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APPLICATION OF LIE SYMMETRIES TO SOLVING MODIFIED BLACK-SCHOLES EQUATION

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ABSTRACT. We perform Lie symmetry analysis on the modified Black-Scholes model described as a partial differential equation (PDE). As a result, a new complete Lie symmetry group and infinitesimal generators of the one-dimensional modified fractional Black-Scholes model are derived. Furthermore, we compute a family of exact invariant solutions that constitute the modified fractional Black-Scholes model using the associated infinitesimal generators and the corresponding similarity reduction equations. Using known solutions, more solutions are generated via group point transformations.

Until 1973, when Black-Scholes developed the Black-Scholes partial differential equation (Black-Scholes PDE), the financial option world faced uncertainties and risks that were impossible to assess. According to McKay [18], this equation was utilized as a model for option pricing and is now well-known in the

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history of modern finance. Robert Merton [13], an economist, coined the term “Black-Scholes”. However, the nature of the Black-Scholes equation

$$(1) \quad \frac{\partial u}{\partial t} + \frac{x^2 \sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + rx \frac{\partial u}{\partial x} - ru = 0,$$

is named after two scientists: Fisher Black and Myron Scholes. These three scientists (Fisher Black, Myron Scholes, and Robert Merton) collaborated, according to Henderson [10]. After years of study culminating in the release of the Black-Scholes model in 1973, additional scholars began investigating the existence of solutions to the Black-Scholes model(1) using various methodologies [1, 2, 4, 7]. By focusing on diverse science and engineering disciplines, fractional calculus began to demonstrate its importance [8, 19]. As a result, numerous researchers fully engaged in this subject and contributed significantly. Furthermore, the book by Mir Sajjad Hashemi and Dumitru Baleanu [9] has had an essential role in the conceptualization of the subject, particularly in group analysis and accurate solutions of fractional partial differential equations. By combining fractional calculus, Black-Scholes, and Lie symmetries, researchers such as [9, 13, 14, 20] could find a relationship between different methods employed to solve the Black-Scholes equation, which led to the implementation of mathematical procedures for analysis incorporating these methods. Wyss et al. [21] employed a fractional Black-Scholes equation with fractional derivatives to price the European call option. Lina Song and Weiguo Wang [20] combined Lie symmetries with finite difference method methodology to evaluate the solution of the fractional Black-Scholes option pricing model, which Wyss considered. Lina Song and Weiguo Wang [20] investigated option price $u = u(x, t)$ due to time-fractional Black-Scholes in the following form:

$$(2) \quad \frac{\partial^\alpha u}{\partial t^\alpha} = (ru - rx \frac{\partial u}{\partial x}) \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(1+\alpha)}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2},$$

where $t > 0$, $0 < \alpha \leq 1$.

r, α, Γ is the risk-free interest rate, volatility, and Gamma function, respectively. The Reimann Liouville[9] is represented by $\frac{\partial^\alpha u}{\partial t^\alpha}$. This work aims to extend and utilize the Lie symmetry technique to solve the modified fractional Black-Scholes model in financial mathematics using the same equation (2). Group invariants, particularly the Lie points symmetry, have recently been introduced into the financial market [3, 6, 16]. However, because of the computing techniques and procedures employed in solving Black-Scholes equations, fractional

derivatives are of greater interest for determining the application of Lie symmetry.

Fractional derivatives, unlike classical derivatives, are computed using the Riemann–Liouville definition. Huang et al. [11] used the time-fractional Harry-Dym equation with Riemann–Liouville derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$ to implement Lie symmetry and accurate solutions. The use of Lie symmetries to solve PDEs is considered an alternate method. The Finite Difference Method (FDM), Finite Element Method (FEM), the Adomain Decomposition Method (ADM), and methods of lines are only a few of the approaches utilized to solve PDEs. According to Qiu et al. [5], non-linear PDEs are more complex but exciting in this discipline. Because of the sensational scope and applications in several studies, there is an urgent need to conduct research to analyze solutions, particularly in fractional Black-Scholes.

Fascinating problems are observed in solving non-linear PDEs in financial mathematics, such as Kumar et al. [15] combination of the Laplace perturbation method and homotopy perturbation. This combination's efficacy resulted in accurate and precise interpretable results for the fractional Black-Scholes equation with boundary conditions for the European option pricing problem. The reduction of the compound and complicated numerical computations was caused by round-off errors. To determine their method's effectiveness, they used two examples and used Hes polynomials as well as converging power series considering Black-Scholes equation (1) in two non-linear fractional differential equations.

