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MONTE CARLO SIMULATION OF PRESENT VALUE OF CASH FLOWS

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Abstract. This paper is concerned with financial application of Monte Carlo simulation of infinite series. In fact, present value of a cash flow is represented as infinite series. Hence, the Monte Carlo may be applied to approximate the present value. It is seen that the Monte Carlo method is sample approximation of the mean of cash flow computed at a stopping time. The stopping time, first, is considered as an up-crossing time point of sequence of uniform random variables and then it is replaced by up-crossing of macro-economic variables (called as factor) which brings the dependency structure to the problem which is modeled by auto-regressive to any things (ARTA) models. Here, two different Monte Carlo simulation methods are presented. A financial application is also given. Continuous time extensions of results are also proposed. Finally, a concluding remark section is given.

1. Introduction. Present value (PV) is the current value of a future sum of money or stream of cash flows given a specified rate of return. Future cash flows are discounted at the discount rate, and the higher the discount rate, the lower the present value of the future cash flows, see [2]. Determining the

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appropriate discount rate is the key to properly valuing future cash flows, whether they are earnings or obligations. In its classic form, it is a type of infinite series. Solomon [8] proposed a Monte Carlo framework for simulation of infinite series $\sum_{n=1}^{\infty} a_n$, assuming it converges. Monte Carlo simulations are used to model the probability of different outcomes in a process that cannot easily be predicted due to the intervention of random variables. It is a technique used to understand the impact of risk and uncertainty in prediction and forecasting models. Monte Carlo simulation can be used to tackle a range of problems in virtually every field such as finance, engineering, supply chain, and science. Monte Carlo simulation is also referred to as multiple probability simulation, see [7].

(i) Solomon work. To review work of Solomon [8], for example, when $a_n = \frac{c_n}{2^n}$, $n \geq 1$, let $q_n = \frac{1}{2^n}$, $n \geq 1$ it is easy to see that $q_n \in (0,1)$ and $\sum_{n=1}^{\infty} q_n = 1$. Therefore,

$$\sum_{n=1}^{\infty} \frac{c_n}{2^n} = E^Q\left(c_N\right),$$

where $Q = (q_1, q_2, \ldots)$ be the corresponding probability mass function. Throughout the current paper, roughly speaking, Q is regarded as probability measure. Therefore, notation E^Q stands for expectation with respect to probability measure Q. Let $U_i's$ be independent and identically distributed random variables (iid) uniformly distributed on $(0,1)$ (i.e., $U(0,1)$) and consider the stopping time

$$N = \inf\left\{n \mid \max_{1\leq i\leq n} U_i \leq 0.5\right\}.$$

Thus, $q_n = P(N = n)$ and Q is the probability measure arising from definition of stopping time N.

Monte Carlo simulation. Generating R sequences of samples of large sizes of U(0,1) distributed random variables and deriving $N_1, \ldots, N_R$, then, the sample mean

$$\frac{\sum_{r=1}^{R} c_{N_r}}{R},$$

approximates $E^Q\left(c_N\right)$. For a comprehensive review of this approach, see [6].

Remark 1. In this paper, there are two probability measures. The first one relates to state variables (say the innovation of ARTA model (Subsection

2.1 or return sequence of Section 3) which is denoted by P, if it is necessary somewhere of paper, and second the probability measures Q's related to different stopping times which are defined rigorously, at each part of paper. To avoid mismatch, exceptions are written by superscripts E^Q or E^P.

(ii) Financial application. For a financial application of this approach, consider the present value (pv) at time zero of deterministic cash flow series $\{c_n\}_{n\geq 1}$ which is

$$pv = \sum_{n=1}^{\infty} \frac{c_n}{(1+r)^n},$$

for some positive discount rate $r > 0$. Here, first the cash flow $\{c_n\}_{n\geq 1}$ is supposed to be deterministic sequence.

