t_i)\| < \frac{\varepsilon}{4(L+1)T}$$

due to (3.11). Therefore, (3.10), applied for $t := t_i$, $x' := x(t)$ and $y' := y(t)$, implies that

$$\begin{aligned} (3.14) \quad \emptyset &\neq \left((x(t), y(t)) + \lambda \left((v(t_i), w(t_i)) + \frac{\varepsilon}{4(L+1)T} \bar{\mathbf{B}} \right) \right) \cap \text{Gr } F \\ &\subset \left((x(t), y(t)) + \lambda \left((v(t), w(t)) + \frac{\varepsilon}{2(L+1)T} \bar{\mathbf{B}} \right) \right) \cap \text{Gr } F \end{aligned}$$

for all $t \in \bar{E} \setminus E_0$ and $\lambda \in (0, \delta)$.

Next, let us choose an arbitrary $t \in ([0, T] \setminus \bar{E}) \cup E_0 =: E_1$. Using the triangle inequality and the Lipschitz continuity of F , we estimate

$$\begin{aligned} (3.15) \quad &d(y(t) + \lambda w(t), F(x(t) + \lambda v(t))) \\ &\leq d(y(t) + \lambda w(t), y(t)) + d(y(t), F(x(t) + \lambda v(t))) \\ &\leq \lambda \|w(t)\| + d(F(x(t)), F(x(t) + \lambda v(t))) \leq \lambda \|w(t)\| + \lambda L \|v(t)\|. \end{aligned}$$

Let us define the measurable function $\mu : [0, T] \rightarrow \mathbb{R}$ as

$$(3.16) \quad \mu(t) = \|w(t)\| + L\|v(t)\| + L$$

for all $t \in [0, T]$ and define the set-valued mapping $H : [0, T] \rightarrow \mathbb{R}^{2n}$ as follows

$$H(t) := \begin{cases} \left((x(t), y(t)) + \lambda \left((v(t), w(t)) + \frac{\varepsilon}{2(L+1)T} \bar{\mathbf{B}}_{\mathbb{R}^{2n}} \right) \right) \cap \text{Gr } F, \\ \hspace{15em} \text{for } t \in \bar{E} \setminus E_0 \\ (x(t) + \lambda v(t), y(t) + \lambda(w(t) + \mu(t)\bar{\mathbf{B}}_{\mathbb{R}^n})) \cap \text{Gr } F, \text{ for } t \in E_1 \end{cases}$$

We have that $H(t)$ is nonempty for all $t \in \bar{E} \setminus E_0$ due to (3.14), and $H(t)$ is nonempty for all $t \in E_1$ due to (3.15). It is straightforward to check that H

is measurable (cf., for example, [7] and [11]). Then, Theorem III.6 from [7] implies that there exists a measurable selection $h : [0, T] \rightarrow \mathbb{R}^{2n}$ of H . We define $(v'(t), w'(t)) \in \mathbb{R}^{2n}$ for each $t \in [0, T]$ from the equality

$$h(t) = (x(t) + \lambda v'(t), y(t) + \lambda w'(t)).$$

It is clear that $(x(t) + \lambda v'(t), y(t) + \lambda w'(t)) \in \text{Gr } F$, v' and w' are measurable. Moreover, the first component is essentially bounded and the second component is close to a summable function, and hence $(v', w') \in X \times Y$.

If we show that $\|(v, w) - (v', w')\| < \varepsilon$, the proof will be complete. Since $[0, T] \setminus E_1 = \bar{E} \setminus E_0$, $v(t) = v'(t)$ for each $t \in E_1$ and

$$\|v(t) - v'(t)\| \leq \frac{\varepsilon}{2(L+1)T} < \frac{\varepsilon}{2LT} \text{ for each } t \in \bar{E} \setminus E_0,$$

we have that

$$\|v - v'\|_\infty < \frac{\varepsilon}{2LT}.$$

Moreover, using (3.10), (3.16), the definition of $\bar{\varepsilon}$ and the fact that

$$\mu(E_1) \leq \mu([0, T] \setminus \bar{E}) + \mu(E_0) < \bar{\varepsilon},$$

we obtain that

$$\begin{aligned} \int_0^T \|w(t) - w'(t)\| dt &\leq \int_{\bar{E} \setminus E_0} \|w(t) - w'(t)\| dt + \int_{E_1} \|w(t) - w'(t)\| dt \\ &\leq \int_0^T \frac{\varepsilon}{2(L+1)T} dt + \int_{E_1} (\|w(t)\| + L\|v(t)\| + L) dt \\ &= \frac{\varepsilon}{2(L+1)} + \int_{E_1} \|w(t)\| dt + L \int_{E_1} \|v(t)\| dt + L\mu(E_1) \\ &\leq \frac{\varepsilon}{2(L+1)} + \frac{\varepsilon}{8} + L(\|v\|_\infty + 1)\mu(E_1) \leq \frac{\varepsilon}{2(L+1)} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8}. \end{aligned}$$