In the financial industry, the issue of risks and uncertainty is not only uncontrollable but also complex and impossible to analyze. When it comes to suitable asset trading, investment, and risk management, options in finance play an essential role. As a result, this problem has heightened the need for research into fractional Black-Scholes solutions using various mathematical tools and methods. This work aims to extend and implement the Lie symmetry technique in financial mathematics to solve the modified fractional Black-Scholes model using equation (2). We consider the use of option price u subject to the time-fractional Black-Scholes equation, following the work of Song et al. [20].

This paper is structured as follows. In Section 1, we find the determining systems of equations of (2). We solve each equation using a Maple package FracSym to find infinitesimals and then find the Lie algebra of obtained commutators. In Section 2, we compute the invariant solution of obtained symmetries and give graphical representations of our solutions and interpretations of the results obtained in Section 3, followed by a conclusion in Section 4.

1. Methods.

1.1. Determination of Lie symmetries of equation (2). The Lie point symmetries for (2) are given by the vector field:

$$(3) \quad X = \xi^1(x, t, u) \frac{\partial}{\partial t} + \xi^2(x, t, u) \frac{\partial}{\partial x} + \phi(x, t, u) \frac{\partial}{\partial u},$$

arising from the following change of variables

$$\begin{aligned} u_1 &= u_1(x, t, u, \epsilon) & x_1 &= x_1(x, t, u, \epsilon); & t_1 &= t_1(x, t, u, \epsilon) \\ u &= u_1(x, t, u, 0); & x &= x_1(x, t, u, 0); & t &= t_1(x, t, u, 0) \\ \frac{\partial u_1(x, t, u, 0)}{\partial \epsilon} &= \phi(x, t, u); & \frac{\partial x_1(x, t, u, 0)}{\partial \epsilon} &= \xi^2(x, t, u); & \frac{\partial t_1(x, t, u, 0)}{\partial \epsilon} &= \xi^1(x, t, u), \end{aligned}$$

if and only if

$$(4) \quad Pr^{\alpha,2} X \left(\frac{\partial^\alpha u}{\partial t^\alpha} - \left((ru - rx \frac{\partial u}{\partial x}) \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(1+\alpha)}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} \right) \right) \Big|_{(2)} = 0,$$

where the superscripts in equation (3) above are indexes, subscript (2) in equation (4) represents an equation $\frac{\partial^\alpha u}{\partial t^\alpha} = (ru - rx \frac{\partial u}{\partial x}) \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(1+\alpha)}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}$, and $Pr^{\alpha,2}$ is the second prolongation of X defined as

$$\begin{aligned} Pr^{\alpha,2} &= X + \phi_\alpha^0 \frac{\partial}{\partial t^\alpha u} + \phi^x \frac{\partial}{\partial u_x} + \phi^{xx} \frac{\partial}{\partial u_{xx}} \\ &= \xi^1(x, t, u) \frac{\partial}{\partial x} + \xi^2(x, t, u) \frac{\partial}{\partial u} + \phi(x, t, u) \frac{\partial}{\partial t} \\ &\quad + \phi_\alpha^0 \frac{\partial}{\partial t^\alpha u} + \phi^x \frac{\partial}{\partial u_x} + \phi^{xx} \frac{\partial}{\partial u_{xx}}. \end{aligned} \quad (5)$$

Here ϕ^x, ϕ^{xx} , are given by

$$(6) \quad \phi^x = D_x(\phi) - u_t D_x(\xi^1) - u_x D_x(\xi^2),$$

$$(7) \quad \phi^{xx} = D_x(\phi^x) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2),$$

Moreover, the total derivatives are:

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial v} + u_{tt} \frac{\partial}{\partial u_t} + u_{xt} \frac{\partial}{\partial u_x} + \dots \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \dots \end{aligned}$$

ξ^1, ξ^2 and ϕ are infinitesimals to be determined. Also α^{th} infinitesimal has the form:

$$(8) \quad \phi_\alpha^0 = D_t^\alpha + \xi^2 D_t^\alpha(u_x) - D_t^\alpha(\xi^2 u_x) + D_t^\alpha(D_t(\xi^1)u) \\ - D_t^{\alpha+1}(\xi^1 u) + \xi^1 D_t^{\alpha+1}(u),$$

where D_t^α is the total fractional derivative operator. By Leibniz rule[19], we have that (8) reduces to:

$$(9) \quad \phi_\alpha^0 = \frac{\partial^\alpha \phi}{\partial t^\alpha} + (\phi_u - \alpha D_t(\xi^1)) \frac{\partial^\alpha u}{\partial t^\alpha} - v \frac{\partial^\alpha \phi_u}{\partial t^\alpha} \\ + \sum_{n=1}^{\infty} \left[\binom{a}{n} \frac{\partial^\alpha \phi_u}{\partial t^\alpha} - \binom{a}{n+1} D_t^{n+1}(\xi^1) \right] D_t^{\alpha-n} u - \sum_{n=1}^{\infty} \binom{a}{n} D_t^n(\xi^2) D_t^{\alpha-n} u_x + \mu,$$

where,

$$\mu = \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{a}{n} \binom{n}{n} \binom{k}{r} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} (-u)^r \frac{\partial^m}{\partial t^m} (u^{k-r}) \frac{\partial^{n-m+k}}{\partial t^{n-m} \partial u^k},$$

note that $\mu = 0$ since $\frac{\partial^k \phi}{\partial u^k} = 0$ for $k \geq 2$, here the infinitesimal ϕ is linear in the variable u and μ . Making use of the Reimann Liouville [9] given by an equation

$$D_t^\alpha u(t, x) = \left\{ \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{u(\xi, x)}{(t-\xi)^{\alpha+1-m}} d\xi, \right.$$

for $0 < m-1 < \alpha \leq m, m \in N$ and the vector field given by equation (3), we notice that, upon substituting equation (2) into equation (5), we get the following invariant;

$$(10) \quad \phi_\alpha^0 - (ru - rxu_x)(1-\alpha) \frac{t^{-\alpha}}{\Gamma(2-\alpha)} \xi^1 - rx \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} ru_x \phi^x \\ + \frac{\Gamma(1+\alpha)}{2} \sigma^2 x^2 \phi^{xx} + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} r\phi - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} ru_x \xi^2 - \Gamma(1+\alpha) \sigma^2 xu_{xx} \xi^2 = 0.$$

Now substituting equations (6), (7), and (9) into equation (10) yield the following:

$$\begin{aligned}
(11) \quad & \frac{\partial^\alpha \phi}{\partial t^\alpha} + (\phi_u - \alpha D_t(\xi^1)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \phi u}{\partial t^\alpha} \\
& + \sum_{n=0}^{\infty} \left[\binom{a}{n} \frac{\partial^\alpha \phi_u}{\partial t^\alpha} - \binom{a}{n+1} D_t^{n+1}(\xi^1) \right] D_t^{\alpha-n} u \\
& \times \sum_{n=1}^{\infty} \binom{a}{n} D_t^n(\xi^2) D_t^{\alpha-n} u_x - (ru - rxu_s)(1-\alpha) \frac{t^{-\alpha}}{\Gamma(2-\alpha)} \\
& - \frac{rxt^{1-\alpha}}{\Gamma(2-\alpha)} [\phi_x + u_x(\phi_u - \xi_x^2) - u_x^2 \xi_u^2 - u_t \xi_x^1 - u_x u_t \xi_u^1] \\
& + \frac{\Gamma(1+\alpha) \sigma^2 x^2}{\Gamma(2-\alpha)} [\phi_{xx} + u_x(2\phi_{xu} - \xi_{xx}^2) + u_{xx}(\phi_u - 2\xi_x^2) + u_x^2(\phi_{uu} - 2\xi_{xu}^2) - u_x^3 \xi_{uu}^2 \\
& - \xi_u^1(u_t u_{xx} + 2u_x u_{xt}) - 3u_x u_{xx} \xi_u^2 - 2u_{xt} \xi_u^1 - u_t \xi_{xx}^1 - 2u_x u_t \xi_{xu}^1 - u_x^2 u_t \xi_{uu}^1] \\
& + \frac{t^{1-\alpha} r \xi}{\Gamma(2-\alpha)} - \frac{t^{1-\alpha} r u_x \phi^2}{\Gamma(2-\alpha)} - \Gamma(1+\alpha) \sigma^2 x u_{xx} \xi^2 = 0.
\end{aligned}$$