Uniform-based N's. Motivated by Solomon [8], define the stopping time N as follows,

$$N = \inf\left\{ n \mid \max_{1\leq i\leq n-1} U_i \leq \frac{1}{1+r},\ U_n > \frac{1}{1+r} \right\}.$$

One can see that

$$\text{pv} = \frac{1}{r} E^Q\left(c_N\right),$$

where $q_n = \dfrac{r}{(1+r)^n}$ and $Q = (q_1, q_2, \ldots)$. Here, again, Monte Carlo approach is applicable, to approximate pv.

Economic-based N's. Next, suppose that the stopping time N depends on macro-economic random variable (called cause factor) $X_n, n \geq 1$, where, it is too important to know if X_n passes threshold a or not? Define

$$N = \inf\left\{ n \mid \max_{1\leq i\leq n-1} X_i \leq a,\ X_n > a \right\}.$$

Assuming X_i's are independent and identically distributed with common continuous distribution function F, then q_n is changed to

$$q_n = P\left(N = n\right) = F^{n-1}\left(a\right)\left(1 - F\left(a\right)\right) = \frac{r}{(1+r)^n},$$

where, $F\left(a\right) = \dfrac{1}{1+r}$, equivalently, $r = \dfrac{1-F(a)}{F(a)}$. Indeed, by taking expectation of c_N with respect to probability measure Q, it is seen that

$$\frac{1}{r} E^Q\left(c_N\right) = \sum_{n=1}^{\infty} \frac{c_n}{(1+r)^n} = pv\left(c\right).$$

It is also easy to see that

$$\begin{cases} \frac{1}{r}E^Q\left(c_N^2\right) = pv\left(c^2\right), \\ \text{var}^Q\left(c_N\right) = r\{pv\left(c^2\right) - r(pv(c))^2\}. \end{cases}$$

The next proposition summarizes the above discussion.

Proposition 1. *Let $c = \{c_n\}_{n\geq 1}$ be deterministic cash flow series and N is a stopping time defined as the first time at which macro-economic variable X_i passes threshold a, i.e.*

$$N = \inf\left\{n \mid \max_{1\leq i\leq n-1} X_i \leq a,\ X_n > a\right\}.$$

Then, the following parities hold

$$E^Q\left(c_N\right) = r \times pv(c),$$

$$\text{var}^Q\left(c_N\right) = r\left\{pv\left(c^2\right) - r(pv(c))^2\right\},$$

where $r = \dfrac{1-F(a)}{F(a)}$, at which F is distribution function of X_i. Notice that $q_n = \dfrac{r}{(1+r)^n}$ and $Q = (q_1, q_2, \ldots)$.

Again, this proposition helps us to simulate $E^Q\left(c_N\right)$ and $\text{var}^Q\left(c_N\right)$, using the Monte Carlo simulation.

The rest of paper is organized as follows. In the next section, the dependency structure is considered among factors and the same theoretical results about c_N is proposed. To this end, two procedures are presented to obtain $P\left(N = n\right)$. A financial application of theoretical results is given in Section 3. Finally Section 4 concludes.

2. Dependent causes. In the previous section, it was assumed that factors X_i's were independent. However, it is not true in practice. Considering a dependency structure among U_i's, where $U_i = F(X_i)$, it reminds the copula approach to analyze this problem. However, to avoid the complexity of copula functions, an alternative method, i.e., the auto-regressive to anything (ARTA) method of Cario and Nelson [3] is advised. Since X_i's have continuous distribution, thus U_i's have uniform distribution and then $W_i = \Phi^{-1}\left(F\left(X_i\right)\right)$,$i = 1, \ldots, n$ have normal distributions and ARTA models are ready to be fitted, in this case.