Since $L \geq 1$, we obtain that

$$\|(v, w) - (v', w')\| = LT\|v - v'\|_\infty + \|w - w'\|_1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \varepsilon,$$

which completes the proof. $\square$

4. Differential inclusions. Let us remind that we are working with the differential inclusion (2.1) whose right-hand side $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and the sets A and B are defined in the beginning of Section 2. Then the following result holds true:

Theorem 4.1. *Let the absolutely continuous function $\bar{x}$ be a solution of (2.1). Then the sets A and B are subtransversal at $(\bar{x}, \dot{\bar{x}}) \in A \cap B$.*

Proof. We set $\delta = 1$. Let us fix a point $(x, y) \in A$ from a δ -neighborhood of $(\bar{x}, \dot{\bar{x}})$. We have that $x(0) = \mathbf{0}$ and $y(t) = \dot{x}(t)$ a.e. on $[0, T]$ and estimate

$$\begin{aligned} p(t) &:= d(\dot{x}(t), F(x(t))) = d(y(t), F(x(t))) \leq d(y(t), F(\bar{x}(t))) + d(F(\bar{x}(t)), F(x(t))) \\ &\leq d(y(t), F(\bar{x}(t))) + L\|x(t) - \bar{x}(t)\| \leq \|y(t) - \dot{\bar{x}}(t)\| + L\|x(t) - \bar{x}(t)\| \end{aligned}$$

a.e. on $[0, T]$. The function p is summable since

$$\begin{aligned} \int_0^T p(t)dt &\leq \int_0^T \|y(t) - \dot{\bar{x}}(t)\|dt + L \int_0^T \|x(t) - \bar{x}(t)\|ds \\ &\leq \|y - \dot{\bar{x}}\|_1 + LT\|x - \bar{x}\|_\infty \leq \delta. \end{aligned}$$

By using the Filippov theorem for Lipschitzian differential inclusions (Theorem 1 in [15], see also Theorem 2.1 in [3] and Lemma 4.8.3 in [11]), there exists a solution $\tilde{x}$ of (2.1) with $\tilde{x}(0) = \mathbf{0}$ such that

$$\|\tilde{x}(t) - x(t)\| \leq \xi(t) := \int_0^t e^{L(t-s)} p(s)ds, \quad \text{for all } t \in [0, T]$$

and

$$\|\dot{\tilde{x}}(t) - \dot{x}(t)\| \leq L\xi(t) + p(t) \text{ a.e. on } [0, T].$$

Obviously, $\xi(t) \leq e^{Lt} \int_0^t p(s)ds$, for all $t \in [0, T]$ and

$$\begin{aligned} \|\dot{x} - \dot{\tilde{x}}\|_1 &= \int_0^T (L\xi(t) + p(t))dt \leq Le^{LT} \int_0^T \int_0^t p(s)dsdt + \int_0^T p(t)dt \\ &\leq (LT e^{LT} + 1) \int_0^T p(t)dt. \end{aligned}$$

Therefore, we have that

$$\begin{aligned}
 d((x, \dot{x}), A \cap B) &\leq \|(x, \dot{x}) - (\tilde{x}, \dot{\tilde{x}})\| \leq LT\|\xi\|_\infty + \|\dot{x} - \dot{\tilde{x}}\|_1 \\
 &\leq LTe^{LT} \int_0^T p(t)dt + (LTe^{LT} + 1) \int_0^T p(t)dt \\
 &= (2LTe^{LT} + 1) \int_0^T p(t)dt.
 \end{aligned}$$

Let (x_B, y_B) be an arbitrary element of B . Then

$$\begin{aligned}
 \int_0^T p(t)dt &= \int_0^T d(\dot{x}(t), F(x(t)))dt \\
 &\leq \int_0^T [\|\dot{x}(t) - y_B(t)\| + d(F(x_B(t)), F(x(t)))] dt \\
 &\leq \|\dot{x} - y_B\|_1 + LT\|x_B - x\|_\infty = \|(x - x_B, \dot{x} - y_B)\|.
 \end{aligned}$$

Since $(x_B, y_B) \in B$ is arbitrary, the above written inequalities imply that

$$\int_0^T p(t)dt \leq \inf_{(x_B, y_B) \in B} \|(x - x_B, \dot{x} - y_B)\| = d((x, \dot{x}), B).$$

Hence,

$$d((x, \dot{x}), A \cap B) \leq (2LTe^{LT} + 1) \int_0^T p(t)dt \leq (2LTe^{LT} + 1)d((x, \dot{x}), B).$$

This completes the proof. $\square$

Using Proposition 2.4 and the fact that A is a linear subspace of $X \times Y$, we obtain

Corollary 4.2. *Let the absolutely continuous function $\bar{x}$ be a solution of (2.1). Then it holds true that*

$$\hat{T}_Q(\bar{x}, \dot{\bar{x}}) \supset A \cap \hat{T}_B(\bar{x}, \dot{\bar{x}}).$$

The above written corollary and Theorem 3.1 imply Theorem 1.2.

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