To get a system of determining equations and symmetry for the FDE of equation (2), we used the FracSym worksheet package, Maple [17], Maple Package FracSym routine, FracDet, and DESOLVEII [12]. The main goal is to provide and obtain a good balance between the information needed to solve the determining equation and the speed with which it is obtained. Also, note: the fourth argument, an integer greater or equal to 1, specifies the number of terms to be "peeled off" from the sums that occur in the extended infinitesimal function for the fractional derivative [12]. The following is a list of linear and homogeneous PDE-determining equations, where the subscript denotes the derivative for the given variable:

$$(12) \quad \phi_{uu} = 0,$$

$$(13) \quad \alpha \xi_t^1 = 0,$$

$$(14) \quad \alpha \xi_u^1 = 0,$$

$$(15) \quad x^2 \sigma^2 \Gamma(\alpha+1) \xi_u^2 = 0,$$

$$(16) \quad x^2 \sigma^2 \Gamma(\alpha+1) \xi_x^2 = 0,$$

$$(17) \quad x^2 \sigma^2 \Gamma(\alpha+1) \xi_u^1 = 0,$$

$$(18) \quad x^2 \sigma^2 \Gamma(\alpha+1) \xi_{uu}^2 = 0,$$

$$(19) \quad x^2 \sigma^2 \Gamma(\alpha+1) \xi_{uu}^1 = 0,$$

$$(20) \quad \alpha \xi_u^2 (\alpha-1) = 0,$$

$$(21) \quad \alpha \xi_u^1 (\alpha - 1) = 0,$$

$$(22) \quad \alpha \xi_{tu}^2 (\alpha - 1) = 0,$$

$$(23) \quad \alpha \xi_{tu}^1 (\alpha - 1) = 0,$$

$$(24) \quad \alpha \xi_{uu}^1 (\alpha - 1) = 0,$$

$$(25) \quad \alpha \xi_{uu}^2 (\alpha - 1) = 0,$$

$$(26) \quad \alpha \xi_{tt}^1 (\alpha - 1) = 0,$$

$$(27) \quad \alpha \xi_u^2 (\alpha - 1)(\alpha - 2) = 0,$$

$$(28) \quad x^2 \sigma^2 \Gamma(\alpha + 1) \xi_u^2 (\alpha - 1) = 0,$$

$$(29) \quad \alpha \xi_{tu}^2 (\alpha - 1)(\alpha - 2) = 0,$$

$$(30) \quad \alpha \xi_{uu}^2 (\alpha - 1)(\alpha - 2) = 0,$$

$$(31) \quad \alpha (\xi_{tuu}^2) (\alpha - 1)(\alpha - 2) = 0,$$

$$(32) \quad \alpha (-\alpha \xi_{tt}^2) + 2\phi_{tu} + \xi_{tt}^2 (\alpha - 1)(\alpha - 2) = 0,$$

$$(33) \quad \alpha (\xi_{uuu}^2) (\alpha - 1)(\alpha - 2) = 0,$$

$$(34) \quad x \sigma^2 \Gamma(\alpha + 1) (\alpha x (\xi_t^2) - 2\xi_x^1 + 2\xi^2) = 0,$$

$$(35) \quad (\alpha - 1) \alpha (-\alpha \xi_{ttt}^2) + 3\phi_{tu} + 2\xi_{ttt}^2 = 0,$$

$$(36) \quad \frac{1}{\Gamma(2 - \alpha)} [-2x^2 \sigma^2 (\Gamma(\alpha + 1) \phi_{ux} \Gamma(2 - \alpha) + x^2 \sigma^2 \Gamma(\alpha + 1) \xi_{xx}^1 \Gamma(2 - \alpha) \\ + 2\xi^2 r x t^{-\alpha} \alpha - 2(\alpha) r x t^{1-\alpha} \xi_t^2 - 2\xi^2 r x t^{-\alpha} + 2r x t^{1-\alpha} \xi_x^1 - 2\xi^1 r t^{1-\alpha})] = 0,$$

$$(37) \quad \frac{1}{\Gamma(2 - \alpha)} [x (-\Gamma(2 - \alpha) \Gamma(\alpha + 1) \phi_{uu} \sigma^2 x + 2\Gamma(2 - \alpha) \Gamma(\alpha + 1) \xi_{ux}^1 \sigma^2 x \\ + 2t^{1-\alpha} \xi_u^1 r)] = 0,$$

$$(38) \quad \frac{1}{\Gamma(2 - \alpha)} x [-\Gamma(2 - \alpha) \Gamma(\alpha + 1) \xi_{ux}^2 \sigma^2 x + t^{1-\alpha} \xi_u^2 \alpha r - t^{1-\alpha} \xi_u^2 r] = 0,$$

$$(39) \quad \frac{1}{\Gamma(2 - \alpha)} [x^2 \sigma^2 \Gamma(1 + \alpha) \xi_{xx}^2 \Gamma(2 - \alpha) + 2\alpha r u t^{1-\alpha} \xi_u^2 + 2r x t^{1-\alpha} \xi_x^2] = 0.$$