2.1. ARTA model. To apply the ARTA model to W_i's, it is necessary they are mean corrected. Let W'' be the sample mean of W_i's and suppose that

mean corrected quantities $V_i = W_i - W''$ are represented by an ARTA model, as follows

$$V_i = \rho V_{i-1} + \zeta_i, i \geq 1,$$

where $|\rho| < 1$, Φ^{-1} is the inverse distribution function of standard normal distribution and ζ_i's are assumed to have common normal distribution with zero mean and variance $\sigma_\zeta^2 < \infty$. Let $Z_i = \dfrac{V_i}{\sigma_\zeta}$. Then,

$$Z_i = \rho Z_{i-1} + \varepsilon_i, i \geq 1,$$

where ε_i's are independent and standard normally distributed random variables. Consider

$$b_n = P\left(\max_{1\leq i\leq n} X_i \leq a\right) = P\left(\max_{1\leq i\leq n} Z_i \leq l\right),$$

where $l = \dfrac{\Phi^{-1}(F(a))}{\sigma_\zeta}$ and notice that

$$N = \inf\left\{n \mid \max_{1\leq i\leq n-1} Z_i \leq l, Z_n > l\right\}.$$

Then,

$$b_{n-1} = b_n + P(N = n).$$

Thus, to find $P(N = n)$, it is enough to find b_n and compute the successive backward difference

$$P(N = n) = b_{n-1} - b_n.$$

As soon as, $P(N = n)$ is found, then letting

$$P(N = n) = \frac{1}{1 + r_n},$$

then, $E^Q(c_N)$ is the present value of $\{c_n\}_{n\geq 1}$ with discount rate $\{r_n\}_{n\geq 1}$,

$$r_n = \frac{1 - P(N = n)}{P(N = n)}.$$

2.2. Computation of b_n. Hereafter, two procedures including Monte Carlo simulation and chi-squared approximation methods are proposed to find b_n's, which were necessary in fitting ARTA models.

Procedure 1 (Monte Carlo simulation)**.** Suppose that $X_0 = x_0$ is observed and let

$$z_0 = \Phi^{-1}(F(x_0))/\sigma_\zeta$$

be the initial value of above-mentioned first order auto-regressive process. By recursive solution of $Z_i = \rho Z_{i-1} + \varepsilon_i$, $i \geq 1$, then

$$Z_i = \varepsilon_i + \rho\varepsilon_{i-1} + \cdots + \rho^{i-1}\varepsilon_1 + \rho^i z_0, \quad i \geq 1.$$

Notice that $(Z_1, \ldots, Z_n)$ has multivariate normal distribution with mean vector $(\rho z_0, \rho^2 z_0, \ldots, \rho^n z_0)$ and covariance matrix Ω, where

$$\Omega_{ij} = \rho^{|i-j|}, i, j \geq 1.$$

The mean, variances and covariance's are derived with respect to probability measure P. To find, distribution of

$$\max_{1\leq i\leq n} Z_i$$

and consequently b_n's, the Monte Carlo simulation is used as follows.

Monte Carlo algorithm. (a) Generate R samples of n-variate normal distribution with mean $(\rho z_0, \rho^2 z_0, \ldots, \rho^n z_0)$ and covariance matrix Ω, where

$$\Omega_{ij} = \rho^{|i-j|}, v \quad i, j \geq 1.$$

(b) For each Monte Carlo sample (r-th sample), compute $\max\limits_{1\leq i\leq n} Z_i$ and use the empirical distribution of maximums to find b_n.

Remark 2. The above mentioned Monte Carlo procedure is critical for finding b_n's. However, it is a time consuming method, since Monte Carlo methods should be repeated to find distribution $\max\limits_{1\leq i\leq n} Z_i$ for each sample size n. To overcome this difficulty, a well-known parametric candidate distribution is suggested for $\max\limits_{1\leq i\leq n} Z_i$ and the parameters of candidate distribution are estimated using the Monte Carlo method for some selected n's, and estimated parameters are represented as functions of sample size n. This procedure is referred as response surface methodology (RSM) technique. Procedure 2 describes this method.

Procedure 2 (Chi-squared Approximation)**.** Here, following Conniffe and Spencer [4], the candidate distribution for squared

$$M_n = \left(\max_{1\leq i\leq n} Z_i\right)^2,$$

is a ϕ scaled chi-square $\chi^2_{(k)}$ distribution with k degrees of freedom *i.e.*, $\phi \times \chi^2_{(k)}$. Let $E^P(M_n) = \mu_M$ and variance $var^P(M_n) = \sigma^2_M$. Notice that using the method of moment estimate approach, then,

$$\mu_M = k\phi \ \text{ and } \ \sigma^2_M = 2k\phi^2.$$

Therefore, $\phi = \dfrac{\sigma_M^2}{2\mu_M}$,$k = \dfrac{\mu_M}{\phi}$. To find parameters k,ϕ, it is enough to estimate μ_M and σ_M^2 using the Monte Carlo method for each sample size n, then compute k, ϕ, for each sample size n. Finally, fitting a function between k, ϕ and sample size n, gives the corresponding RSM functions. Alternatively, a RSM technique may be used, in practice, to find the functional form of $\mu_M(n)$,$\sigma_M^2(n)$ (i.e., the mean and variance of M_n, for various selection of sample size n's) and using

$$\phi_n = \frac{\sigma_M^2(n)}{2\mu_M(n)}, \quad k_n = \frac{\mu_M(n)}{\phi}$$

functions ϕ_n and k_n are derived.

The following remark gives more descriptions.

Remark 3. In some cases, the theoretical model that relates predictor variables to a response variable is not available. In these situations, the relation between predictors and response should be obtained, empirically, usually by running a Monte Carlo simulation. This method is called the RSM. For more details, see [5].

3. Financial application. Consider a bond issued by *Amazon Co.* with known face value. The face value is assumed to be one dollar without loss of generality and deterministic coupon series A_i, $i \geq 1$. Let N be the first time at which return process R_i's of stock price of *Amazon* passes the threshold a, say $a = 0.02$. These thresholds are too important for traders of stocks, which this application is far from of our scope in this section. Instead, it is interested to approximate the expected present value of stopped coupon process A_N. Dataset contains 504 observations of daily stock return of *Amazon.* for period of 21 May 2018 to 20 May 2020. The *skew* and $kurtosis - 3$ measures of returns are 0.00246 and 3.106, respectively which indicates that R_i's have light-tailed distribution and it is not normal. To filt ARTA model, it is necessary R_i's have common normal distribution. As mentioned in section 2, one way is to find F i.e., distribution of R_i and use the transformation $\Phi^{-1}(F(X_i))$ to reach to normally distributed variables. Instead, a simple way is to consider zero mean variables

$$V_i = \Phi^{-1}(|R_i|), \quad i = 1, \ldots, 504,$$

and notice that *skew* and $kurtosis - 3$ measures are negligible for V_i's. The p-value of Jarque-Bera normality test for V_i's is 0.23 which verifies the normality of these variables.

Fitting an ARTA model to V_i's shows that $\rho = 0.1$ and $\sigma_\zeta = 0.433$. Using the Procedure 2, by running 1000 repeated Monte Carlo simulation, it is seen that

$\mu_M = 1.71$ and $\sigma_M^2 = 0.175$. The p-value of Kolmogorov–Smirnov test is 0.28 which implies the goodness of fit of chi-square distribution for M_n. Therefore, $\phi = 0.0517$, $k = 33.42$. These values are computed for $n = 504$ sample size. Before running a RSM technique, plotting scatter plots of $\mu_M(n)$ and $\sigma_M^2(n)$ against n suggest logarithmic candidate functions of n. In this case, it is seen that

$$\begin{cases} \mu_M(n) = 0.274 \log(n), \\ \sigma_M^2(n) = -0.05 \log(n) + 0.374. \end{cases}$$

It is assumed that $n \leq 1772$ to make sure that $\sigma_M^2(n) \geq 0$. For $n \geq 1773$, another RSM should be fitted for $\sigma_M^2(n)$ which is omitted, here. Coefficients 0.274, -0.05, and 0.374 are slopes and intercept of RSM's. Say, as $\log(n)$ one unit increases, then $\mu_M(n)$ increases in size of 0.274. For $a = 0.02$, the following plot shows the plot of $P(N = n)$,$n = 1, \ldots, 30$. As n increases, the $P(N = n)$,probability of passing this threshold at n-th time point decreases. Here, a geometric distribution is fitted to $P(N = n) = p(1-p)^{n-1}$. Notice that $E^Q(N) = 410.16$. Thus, $\hat{p} = 0.0024$.