The Auxiliary conditions, which include the sums and fractional derivative terms from this system is obtained using FracSym, and the resulting equations are as follows;

$$(40) \quad \xi^2(x, 0, u) = 0.$$

$$(41) \quad \frac{1}{2t^\alpha \Gamma(2-\alpha)} \left[x^2 \sigma^2 \Gamma(\alpha+1) \frac{\partial^2}{\partial x^2} \phi t^\alpha \Gamma(2-\alpha) - 2\alpha r u t \frac{\partial}{\partial x} \xi^2 \right. \\ \left. - 2u \frac{\partial^{\alpha+1}}{\partial t^2 \partial u} \phi t^\alpha \Gamma(2-\alpha) + 2\phi^2 r u \alpha + 2 \frac{\partial}{\partial u} \phi r u t \right. \\ \left. + 2r u x t \frac{\partial}{\partial x} \phi + \frac{\partial^\alpha}{\partial t^\alpha} \phi t^\alpha \Gamma(2-\alpha) - 2\xi^2 r u - 2\phi \right] = 0.$$

$$(42) \quad \sum_{n=3}^{\infty} \left[-\frac{1}{n+1} \binom{\alpha}{n} D_{t^{\alpha-n}} u D_{t^{n+1}} \xi^2 \alpha - D_{t^{\alpha-n}} u D_{t^{n+1}} \xi^2 n + D_{t^{\alpha-n}} \frac{\partial}{\partial x} u D_{t^n} \xi^1 n \right. \\ \left. + D_t^{\alpha-n} \frac{\partial}{\partial x} u D_{t^n} \xi^1 \right] + \sum_{n=3}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^{n+1}}{\partial t^n \partial u} \xi D_{t^{\alpha-n}} u \right] = 0.$$

Using DESOLVII to solve equations (12)–(39) we get the following infinitesimals for our FDE:

$$(43) \quad \xi^1 = x c_1,$$

$$(44) \quad \xi^2 = 0,$$

$$(45) \quad \phi = \beta(x, t) + u c_4.$$

Upon substitution of equations (43)–(45) into equations (41) and (42) to check whether auxiliary conditions are satisfied, we get;

$$(46) \quad \xi^2(x, 0, u) = 0.$$

$$(47) \quad \frac{1}{2t^\alpha \Gamma(2-\alpha)} \left[x^2 \sigma^2 \Gamma(\alpha+1) \frac{\partial^2}{\partial x^2} (\beta(x, t) + u c_4) t^\alpha \Gamma(2-\alpha) \right. \\ \left. - 2\alpha r u t \frac{d}{dt} 0 - 2u \frac{\partial^{\alpha+1}}{\partial t^\alpha \partial u} B(x, t) + u c_4 t^\alpha \Gamma(2-\alpha) + 2 \frac{\partial}{\partial u} (\beta(x, t) + u c_4) r u t \right. \\ \left. + 2r x t \frac{\partial}{\partial u} (\beta(x, t) + u c_4) + 2 \frac{\partial^\alpha}{\partial t^\alpha} (B(x, t) + u c_4) t^\alpha \Gamma(2-\alpha) - 2(B(x, t) + u c_4) r t \right] = 0.$$

$$(48) \quad \sum_{n=3}^{\infty} \left[-\left(\frac{1}{n+1} \right) \left[\binom{\alpha}{n} \left[D_{t^{\alpha-n}}(u(x, t)) D_{t^{n+1}}(0) \alpha - (D_{t^{\alpha-n}}(u(x, t)) D_t^{n+1}(0) n) \right. \right. \right. \\ \left. \left. + D_{t^{\alpha-n}} \left(\frac{\partial}{\partial x} (u(x, t)) \right) D_{t^n}(x c_1) n + D_{t^{\alpha-n}} \left(\frac{\partial}{\partial x} u(x, t) \right) D_t^n(x c_1) \right] \right] \\ \left. + \sum_{n=3}^{\infty} \binom{\alpha}{n} \frac{\partial^{n+1}}{\partial t^n \partial u} (\beta(x, u) + u c_4) D_{t^{\alpha-n}}(u(x, t)) \right] = 0.$$