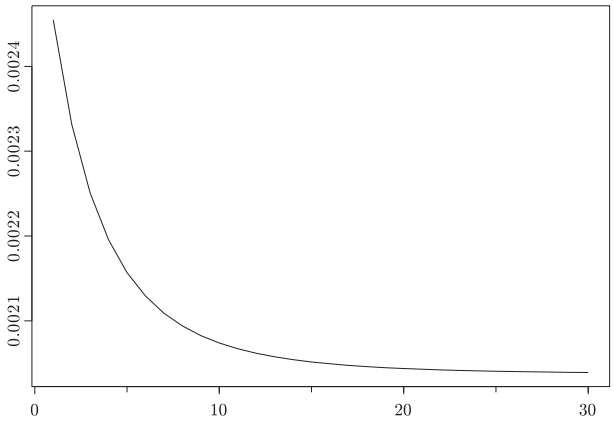

Fig. 1. Plot of $P(N = n)$

4. Continuous time setting. In previous sections, on a discrete time grid, some dualities between an amount payable once at a random time and a discounted cash flow are studied. These results have continuous time analogues for the event-based instruments. For example, for default-able derivatives, the impact of the random time appears increasing the discount rate by the intensity of the stopping time, see [1]. These results are studied in this sub-section.

4.1. Deterministic income stream. The present value of deterministic continuous income stream I_t,$t \geq 0$, assuming convergence, with continuous compounding with rate $r > 0$, is given by

$$pv = \int_0^\infty e^{-rt} I_t dt.$$

Notice that pv is the Laplace transform of I_t, i.e., $pv = \mathcal{L}(I_t)_r$. Hence, numerical methods for approximation the Laplace transforms are applicable, here, which is not considered in this paper. Instead, let T be exponentially distributed random variable with density $f_T(t) = re^{-rt}$. Therefore,

$$pv = \frac{1}{r} E^T (I_T).$$

Here, E^T stands for the expectation under distribution of T. By generating $T_i, i = 1, \ldots, M$ sample from exponential distribution, then $\dfrac{1}{rM} \sum_{i=1}^{M} I_{T_i}$ is the Monte Carlo estimate of pv. The exponential distribution is often used to model the waiting time between occurrences of two consequences events in a Poisson process. In this way, the exponential distribution is simulated like previous sections. Another method to simulate exponential distribution in this case is to use transformation $-\log(U)$, where U is uniformly distributed random number on $(0, 1)$.

4.2. Stochastic I_t's. Suppose that I_t is the price of financial asset for $t \in [0, H]$ satisfying $dIdt = 0$. For example, when I_t is the Ito process, this property is held. Using the integration by parts, it is seen that

$$pv = \frac{\int_0^H e^{-rt} dI - e^{-rH} I_H + I_0}{r}.$$

It is easy to see that

$$pv = r^{-1} \left\{ \int_0^H (e^{-rt} - e^{-rH}) dI + (1 + e^{-rH}) I_0 \right\}.$$

Assuming $I = B$, standard Brownian motion on $[0, T]$, then

$$pv = \int_0^H \frac{(e^{-rt} - e^{-rH})}{r} dB,$$

which is normally distributed with zero mean and variance $\displaystyle\int_0^T \frac{(e^{-rt} - e^{-rH})^2}{r^2} dt$. Generating samples from the mentioned normal distribution, all quantities such as $P(pv > L)$ is estimated using the Monte Carlo simulation.

5. Concluding remarks. Usually, a macro-economical or financial factor exists that the value of cash flow relates to this factor. The Monte Carlo method may be used to simulate present value of a financial cash flow stopped at some stopping time. This stopping time is defined as the first time at which macro-economical factor passes a threshold. The stochastic inherent of this factor and its behavior, most of time, may be modeled by an ARTA process. There are two approaches for running the Monte Carlo simulation, first running the direct Monte Carlo simulation and the next to use chi-squared approximation. Results are also extended in Continuous time setting.

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