$B(x, t)$ is an arbitrary solution of equation (47). Now evaluating Equations (47) and (48), we get (49) and (50), respectively;

$$(49) \quad \frac{1}{2t^\alpha \Gamma(2-\alpha)} \left[x^2 \sigma^2 \Gamma(\alpha+1) \frac{\partial^2}{\partial x^2} (B(x, t)) t^\alpha \Gamma(2-\alpha) \right. \\ \left. - 2\alpha u c_4 \text{pochhammer}(1-\alpha, \alpha) t^\alpha \Gamma(2-\alpha) + 2c_4 r u t + 2r x t \frac{\partial}{\partial x} B(x, t) \right. \\ \left. + 2 \left(\frac{\partial^\alpha}{\partial t^\alpha} (B(x, t) + u c_4 \text{pochhammer}(1-\alpha, \alpha)) t^\alpha \Gamma(2-\alpha) + u c_4 \right) r t \right] = 0.$$

$$(50) \quad - \frac{1}{\alpha(\alpha-1)(\alpha-2)} 6 \binom{\alpha}{3} \left(D_{t^n}(x c_1) \left(-\frac{\alpha^2}{2} + 2^\alpha - \frac{\alpha}{2} - 1 \right) D_{t^{\alpha-n}} \left(\frac{\partial}{\partial x} u(x, t) \right) \right. \\ \left. + D_{t^{n+1}}(0) \left(D_{t^{\alpha-n}}(u(x, t)) \left(-\frac{\alpha^3}{6} + 2^\alpha - \frac{5\alpha}{6} - 1 \right) \right) \right) = 0.$$

Where **pochhemmer** is defined as rising or ascending factorial, and it can be presented symbolically as:

$$(51) \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

The term $(a)_n$ follows from the hypergeometric series or functions, thus equation (46) is satisfied. Equations (49) and (50) are satisfied for total derivatives $D_{t^{n_1}}$ and $D_{t^{1+n_1}}$ for some n_1 . Therefore the final symmetry of our FDE is obtained as follows:

$$(52) \quad \xi^1 = x c_1,$$

$$(53) \quad \xi^2 = 0,$$

$$(54) \quad \phi = \beta(x, t) + u c_4,$$

where c_1, c_4 are arbitrary constants and $\beta(x, t)$ is an arbitrary solution of equation (2). Now, imposing equations (52)–(54) into equation (3), we have the symmetry generator given by:

$$(55) \quad X = x c_1(x, t, u) \frac{\partial}{\partial x} + (\beta(x, t) + u c_4) \frac{\partial}{\partial u}.$$

The symmetries are obtained by setting one constant to 1 and the rest to zeros, so in vector form, equation (2) is spanned by the vector fields:

$$(56) \quad X_1 = x \frac{\partial}{\partial x},$$

$$(57) \quad X_2 = u \frac{\partial}{\partial u}.$$

Infinite symmetry:

$$(58) \quad X_\infty = \beta(x, t) \frac{\partial}{\partial u}.$$

1.1.1. Assumptions of the solution process. Upon using DESOLV11 package, the following non zero and linearly independent assumptions are obtained:

Non-zero assumptions:

$\alpha, r, \sigma, t, x, t^\alpha, \alpha(\alpha - 1), t^\alpha \Gamma(2 - \alpha), \alpha(\alpha - 1)(\alpha - 2), x\sigma\Gamma(\alpha + 1), x\alpha \ln(x), \sigma x(\alpha - 1)\Gamma(\alpha + 1), \alpha - 2, \alpha - 1, \Gamma(2 - \alpha), \Gamma(\alpha + 1).$

Linearly independent assumptions:

$1, t, t^\alpha$

1.2. Lie algebra. Given a vector space over a field F , we define a Lie algebra as $[\cdot, \cdot]$ and it is denoted by $\mathfrak{L}$ [9]. Given a bilinear commutation law, the following properties have to be satisfied:

1. *Closure*: For $X, Y \in \mathfrak{L}$, we have that $[X, Y] \in \mathfrak{L}$.

2. *Bilinearity*: $[X, \alpha Y + \beta Z] = \alpha [X, Y] + \beta [X, Z]$, where $\alpha, \beta \in F$ and $X, Y, Z \in \mathfrak{L}$.

3. *Jacobi identity*: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

4. *Skew-symmetry*: $[X, Y] = -[Y, X]$

1.2.1. Obtaining Lie algebra. Below is a summary table of Lie algebra obtained from commutators (56), (57), and (58). We represent this in a form $[X_i, X_j]$ for some $i, j = 1, 2, \infty$;

Table 1. Lie algebras

$[X_i, X_j]$	X_1	X_2	X_∞
X_1	0	0	$x\beta_x \frac{\partial}{\partial u}$
X_2	0	0	$-\beta(x, t)$
X_∞	$-x\beta_x \frac{\partial}{\partial u}$	$\beta(x, t)$	0

2. Invariant solution.

2.1. Group invariant solution. Just like partial differential equations (PDEs), the invariant solution of a fractional differential equation is defined by considering a function $u = u(x, t)$ which is said to be an invariant solution under

fractional differential equation concerning the infinitesimal operator (3) if and only if

$$\xi^1(u, x, t)u_t + \xi^2(u, x, t)u_x = \phi(u, t, x).$$

With the assumption that ξ^1 , and ξ^2 not being zeros, we make use of a characteristic method;

$$(59) \quad \frac{dt}{\xi^1} = \frac{dx}{\xi^2} = \frac{du}{\phi}.$$

Letting two arbitrary differentiable functions $p(u, x, t)$ and $q(u, x, t)$ with $q_u \neq 0$ be independent first integral functions of (59), we turn to have a general solution of the invariant condition as $q = F(p)$. We then solve for F by substituting this solution into our original FDE (2). One should note that the resulting equation after substitution may either be solvable or not solvable depending on the nature of the equation obtained.

A function $u = \theta(x, t)$ is said to be an invariant solution of

$$(60) \quad \frac{\partial^\alpha u}{\partial t^\alpha} = F(x, t, u, u_x, u_{xx}, \dots), \quad 0 < \alpha \leq 1.$$

to the infinitesimal (3) if and only if the following hold:

$$u = \theta(x, t) \text{ satisfies equation (60).}$$

2.1.1. Characteristic equation. Considering the combination of X_1 and X_2 from equations (56) and (57), that is, $X_1 + X_2 = x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}$, we have the following characteristics equation:

$$(61) \quad \frac{dt}{0} = \frac{dx}{x} = \frac{du}{u}.$$

From the second and third ratios on (61), we have that

$$\ln(u) - \ln(x) = \ln(c_1) \implies \ln\left(\frac{u}{x}\right) = \ln(c_1) \implies \frac{u}{x} = c_1 \implies u = xc_1,$$

where c_1 is an arbitrary constant. From the first ratio on (61), we have that $t = c_2$ for some constant c_2 , therefore

$$(62) \quad u = xg(c_2) = xg(t).$$

To solve for $g(t)$, we substitute equation (63) into equation (2) and get the following result

$$(63) \quad \frac{\partial^\alpha g(t)}{\partial t^\alpha} = 0.$$

With the use of the information from [12]. Considering the Laplace transform method to solve for $g(t)$ in (63).

$$(64) \quad \mathcal{L} \left\{ \frac{\partial^\alpha g(t)}{\partial t} \right\} = 0 \implies s^\alpha G(S) - D^{-(1-\alpha)} y(0) = 0, \quad 0 < \alpha \leq 1,$$

where $G(S)$ is the Laplace transform of $g(t)$. Assume $D^{-(1-\alpha)} y(0)$ exists such that $D^{-(1-\alpha)} y(0) = c_3$, thus (64) can be written as

$$(65) \quad S^\alpha G(S) - c_3 = 0 \implies G(S) = \frac{c_3}{s^\alpha}.$$

Taking the inverse Laplace transform, we get

$$(66) \quad g(t) = \frac{c_3 t^{\alpha-1}}{\Gamma(\alpha)}.$$

Considering the symmetry X_1 given by (56), we have the following characteristic equation.

2.2. Characteristic equation.

$$(67) \quad \frac{dt}{0} = \frac{dx}{x} = \frac{du}{0}.$$

The first ratio and the last ratio in equation (67) above do not necessarily mean division by zero. Since t and u have similar variables from equation (67), it suffices to write $u = g(t)$. Substitution of u into equation (2) yields

$$(68) \quad D_t^\alpha g(t) = r g(t) \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}.$$

Solving for $g(t)$ from (68), the solution is trivial for $\alpha = 1$, that is

$$(69) \quad g(t) = k e^{rt}, \quad \text{for some constant } k.$$

However, we are interested in finding the solution for all values of α such that $0 < \alpha \leq 1$. To my knowledge and findings, equation (68) is considered a numerical solution.

3. Results and discussion.

3.1. Graphical solutions. Graphical solutions of $g(t) = \frac{c_3 t^{\alpha-1}}{\Gamma(\alpha)}$ and $g(t) = ke^{rt}$ in the xy plane and in 3D, respectively. Figures 1 and 2 depict the graphical solution of equation (66) obtained using the Lie symmetry method at $0 < t \leq 10$. Where $0 < \alpha \leq 1$, $r = 0.8$, and the stock price range from 0 to 5 in 2D and 3D, respectively. Figures 1 and 2's development trend is similar to that of a put option in a classical Black-Scholes equation(1) and the numerical solution graph in [20]. Also, Figures 3 and 4 depict the graphical solution of equation (69) obtained using the Lie symmetry method at $0 < k \leq 1$, and $r = 0.8$ in 2D and 3D, respectively. The development trend and shape of these figures are similar to that of a call option in a classical Black-Scholes (1).

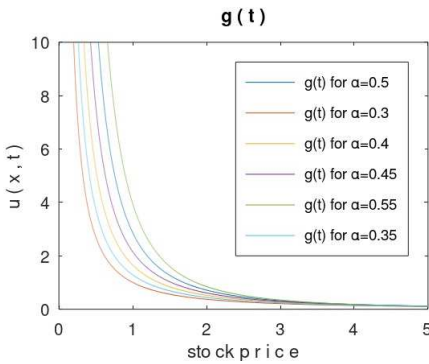


Fig. 1. A 5×5 plot of (66) with $0 < \alpha \leq 1$

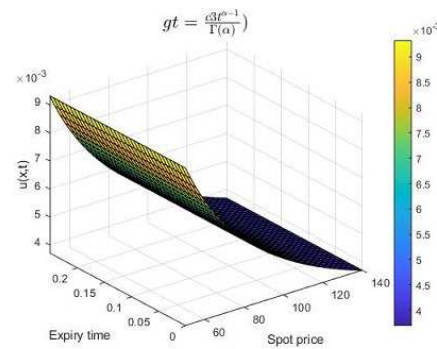


Fig. 2. 60×60 solution of (66) with $\alpha = 0.5$

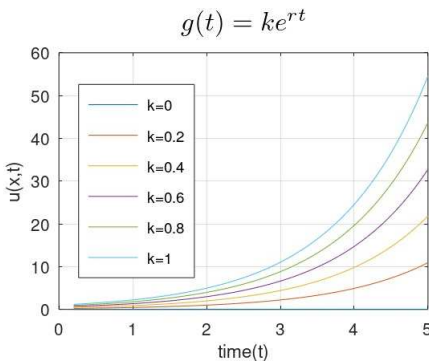


Fig. 3. Solution of (69) with $r = 0.8$

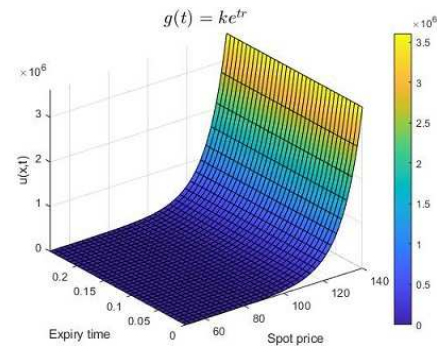


Fig. 4. 60×60 solution of (69) with $r = 0.8$

4. Conclusion. The Lie symmetry technique was used to solve the modified fractional Black-Scholes equation. Our goal was met because we could compute the determining equation using the Maple package `fracSym`. Furthermore, after solving each determining equation obtained, we constructed the infinitesimals, resulting in two invariant solutions. The obtained invariant solutions are built using a characteristic equation system. We first considered the combination of X_1 and X_2 , which gave us a solvable equation and, thus, fascinating results. However, considering X_1 alone provided a trivial solution when $\alpha = 1$, indicating a positive trend about the call option of classical Black-Scholes, but our main focus was on the fractional order of derivatives; that is, we considered $0 < \alpha \leq 1$. As a result, the result obtained for this interval $0 < \alpha \leq 1$ was not solvable. Considering the value $\alpha = 1$ in equation (2) resulted in an underlying classical Black-Scholes equation (1). However, the graphs and results show that as we increase α , the risk-free interest rate r rises. Hence, a decrease in α led to a decrease in risk r .

Moreover, the risk increases when the stock price falls because volatility rises when $\alpha \leq 1$. So this results in a negative skewness in stock returns, so, as we decrease α , the local volatility rises and indicates that x and u are negatively correlated to each other, which also results in negative skewness in stock return. As we increase the stock price x , the leverage effect increases. The comparison of $g(t)$ in (66) with numerical solutions from [20] demonstrates the power of fractional derivatives. The successful application of Lie symmetry to the fractional Black-Scholes equation demonstrates how effective and computationally light our method is.

Conflict of interests. The authors declare that there is no conflict of interest regarding the publication of this paper.

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